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Inequalities for Eigenvalues of a General Clamped Plate Problem

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Abstract. Let D be a connected bounded domain in $\mathbb{R}^n$. Let $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \cdots$ be the eigenvalues of the following Dirichlet problem:

$$\begin{cases} \Delta^2 u(x) + V(x)u(x) = \mu\rho(x)u(x), & x \in D \\ u|_{\partial D} = \frac{\partial u}{\partial n}|_{\partial D} = 0, \end{cases}$$

where $V(x)$ is a nonnegative potential, and $\rho(x) \in C(\bar{D})$ is positive. We prove the following inequalities:

$$\mu_{k+1} \leq \frac{1}{k}\sum_{i=1}^{k}\mu_i + \Big[\frac{8(n+2)}{n^2}\Big(\frac{\rho_{\max}}{\rho_{\min}}\Big)^2\Big]^{1/2} \times \frac{1}{k}\sum_{i=1}^{k}[\mu_i(\mu_{k+1}-\mu_i)]^{1/2},$$

$$\frac{n^2k^2}{8(n+2)} \leq \Big(\frac{\rho_{\max}}{\rho_{\min}}\Big)^2\Big[\sum_{i=1}^{k}\frac{\mu_i^{1/2}}{\mu_{k+1}-\mu_i}\Big] \times \sum_{i=1}^{k}\mu_i^{1/2}.$$

1 Introduction

Let $\mathbb{R}^n$ denote an n-dimensional Euclidean space and let D be a connected bounded domain in $\mathbb{R}^n$. In order to describe vibrations of a clamped plate, we must consider an eigenvalue problem for a Dirichlet biharmonic operator, called a clamped plate problem:

$$\text{(I)} \qquad \begin{cases} \Delta^2 u(x) = \mu u(x), & x \in D \\ u|_{\partial D} = \frac{\partial u}{\partial n}|_{\partial D} = 0, \end{cases}$$

where Δ is the Laplacian in $\mathbb{R}^n$ and Δ^2 is the biharmonic operator in $\mathbb{R}^n$. Let the eigenvalues of the clamped plate problem be designated by

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \dots,$$

with corresponding real eigenfunctions $u_1, u_2, \dots, u_k, \dots$, normalized such that

$$\int_D u_i u_j = \delta_{ij}, \quad i, j = 1, 2, 3, \dots.$$

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For this clamped plate problem, in 1956 Payne, Pólya, and Weinberger [6] established an inequality for the biharmonic operator Δ^2. Indeed, they proved

$$\text{(i)} \qquad \mu_{k+1} \leq \mu_k + \frac{8(n+2)}{n^2}\frac{1}{k}\sum_{i=1}^{k}\mu_i.$$

As a generalization of their result, in 1984 Hile and Yeh [4] obtained

$$\text{(ii)} \qquad \sum_{i=1}^{k}\frac{\mu_i^{\frac{1}{2}}}{\mu_{k+1}-\mu_i} \geq \frac{n^2k^{\frac{3}{2}}}{8(n+2)}\Big(\sum_{i=1}^{k}\mu_i\Big)^{\frac{-1}{2}}.$$

Furthermore, in 1990, Hook [5] and Chen and Qian [1] proved, independently, that

$$\text{(iii)} \qquad \frac{n^2k^2}{8(n+2)} \leq \Big[\sum_{i=1}^{k}\frac{\mu_i^{1/2}}{\mu_{k+1}-\mu_i}\Big]\sum_{i=1}^{k}\mu_i^{1/2}.$$

In 2005, Q. M. Cheng and H. Yang [2] proved that

$$\text{(iv)} \qquad \mu_{k+1} \leq \frac{1}{k}\sum_{i=1}^{k}\mu_i + \Big[\frac{8(n+2)}{n^2}\Big]^{1/2} \times \frac{1}{k}\sum_{i=1}^{k}[\mu_i(\mu_{k+1}-\mu_i)]^{1/2}.$$

In this paper we generalize these results by considering the eigenvalue problem for a generalized clamped plate of the form:

$$\text{(II)} \qquad \begin{cases} \Delta^2 u(x) + V(x)u(x) = \mu\rho(x)u(x), & x \in D \\ u|_{\partial D} = \frac{\partial u}{\partial n}|_{\partial D} = 0, \end{cases}$$

where $V(x)$ represents a nonnegative potential and $\rho(x)$ is a positive continuous function on $\bar{D}$.

This problem has eigenvalues as above, which we shall continue to denote by $\{\mu_i\}_{i=1}^{\infty}$ such that $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \cdots$. Moreover, the corresponding real eigenfunctions $\{u_i\}_{i=1}^{\infty}$ form an orthogonal basis for $L^2(D,\rho)$, that is,

$$\int_D \rho(x)u_iu_j = \delta_{ij}, \quad i,j = 1,2,3,\dots.$$

Our goal in this paper is to generalize the inequalities mentioned above to the general biharmonic clamped plate problem (II). Indeed we prove the following theorems.

Theorem 1.1 *The eigenvalues of the clamped plate problem* (II) *satisfy the following inequality:*

$$\mu_{k+1} \leq \frac{1}{k}\sum_{i=1}^{k}\mu_i + \Big[\frac{8(n+2)}{n^2}\Big(\frac{\rho_{\max}}{\rho_{\min}}\Big)^2\Big]^{1/2} \times \frac{1}{k}\sum_{i=1}^{k}[\mu_i(\mu_{k+1}-\mu_i)]^{1/2},$$

where ρ is a positive and continuous function on $\bar{D}$ and $\rho_{\max}, \rho_{\min}$ denote the obvious quantities.

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The first inequality of Theorem 1.1 is implicit in terms of μ_{k+1}. We can conclude an explicit inequality as follows.

Corollary 1.2 *If the assumptions of Theorem* 1.1 *hold, then we have*

$$\mu_{k+1} \le \Big[1+\frac{4(n+2)}{n^2}(\frac{\rho_{\max}}{\rho_{\min}})^2\Big]\frac{1}{k}\sum_{i=1}^{k}\mu_i + \Big\{\Big[\frac{4(n+2)}{n^2}\Big(\frac{\rho_{\max}}{\rho_{\min}}\Big)^2\frac{1}{k}\sum_{i=1}^{k}\mu_i\Big]^2 - \frac{8(n+2)}{n^2}\Big(\frac{\rho_{\max}}{\rho_{\min}}\Big)^2\frac{1}{k}\sum_{i=1}^{k}\Big[\mu_i-\frac{1}{k}\sum_{j=1}^{k}\mu_j\Big]^2\Big\}^{1/2}$$

Theorem 1.3 *Under the assumptions of Theorem* 1.1, *the eigenvalues of the biharmonic problem* (II) *satisfy the following inequality:*

$$\frac{n^2k^2}{8(n+2)} \le \Big(\frac{\rho_{\max}}{\rho_{\min}}\Big)^2\Big[\sum_{i=1}^{k}\frac{\mu_i^{1/2}}{\mu_{k+1}-\mu_i}\Big]\times\sum_{i=1}^{k}\mu_i^{1/2}.$$

2 Proofs of the Results

Now we are in a position to prove the main theorems.

Proof of Theorem 1.1 Define the self-adjoint operator $\frac{1}{\rho}T$ with respect to the weighted inner product $\int_D \rho uv$, where $T=\Delta^2+V(x)$. Let $\{u_i\}_{i=1}^{\infty}$ be the eigenfunctions of $\frac{1}{\rho}T$. Orthogonality of the eigenfunctions $\{u_i\}_{i=1}^{\infty}$ with respect to the weighted inner product $\int_D \rho u_i u_j$ implies that the test functions

$$\phi_i = xu_i - \sum_{j=1}^{k} a_{ij}u_j$$

are orthogonal to u_j for $1\le i,j\le k$, where x represents any Cartesian coordinate x_l for $1\le l\le n$, and $a_{ij}=\int_D x\rho u_i u_j = a_{ji}$. In order to find an upper bound for μ_{k+1}, we use the Rayleigh–Ritz inequality [3] for the self-adjoint operator $\frac{1}{\rho}T$, *i.e.*,

$$\mu_{k+1} \le \frac{\int_D \phi_i T\phi_i}{\int_D \rho\phi_i^2}. \tag{2.1}$$

By definition of the linear transformation T, we have

$$T\phi_i = T\Big(xu_i-\sum_{j=1}^{k}a_{ij}u_j\Big) = T(xu_i)-\sum_{j=1}^{k}a_{ij}Tu_j$$
$$= xTu_i + 4(\Delta u_i)_x - \sum_{j=1}^{k}a_{ij}Tu_j = x\rho\mu_i u_i + 4(\Delta u_i)_x - \sum_{j=1}^{k}a_{ij}\rho\mu_j u_j.$$

Using integration by parts we can easily see that

$$(2.2)\qquad 4\int_D xu_i(\Delta u_i)_x = 2\int_D |\nabla u_i|^2 + 4\int_D (u_i)_x^2.$$

Using (2.2) and the orthogonality of ϕ_i and u_j with respect to the weighted inner product we obtain

$$(2.3)\qquad \int_D \phi_i T\phi_i = \mu_i \int_D \rho\phi_i^2 + 4\int \phi_i(\Delta u_i)_x$$
$$= \mu_i \int_D \rho\phi_i^2 + 2\int_D |\nabla u_i|^2 + 4\int_D (u_i)_x^2 - 4\sum_{j=1}^k a_{ij}b_{ij},$$

where $b_{ij} = \int_D (\Delta u_i)_x u_j = -b_{ji}$. We also have

$$(2.4)\qquad 4b_{ij} = 4\int_D (\Delta u_i)_x u_j = \int_D [\Delta^2(xu_i) - x(\Delta^2 u_i)]u_j$$
$$= \int_D [xu_i\Delta^2 u_j - xu_j\Delta^2 u_i] = \int_D [xu_i Tu_j - xu_j Tu_i]$$
$$= -(\mu_i - \mu_j)a_{ij}.$$

Now combining (2.3) and the Rayleigh–Ritz inequality (2.1), we have

$$(2.5)\qquad (\mu_{k+1} - \mu_i)\int_D \rho\phi_i^2 \le 2\int_D |\nabla u_i|^2 + 4\int_D (u_i)_x^2 - 4\sum_{j=1}^k a_{ij}b_{ij}.$$

On the other hand, by using integration by parts, we find

$$-2\int_D \phi_i(u_i)_x = -2\int_D \Big[xu_i - \sum_{j=1}^k a_{ij}u_j\Big](u_i)_x = \int_D u_i^2 + 2\sum_{j=1}^k a_{ij}c_{ij},$$

where $c_{ij} = \int_D (u_i)_x u_j = -c_{ji}$. Orthogonality of ϕ_i and u_j implies that

$$(2.6)\qquad \int_D u_i^2 + 2\sum_{j=1}^k a_{ij}c_{ij} = -2\int_D \phi_i(u_i)_x$$
$$= \int_D \rho^{\frac{1}{2}}\phi_i\Big[-2\rho^{\frac{-1}{2}}(u_i)_x + 2\rho^{\frac{1}{2}}\sum_{j=1}^k c_{ij}u_j\Big]$$
$$\le \int_D \Big\{\alpha\rho\phi_i^2 + \frac{1}{\alpha}\Big[-\rho^{\frac{-1}{2}}(u_i)_x + \rho^{\frac{1}{2}}\sum_{j=1}^k c_{ij}u_j\Big]^2\Big\}$$
$$\le \alpha\int_D \rho\phi_i^2 + \frac{1}{\alpha}\Big[\int_D \rho^{-1}(u_i)_x^2 - \sum_{j=1}^k c_{ij}^2\Big],$$

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where α is a positive number. Multiplying both sides of (2.6) by $(\mu_{k+1} - \mu_i)$, and combining with (2.5) we find

$$(2.7)\quad (\mu_{k+1} - \mu_i)\Big[\int_D u_i^2 + 2\sum_{j=1}^k a_{ij}c_{ij}\Big] \le \alpha\Big\{2\int_D |\nabla u_i|^2 + 4\int_D (u_i)_x^2 - 4\sum_{j=1}^k a_{ij}b_{ij}\Big\} + \frac{(\mu_{k+1} - \mu_i)}{\alpha}\Big[\int_D \rho^{-1}(u_i)_x^2 - \sum_{j=1}^k c_{ij}^2\Big].$$

Choosing $\alpha = (\mu_{k+1} - \mu_i)^{\frac{1}{2}}\alpha_1$, where $\alpha_1 > 0$, and taking the sum on i from 1 to k, we have

$$\sum_{i=1}^k (\mu_{k+1} - \mu_i)\int_D u_i^2 + 2\sum_{i=1}^k\sum_{j=1}^k (\mu_{k+1} - \mu_i)a_{ij}c_{ij} \le \alpha_1\sum_{i=1}^k (\mu_{k+1} - \mu_i)^{\frac{1}{2}}\Big\{2\int_D |\nabla u_i|^2 + 4\int_D (u_i)_x^2 - 4\sum_{j=1}^k a_{ij}b_{ij}\Big\} + \frac{1}{\alpha_1}\sum_{i=1}^k (\mu_{k+1} - \mu_i)^{\frac{1}{2}}\Big[\int_D \rho^{-1}(u_i)_x^2 - \sum_{j=1}^k c_{ij}^2\Big].$$

Defining

$$A = \sum_{i=1}^k (\mu_{k+1} - \mu_i)^{\frac{1}{2}}\Big\{\alpha_1\Big[2\int_D |\nabla u_i|^2 + 4\int_D (u_i)_x^2\Big] + \frac{1}{\alpha_1}\int_D \rho^{-1}(u_i)_x^2\Big\},$$

we have

$$(2.8)\quad \sum_{i=1}^k (\mu_{k+1} - \mu_i)\int_D u_i^2 + 2\sum_{i=1}^k\sum_{j=1}^k (\mu_{k+1} - \mu_i)a_{ij}c_{ij} \le A - 4\alpha_1\sum_{i=1}^k\sum_{j=1}^k (\mu_{k+1} - \mu_i)^{\frac{1}{2}}a_{ij}b_{ij} - \frac{1}{\alpha_1}\sum_{i=1}^k\sum_{j=1}^k (\mu_{k+1} - \mu_i)^{\frac{1}{2}}c_{ij}^2.$$

Since $a_{ij} = a_{ji}$, $c_{ij} = -c_{ji}$, we have

$$(2.9)\qquad 2\sum_{i=1}^k\sum_{j=1}^k (\mu_{k+1} - \mu_i)a_{ij}c_{ij} = -\sum_{i=1}^k\sum_{j=1}^k (\mu_i - \mu_j)a_{ij}c_{ij}.$$

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Since $4b_{ij} = -(\mu_i - \mu_j)a_{ij}$, we obtain

$$(2.10)\qquad -4\alpha_1 \sum_{i=1}^{k}\sum_{j=1}^{k}(\mu_{k+1}-\mu_i)^{\frac{1}{2}} a_{ij}b_{ij}$$
$$= \alpha_1 \sum_{i=1}^{k}\sum_{j=1}^{k}(\mu_{k+1}-\mu_i)^{\frac{1}{2}}(\mu_i-\mu_j)a_{ij}^2$$
$$= \frac{\alpha_1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\{(\mu_{k+1}-\mu_i)^{\frac{1}{2}}-(\mu_{k+1}-\mu_j)^{\frac{1}{2}}\}(\mu_i-\mu_j)a_{ij}^2$$
$$= -\frac{\alpha_1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{1}{(\mu_{k+1}-\mu_i)^{\frac{1}{2}}+(\mu_{k+1}-\mu_j)^{\frac{1}{2}}}(\mu_i-\mu_j)^2 a_{ij}^2$$

and

$$(2.11)\qquad -\frac{1}{\alpha_1}\sum_{i=1}^{k}\sum_{j=1}^{k}(\mu_{k+1}-\mu_i)^{\frac{1}{2}}c_{ij}^2 =$$
$$-\frac{1}{2\alpha_1}\sum_{i=1}^{k}\sum_{j=1}^{k}\Big\{(\mu_{k+1}-\mu_i)^{\frac{1}{2}}+(\mu_{k+1}-\mu_j)^{1/2}\Big\}c_{ij}^2.$$

On the other hand, we have

$$\sum_{i=1}^{k}\sum_{j=1}^{k}(\mu_i-\mu_j)a_{ij}c_{ij} \le \frac{\alpha_1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{1}{(\mu_{k+1}-\mu_i)^{\frac{1}{2}}+(\mu_{k+1}-\mu_j)^{\frac{1}{2}}}(\mu_i-\mu_j)^2 a_{ij}^2$$
$$+\frac{1}{2\alpha_1}\sum_{i=1}^{k}\sum_{j=1}^{k}\{(\mu_{k+1}-\mu_i)^{\frac{1}{2}}+(\mu_{k+1}-\mu_j)^{\frac{1}{2}}\}c_{ij}^2$$

Combining (2.8), (2.9), (2.10), (2.11), and (2.4) we conclude

$$(2.12)\qquad \sum_{i=1}^{k}(\mu_{k+1}-\mu_i)\int_D u_i^2 \le A.$$

On the other hand, we have

$$(2.13)\qquad \sum_{i=1}^{k}\frac{(\mu_{k+1}-\mu_i)}{\rho_{\max}} \le \sum_{i=1}^{k}(\mu_{k+1}-\mu_i)\int_D u_i^2,$$

$$(2.14)\qquad \int_D \rho^{-1}(u_i)_x^2 \le \int_D \frac{(u_i)_x^2}{\rho_{\min}},$$

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where $\rho_{\max} = \max_{x\in\bar{D}} \rho(x)$ and $\rho_{\min} = \min_{x\in\bar{D}} \rho(x)$. Since inequality (2.12) is valid for $x = x_l$, $1 \le l \le n$, then using relations (2.13) and (2.14) we find

$$\sum_{i=1}^{k} \frac{(\mu_{k+1} - \mu_i)}{\rho_{\max}} \le \sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^{\frac{1}{2}} \times \Big\{ \alpha_1 \Big[2 \int_D |\nabla u_i|^2 + 4 \int_D (u_i)^2_{x_l} \Big] + \frac{1}{\alpha_1} \int_D \frac{(u_i)^2_{x_l}}{\rho_{\min}} \Big\}.$$

Since $V(x) \ge 0$, $\int_D \rho u_i^2 = 1$, we have

$$\sum_{l=1}^{n} \int_D (u_i)^2_{x_l} = \int_D |\nabla u_i|^2 \qquad \int_D |\nabla u_i|^2 \le \rho_{\min}^{\frac{-1}{2}} \mu_i^{\frac{1}{2}} \tag{2.15}$$

because

$$\begin{aligned} \int_D |\nabla u_i|^2 &= \int_D u_i(-\Delta u_i) = \int_D \rho^{\frac{1}{2}} u_i(-\rho^{\frac{-1}{2}} \Delta u_i) \\ &\le \Big[\int \rho u_i^2 \int \rho^{-1} (\Delta u_i)^2 \Big]^{\frac{1}{2}} \le \Big[\int \rho_{\min}^{-1} u_i \Delta^2 u_i \Big]^{\frac{1}{2}} \le \Big[\int \rho_{\min}^{-1} u_i T u_i \Big]^{\frac{1}{2}} \\ &= \rho_{\min}^{\frac{-1}{2}} \mu_i^{\frac{1}{2}}. \end{aligned}$$

Therefore, by summing on l from 1 to n we obtain

$$n \sum_{i=1}^{k} \frac{(\mu_{k+1} - \mu_i)}{\rho_{\max}} \le \sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^{\frac{1}{2}} \rho_{\min}^{\frac{-1}{2}} \mu_i^{\frac{1}{2}} \Big\{ \alpha_1 [2n + 4] + \frac{1}{\alpha_1 \rho_{\min}} \Big\}.$$

Choosing $\alpha_1 = \rho_{\min}^{\frac{-1}{2}} (2n + 4)^{\frac{-1}{2}}$, we conclude the result

$$\mu_{k+1} - \frac{1}{k} \sum_{i=1}^{k} \mu_i \le \Big[\frac{8(n + 2)}{n^2} \Big(\frac{\rho_{\max}}{\rho_{\min}} \Big)^2 \Big]^{1/2} \times \frac{1}{k} \sum_{i=1}^{k} [\mu_i(\mu_{k+1} - \mu_i)]^{1/2}. \qquad \blacksquare$$

This inequality is the analog of inequality (iv) in this more general setting.

Proof of Corollary 1.2 In order to simplify the calculations, we define

$$M_k = \frac{1}{k} \sum_{i=1}^{k} \mu_i, \quad T_k = \frac{1}{k} \sum_{i=1}^{k} \mu_i^2, \quad \sigma = \frac{\rho_{\max}}{\rho_{\min}}.$$

It follows from the first inequality of Theorem 1.1 that

$$
\begin{aligned}
(\mu_{k+1}-M_k)^2 &\le \frac{8(n+2)}{n^2}\sigma^2\Big\{\frac{1}{k}\sum_{i=1}^{k}[\mu_i(\mu_{k+1}-\mu_i)]^{1/2}\Big\}^2\\
&\le \frac{8(n+2)}{n^2}\sigma^2\frac{1}{k}\sum_{i=1}^{k}\mu_i(\mu_{k+1}-\mu_i)\\
&= \frac{8(n+2)}{n^2}\sigma^2(\mu_{k+1}M_k-T_k).
\end{aligned}
$$

Now direct calculations show that

$$
\begin{aligned}
&\Big\{\mu_{k+1}-\Big[1+\frac{4(n+2)}{n^2}\sigma^2\Big]M_k\Big\}^2\\
&= (\mu_{k+1}-M_k)^2+\frac{16(n+2)^2}{n^4}\sigma^4M_k^2-\frac{8(n+2)}{n^2}\sigma^2(\mu_{k+1}-M_k)M_k\\
&\le \frac{8(n+2)}{n^2}\sigma^2(\mu_{k+1}M_k-T_k)+\frac{16(n+2)^2}{n^4}\sigma^4M_k^2\\
&\qquad -\frac{8(n+2)}{n^2}\sigma^2\mu_{k+1}M_k+\frac{8(n+2)}{n^2}\sigma^2M_k^2\\
&= \frac{8(n+2)}{n^2}\sigma^2M_k^2-\frac{8(n+2)}{n^2}\sigma^2T_k+\frac{16(n+2)^2}{n^4}\sigma^4M_k^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mu_{k+1}-\Big[1+\frac{4(n+2)}{n^2}\sigma^2\Big]M_k \le\\
&\qquad\qquad \Big\{\Big[\frac{4(n+2)}{n^2}\sigma^2M_k\Big]^2-\frac{8(n+2)}{n^2}\sigma^2(M_k-T_k)^2\Big\}^{1/2}.
\end{aligned}
$$

■

Proof of Theorem 1.3 By substituting relations (2.13) and (2.14) in relation (2.7) we find

$$
\begin{aligned}
\frac{1}{\rho_{\max}}+2\sum_{j=1}^{k}a_{ij}c_{ij} \le \frac{\alpha}{(\mu_{k+1}-\mu_i)}\Big\{2\int_D|\nabla u_i|^2+4\int_D(u_i)_x^2-4\sum_{j=1}^{k}a_{ij}b_{ij}\Big\}\\
+\frac{1}{\alpha}\Big[\int_D\frac{(u_i)_x^2}{\rho_{\min}}-\sum_{j=1}^{k}c_{ij}^2\Big].
\end{aligned}
$$

Now if we choose

$$
\alpha=\frac{(\mu_{k+1}-\mu_i)\alpha_2}{\sum_{p=1}^{k}\rho_{\min}^{-1/2}\mu_p^{1/2}},\quad \alpha_2>0,
$$

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then summing on i from 1 to k, and using relation (2.4), we find

$$\frac{k}{\rho_{\max}} + 2\sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}c_{ij} \le \frac{\alpha_2}{\sum_{p=1}^{k}\rho_{\min}^{-1/2}\mu_p^{1/2}}\Big\{\sum_{i=1}^{k}\Big[2\int_D |\nabla u_i|^2 + 4\int_D (u_i)_x^2\Big] + \sum_{i=1}^{k}\sum_{j=1}^{k}(\mu_i-\mu_j)a_{ij}^2\Big\} + \frac{\sum_{p=1}^{k}\rho_{\min}^{-1/2}\mu_p^{1/2}}{\alpha_2}\sum_{i=1}^{k}\frac{1}{(\mu_{k+1}-\mu_i)}\Big[\int_D \frac{(u_i)_x^2}{\rho_{min}} - \sum_{j=1}^{k}c_{ij}^2\Big].$$

From the antisymmetry property of c_{ij} and $(\mu_i-\mu_j)a_{ij}^2$, we have

$$2\sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}c_{ij} = 0, \quad \sum_{i=1}^{k}\sum_{j=1}^{k}(\mu_i-\mu_j)a_{ij}^2 = 0.$$

Moreover,

$$\frac{\sum_{p=1}^{k}\rho_{\min}^{-1/2}\mu_p^{1/2}}{\alpha_2}\sum_{j=1}^{k}c_{ij}^2 > 0.$$

Hence we have

$$\text{(2.16)} \quad \frac{k}{\rho_{\max}} \le \frac{\alpha_2}{\sum_{p=1}^{k}\rho_{\min}^{-1/2}\mu_p^{1/2}}\sum_{i=1}^{k}\Big\{2\int_D |\nabla u_i|^2 + 4\int_D (u_i)_x^2\Big\} + \frac{\sum_{p=1}^{k}\rho_{min}^{-1/2}\mu_p^{1/2}}{\alpha_2}\sum_{i=1}^{k}\int_D \frac{(u_i)_x^2}{\rho_{\min}(\mu_{k+1}-\mu_i)}.$$

Since inequality (2.16) is valid for $x = x_l$, $1 \le l \le n$, then summing on l from 1 to n and using the relations (2.15), we have

$$\text{(2.17)} \quad \frac{nk}{\rho_{\max}} \le \frac{\alpha_2\rho_{\min}^{1/2}}{\sum_{p=1}^{k}\mu_p^{1/2}}\Big\{2n\sum_{i=1}^{k}\rho_{\min}^{-1/2}\mu_i^{\frac{1}{2}} + 4\sum_{i=1}^{k}\rho_{\min}^{-1/2}\mu_i^{\frac{1}{2}}\Big\} + \frac{\rho_{\min}^{-1/2}\sum_{p=1}^{k}\mu_p^{1/2}}{\alpha_2}\sum_{i=1}^{k}\frac{\rho_{\min}^{-1/2}\mu_i^{\frac{1}{2}}}{\rho_{\min}(\mu_{k+1}-\mu_i)}.$$

Simplifying (2.17) implies that

$$\frac{nk}{\rho_{\max}} \le \alpha_2(2n+4) + \frac{\sum_{p=1}^{k}\rho_{\min}^{-1/2}\mu_p^{1/2}}{\alpha_2}\sum_{i=1}^{k}\frac{\rho_{\min}^{-1/2}\mu_i^{1/2}}{\rho_{\min}(\mu_{k+1}-\mu_i)}.$$

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By choosing $\alpha_2 = \frac{nk}{4(n+2)\rho_{\max}}$ we obtain the desired result

$$\frac{n^2k^2}{8(n+2)} \leq \Big(\frac{\rho_{\max}}{\rho_{\min}}\Big)^2 \Big[\sum_{i=1}^{k} \frac{\mu_i^{1/2}}{\mu_{k+1} - \mu_i}\Big] \times \sum_{i=1}^{k} \mu_i^{1/2}. \qquad \blacksquare$$

This inequality is the analog of inequality (iii) in this more general setting.

Remark The inequalities similar to (i) and (ii) in this more general case can be obtained if we replace $\frac{8(n+2)}{n^2}$ by $\frac{8(n+2)}{n^2}\big(\frac{\rho_{\max}}{\rho_{\min}}\big)^2$. Note that this also true of (iii) and (iv).

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