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Local Systems on $\mathbb{P}^1 - S$ for S a Finite Set

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Abstract. I give the necessary and sufficient conditions for the existence of Unitary local systems with prescribed local monodromies on $\mathbb{P}^1 - S$ where S is a finite set. This is used to give an algorithm to decide if a rigid local system on $\mathbb{P}^1 - S$ has finite global monodromy, thereby answering a question of N. Katz. The methods of this article (use of Harder–Narasimhan filtrations) are used to strengthen Klyachko's theorem on sums of Hermitian matrices. In the Appendix, I give a reformulation of Mehta–Seshadri theorem in the SU(n) setting.

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1. Introduction

The following question of N. Katz [7] is the starting point of this article. Let L be a rigid local system on $\mathbb{P}^1 - \{p_1, \dots, p_s\}$ (see the section on rigid local systems for the definitions) with finite monodromies at the punctures. When does the local system L have finite global monodromy (in terms of the Jordan canonical forms of the local monodromies around the punctures)? Clearly the local monodromies should be of finite order. But of course there are more conditions.

The above problem is related to the following problem concerning SU(n): Let $\bar{A}_1, \bar{A}_2 \dots \bar{A}_s$ be conjugacy classes in SU(n). When can we lift to matrices A_i in SU(n) with conjugacy class of $A_i = \bar{A}_i$ and so that $A_1 A_2 \dots A_s = I$? We will see that an affirmative answer for all Gal($\bar{\mathbb{Q}}/\mathbb{Q}$) conjugates of the local monodromies for the second problem is what is needed for the first problem.

The SU(n) problem is related to quantum cohomology as was independently observed by Agnihotri and Woodward [1] and the author (with help from Pandharipande on quantum cohomology). I. Biswas [3] had previously considered and solved this problem for SU(2).

Firstly, by the theorem of Mehta and Seshadri (modified to the SU(n) setting) the existence problem for lifting is the same as the existence of a semistable parabolic vector bundle with prescribed local weights on $\mathbb{P}^1$. Using the openness of semi-stability, this is reduced to checking if the trivial vector bundle with generic flags and the prescribed weights is semistable. This means that no subbundle should contradict semi-stability. The subbundles of a given rank and degree of the trivial vector bundle of rank n form a open subset of a Quot scheme and also of the moduli space of maps from $\mathbb{P}^1$ to an appropriate Grassmann variety. Now the question of existence of a subbundle is translated into existence of a map from $\mathbb{P}^1$ to a Grassmann variety, such that the prescribed points on $\mathbb{P}^1$ go to appropriate generic Schubert cycles. The existence is (with a little bit more work) realized as the nonvanishing of certain Gromov–Witten numbers. The Harder–Narasimhan filtration is used to conclude that only inequalities corresponding to intersections which are numerically one need to be considered.

The above problem is related to one considered by Klyachko [9]. Using the Harder–Narasimhan filtration in that context lets us conclude that only intersections which are numerically one need to be considered. The reduction to this restricted set of inequalities in various contexts was triggered by an electronic message from C. Woodward to the author in which Woodward noticed this (and proved it) in a special case of Klyachko's problem.

In the following article, to make consistent the subscripts and superscripts, I have adopted the following rule. The points of the curve are always subscripts and the indices of the filtration are superscripts. For example, an element of a filtration of a fiber of a vector bundle V at a point x is denoted as V_x^i. A basepoint u on $\mathbb{P}^1$ is chosen and fixed throughout this article. The weights are always taken to be nonincreasing (except in the Appendix where we recall the definition used by Mehta and Seshadri in their paper [10]).

2. A Precise Formulation of the SU(n) Problem

Conjugacy classes in SU(n) form a simplex of dimension $n-1$. We call two matrices A, B in SU(n) conjugate if there is a $g \in \mathrm{SU}(n)$ so that $A = gBg^{-1}$. The equivalence classes under conjugation of SU(n) has a natural structure of a simplex realized as $\Delta(n) = \{(a^1, \ldots, a^n) \colon a^1 \geqslant a^2 \cdots \geqslant a^n \geqslant a^1 - 1, \sum_{i=1}^n a^i = 0\}$ where the correspondence takes (a^i) to the conjugacy class of the diagonal matrix with $\exp(2\pi i a^i)$ on the diagonal. This map from the points of the simplex to conjugacy classes is a homeomorphism (quotient topology on conjugacy classes).

Denote by $\Gamma_n(s)$ the subset of $\Delta(n)^s$ consisting of conjugacy classes $\bar{A}_1, \ldots, \bar{A}_s$ which can be lifted to matrices $A_1, \ldots, A_s$ so that conjugacy class of $A_i = \bar{A}_i$ for all i and $A_1 \ldots A_s = I$.

GEOMETRIC PICTURE 1. $\pi_1(\mathbb{P}^1 - \{p_1, \ldots, p_s\}, u) =$ *free group on generators* $\gamma_1, \ldots, \gamma_s$ *with the relation* $\gamma_1 \cdots \gamma_s = I$ *where the* γ_i *are loops around each puncture (appropriately chosen and oriented). Then we are asking if there is a representation* $\rho \colon \pi_1(\mathbb{P}^1 - \{p_1, \ldots, p_s\}, u) \to \mathrm{SU}(n)$ *so that the local monodromies (being the conjugacy classes of* $\rho(\gamma_i)$*) are the prescribed ones.*

We provide a description of $\Gamma_n(s)$ in terms of inequalities controlled by Quantum Schubert calculus in Section 6.

3. The Theorem of Mehta and Seshadri

The theorem of Narasimhan and Seshadri gives bijective correspondence between irreducible unitary representations of the fundamental group of a compact Riemann surface and the set of stable vector bundles of degree 0. For punctured curves, the space of irreducible unitary representations is also in correspondence with certain algebraic objects on the curve, called stable parabolic bundles of degree 0 (theorem of Mehta and Seshadri). We first set up some notations. Note that we state the theorem for SU(n) representations, which follows from the original form of Mehta and Seshadri by rescaling. (A proof is included in the Appendix.)

We will assume for the definition that X is a projective smooth curve over $\mathbb{C}$ and that $p_1, p_2 \dots, p_s$ are distinct points on X.

DEFINITION 1. A 'complete' parabolic vector bundle on a curve X with s marked points (the p_i's) consists of a vector bundle V of rank n on X and the following additional data:

(1) For all $x \in \{p_1, \dots, p_s\}$ a filtration of the fiber V_x: $V_x = V_x^n \supset V_x^{n-1} \supset \cdots \supset V_x^0 = \{0\}$ by vector subspaces with strict inclusions.
(2) Weights $a_x^1 \geqslant a_x^2 \geqslant \cdots \geqslant a_x^n \geqslant a_x^1 - 1$.

Also, define the parabolic degree of V as

$$\mathrm{Par}(V) = \deg(V) + \sum_x \sum_{i=1}^n a_x^i.$$

Define the slope $\mu = (\mathrm{Par}(V))/(\mathrm{rank}(V))$.

A subbundle W of V gets an induced parabolic structure by intersecting the above filtration of V_x with the fibers of W, looking at breaks in the sequence

$$W_x = W \cap V_x^n \supseteq W_x \cap V_x^{n-1} \supseteq \cdots \supseteq W_x \cap V_x^0 = \{0\}.$$

and assigning the highest weight possible. That is we get from the sequence above a complete flag on W_x and assign to W_x^i the weight $b_x^i = a_x^j$, where j is the smallest number satisfying $W_x^i = W_x \cap V_x^j$.

We can now state the form of the Mehta–Seshadri theorem useful for our purposes Let $a_{p_j}^i$ be numbers satisfying $a_{p_j}^1 \geqslant a_{p_j}^2 \cdots \geqslant a_{p_j}^n \geqslant a_{p_j}^1 - 1$, $\sum_{i=1}^n a_{p_j}^i = 0$ for each p_j.

THEOREM 1 (Mehta–Seshadri [10]). *There exists an irreducible representation* $\rho: \pi_1(\mathbb{P}^1 - \{p_1, \dots, p_s\}, u) \to \mathrm{SU}(n)$ *with the conjugacy class of* $\rho(\gamma_j)$ *given by* $(a_{p_j}^1, \dots, a_{p_j}^n)$ *for* $j = 1, 2, \dots, s$ *if and only if there is a stable parabolic bundle of degree*

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0 with the weights again given by $(a^1_{p_j}, \ldots, a^n_{p_j})$ *for* $j = 1, 2, \ldots, s$. *Moreover, there exists a semistable vector bundle with these weights if and only if there is a (possibly reducible) representation of the fundamental group in* SU(n) *with the corresponding conjugacy classes (as before).*

Recall that a bundle of parabolic degree 0 is called *stable* (resp. *semistable*) if all its subbundles have negative (resp. nonpositive) parabolic degree.

The above is not the form given in [10] but can be obtained from it. This is done in the Appendix.

4. Some Elementary Reductions and Gromov–Witten Invariants

Let $\Gamma_n(s)^{\text{int}}$ be the set of points in $\Gamma_n(s)$ which are in the interior of $\Delta(n)^s$. This corresponds to Jordan canonical forms with distinct eigenvalues. The first lemma is that

LEMMA 1. *The (topological) closure of* $\Gamma_n(s)^{\text{int}}$ *in* $\Delta(n)^s$ *is* $\Gamma_n(s)$.

Proof. We use the fact that in a connected semi-simple group, any nonempty open subset (in the usual topology) of a maximal compact subgroup is Zariski dense.

Let $G = \text{SL}(n)^{s-1}$ and $K = \text{SU}(n)^{s-1}$. Let $T = \{(g_1, \ldots, g_{s-1}) : g_i \in G$, the g_i's have characteristic polynomials with distinct eigenvalues, and also the product $g_1 \ldots g_{s-1}$ has characteristic polynomial with distinct eigenvalues$\}$. T is nonempty and Zariski dense because T is an intersection of nonempty Zariski open subsets.

$\Gamma_n(s)$ is closed in $\Delta(n)^s$ because it is the continuous image of the compact set K. The map sends $g_1, \ldots, g_{s-1}$ to the point (of conjugacy classes) $(g_1, g_2, \ldots, g_{s-1}, g_s)$ where $g_s = (g_1 \cdots g_{s-1})^{-1}$. It suffices to show that if U is an open (in the usual topology) subset of K containing $(g_1, \ldots, g_{s-1})$, then U intersects T. This follows from Zariski density of U and the Zariski-openness of T. □

It is also useful to show that elements of $\Gamma_n(s)$ which come from irreducible representations are dense in it. This follows, since in G (as in the proof above), the elements which have a common eigenspace is a Zariski closed set. That is, if we look at the subset of G formed by $(g_1, \ldots, g_{s-1})$ such that g_i leave a nonzero subspace of $\mathbb{C}^n$ invariant, then this subset is Zariski closed and not all of G. Substitute the complement of this set for T in the above proof and we obtain the assertion above.

The conclusion from the above discussion is that we can restrict ourselves to elements of $\Gamma_n(s)$ which have distinct eigenvalues and correspond to irreducible representations. The closure of the subset we obtain this way will exactly equal $\Gamma_n(s)$. This problem algebraizes by the Mehta–Seshadri theorem. But for technical reasons we will consider semistable bundles too.

Our problem now is therefore, given $a^i_{p_j}$ with sum over i for each p_j zero and $a^1_{p_j} \geqslant \cdots \geqslant a^n_{p_j} \geqslant a^1_{p_j} - 1$, we want to know if there is a stable parabolic vector bundle

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on $\mathbb{P}^1$ with the $a^i_{p_j}$ as the parabolic data. The openness of (semi)stability implies that we may assume the vector bundle to be trivial and with general flags:

THEOREM 2. *If there is a (semi)stable parabolic bundle with the given data, then there is also a (semi)stable parabolic vector bundle with the same data and so that the underlying vector bundle is trivial and the flags at the points p_j are generic.*

Proof. The degree of the vector bundle is 0. It is well known that (Theorem 6.) a vector bundle on $\mathbb{P}^1$ sits in a family where the generic member decomposes as $(\mathcal{O}(a^1) \oplus \cdots \oplus \mathcal{O}(a^n))$ and the distance between a^i and a^j is less than or equal to 1. In our situation the degree $= 0$ forces the a^i to be all zero. We can now vary our original stable vector bundle in such a family with the flags also varying. The openness of stability in families, then, gives us the statement of the theorem. □

THEOREM 3 (Harder and Narasimhan). *Suppose E is a parabolic vector bundle on a curve X. Let μ denote* $\sup\{(\mu(V)$ *: V is subbundle of $E\}$. Let V be a subbundle of maximum rank among subbundles with slope μ. Then V is unique, that is, any subbundle of the same rank as V has smaller parabolic slope.*

Proof. See the paper of Mehta and Seshadri [10]. □

We set up some notation for future use.

DEFINITION 2. (1) Let $Q(d, r, n) =$ the Quot Scheme of degree d, rank $n - r$ quotients of $\mathcal{O}^n$. The set of degree $-d$, rank r subbundles of $\mathcal{O}^n$ is a subscheme of this subscheme (by dualizing).

(2) Let $M(d, r, n) =$ moduli space of maps from $\mathbb{P}^1$ to $\mathrm{Gr}(r, n)$ of degree d

We set up notation for certain subvarieties of Grassmann varieties which generate the cohomology of these varieties [5].

DEFINITION 3. For a conjugacy class $\bar{A}$ of $\mathrm{SU}(n)$,and a subset I of $\{1, 2, \ldots, n\}$ of cardinality r, define $\lambda_I(\bar{A}) = \sum_{i \in I} \lambda_i(\bar{A})$ where $\bar{A} = (\lambda_1, \ldots, \lambda_n) \in \Delta(n)$.

The subsets I above also correspond to Schubert subvarieties of the Grassmann variety:

DEFINITION 4. Let $I = \{i_1 < i_2 < \cdots < i_r\}$. Let $F^\bullet$ be a complete flag in a n-dimensional vector space E. Now let $\Omega_I(F^\bullet) = \{L \in \mathrm{Gr}(r, E) \mid \dim(L \cap F^{i_t}) \geqslant t$ for $1 \leqslant t \leqslant r\}$. We denote the cohomology class of this subvariety by σ_I. The codimension of this subvariety is the number of pairs (j, i) with $i < j$.

We note the following theorem:

THEOREM 4. *Let* $0 \to \mathcal{S} \to \mathcal{O}^n \to \mathcal{Q} \to 0$, *where* $\mathcal{S}$ *is the universal subbundle of rank* r *and* $\mathcal{Q}$ *the quotient be the universal sequence on the Grassmann variety* $\mathrm{Gr}(r, n)$. *Then*

(1) *The dual of* $\bigwedge^r \mathcal{S}$ *is ample on* $\mathrm{Gr}(r, n)$.
(2) $\mathrm{Pic}(\mathrm{Gr}(r, n))$ *is generated by* $-c_1(\mathcal{S})$.
(3) *If* $\rho: \mathbb{P}^1 \to \mathrm{Gr}(r, n)$, *then* $\deg(\rho^*\mathcal{S}) = \rho_*[\mathbb{P}^1] \cap (-c_1(\mathcal{S}))$

Note that $-\rho_*[\mathbb{P}^1] \cap (-c_1(\mathcal{S}))$ *is called the degree of the map* ρ.

Proof. The first statement is clear because under the Plücker embedding $\wedge^r S$ is the pullback of $\mathcal{O}(-1)$.

The second is standard [5].

The third follows from the projection formula

$$\rho_*[\mathbb{P}^1] \cap c_1(\mathcal{S}) = \rho_*([\mathbb{P}^1] \cap c_1(\rho^*\mathcal{S})).$$

Note that ρ_* on dimension 0 cycles is the identity (numerically) and that $[\mathbb{P}^1] \cap c_1(\rho^*\mathcal{S}) = \deg(c_1(\rho^*(\mathcal{S}))$ numerically. □

The inequalities defining $\Gamma_n(s)$ can be written in terms of Gromov–Witten numbers of the Grassmannian. Let $I_1, I_2, \ldots, I_s$ be subsets of $\{1, 2, \ldots, n\}$ of cardinality r and $p_1, \ldots, p_s$ general points of $\mathbb{P}^1$. The Gromov–Witten number $\langle \sigma_{I^1}, \ldots, \sigma_{I^s} \rangle_d$ is defined to be, for generic flags $F^\bullet_{p_i}$ $(i = 1, 2, \ldots, s)$, the number of maps $\mathbb{P}^1 \to \mathrm{Gr}(r, n)$ of degree d such that the image of p_i is in the Schubert variety $\Omega_I(F^\bullet_{p_i})$. We define the number to be zero if this number is infinite (note the genericity assumption on the flags).

The Gromov–Witten invariants also have an interpretation in terms of vector bundles on $\mathbb{P}^1$. Let $V = \mathcal{O}^n$ be a vector bundle on $\mathbb{P}^1$. We have a universal sequence of vector bundles on $\mathrm{Gr}(r, n)$

$$0 \to \mathcal{S} \to \mathcal{O}^n \to \mathcal{Q} \to 0$$

where $\mathcal{S}$ is the universal subbundle of rank r and $\mathcal{Q}$ the quotient. It is now easy to verify that degree d maps $\rho: \mathbb{P}^1 \to \mathrm{Gr}(r, n)$ are in 1–1 correspondence with subbundles of rank r and degree $-d$ of V by pulling back the universal sequence via the map ρ. Also, the image of point p_i under this map is exactly the fiber of this subbundle at p_i. It is useful to fix an n-dimensional space T and identify all fibers of the bundle V with T. To obtain the other direction of this correspondence note that subbundles $\mathcal{S}$ correspond to a family of r dimensional subspaces of T(over $\mathbb{P}^1$).

The number defined above, therefore counts the number (zero if infinite) of subbundles E of V of degree $-d$ and rank r such that the fiber E_{p_i} as a subset of T lies in the Schubert variety $\Omega_I(F^\bullet_{p_i})$.

The Gromov–Witten invariants are computable. We mention two important properties which makes their calculation a finite process: the factorization formula and the associativity of the small quantum cohomology ring. We refer to the

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Mittag–Leffler notes [11]. A. Bertram gives an effective description of the quantum cohomology ring of Grassmann varieties [2].

5. Some Genericity Statements

We collect a few standard facts and supply proofs.

THEOREM 5. *Let S be a scheme over $\mathbb{C}$. Let V be a vector bundle over $\mathbb{P}^1 \times S$. Let $s \in S$ be a geometric point such that V_s decomposes as a direct sum of $\mathcal{O}(a)$ and $\mathcal{O}(a+1)$. Then V_t decomposes in the same way fot t varying in a neighborhood of s.*

Proof. Twist by $\mathcal{O}(-a-1)$ to reduce to the case $a=-1$. Then $h^1(\mathbb{P}^1, \mathcal{O})$, $h^1(\mathbb{P}^1, \mathcal{O}(-1))$ and $h^0(\mathbb{P}^1, \mathcal{O}(-1))$ are zero. The semicontinuity theorem [H] tells us that $h^1(\mathbb{P}^1, V_t)$ and $h^0(\mathbb{P}^1, V_t(-1))$ are 0 in a neighbourhood of s. Also, V_t is a direct sum of $\mathcal{O}(i)$ for various i. The vanishing of the h^1 and the h^0 of the twisted sheaf forces the i's appearing to be either 0 or -1. Hence, the statement.

LEMMA 2 (Serre). *Let X be a smooth complete curve over an algebraically closed field k. Let V be a vector bundle on X generated by its global sections. There is an exact sequence:*

$$0 \longrightarrow \mathcal{O}^{\oplus(n-1)} \longrightarrow V \longrightarrow \det(V) \longrightarrow 0.$$

Proof. Let $x \in X$. We have a surjective map $H^0(X, V) \to V_x$. The kernel has dimension $h^0(V) - n$. Let $P = \mathbb{P}(H^0(X, V))$ and consider the kernel as a subspace of P. As x varies over X the union of these subspaces form a subvariety K of P of codimension $n-1$.

Since P has a transitive action of $\mathrm{PGL}(h^0)$, we can find a linear subspace of dimension $n-2$ which does not intersect the set K. We lift this to a subspace K' of $H^0(X, V)$ of dimension $n-1$. This gives an injection $K' \otimes_{\mathbb{C}} \mathcal{O}_X \to V$ of vector bundles. The quotient is easily seen to be $\det(V)$. □

THEOREM 6. *Let V be a vector bundle on $\mathbb{P}^1$. Then there is a family containing V of vector bundles over an irreducible base whose generic member breaks up as direct sum of $\mathcal{O}(a)$ such that the distance between any two a's appearing is at most 1.*

Proof. Let us assume that V is the direct sum of $\mathcal{O}(a_i)$ for $i = 1, \ldots, n$. Assume (by twisting) that all the a_i's are positive. Then V is generated by its global sections.

Let $d = \sum_{j=1}^n a_i$. Also write $d = nq + r$ where $0 \leqslant r < n$. Let W be the direct sum of $n-r$ copies of $\mathcal{O}(q)$ and r copies of $\mathcal{O}(q+1)$. Then $\det(V) = \det(W)$. W is generated by its global sections.

By the lemma of Serre above, we have exact sequences

$$0 \longrightarrow \mathcal{O}^{\oplus(n-1)} \longrightarrow V \longrightarrow \det(V) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}^{\oplus(n-1)} \longrightarrow W \longrightarrow \det(W) \longrightarrow 0.$$

Hence, we see that V and W are members of the connected irreducible family of extensions $\mathrm{Ext}(\det(V), \mathcal{O}^{\oplus(n-1)})$ of $\det(V) = \det(W)$ by $\mathcal{O}^{\oplus(n-1)}$. □

We now make explicit the genericity assumptions. The following moving lemma is well known.

LEMMA 3 (A Moving lemma). *Let X be a smooth variety over $\mathbb{C}$, G a connected algebraic group acting transitively on X, $\{z_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ two finite collections of locally closed subvarieties of X. Then there exists a dense open subset U of G so that for every $\sigma \in U$*

(1) *all the intersections $\sigma y_j \cap z_i$ are proper,*
(2) *if y_j and z_i are smooth then $\sigma y_j \cap z_i$ is tranverse.*

Proof. For a proof see Kleiman [8]. □

In our situation $X = \prod_{p_i} \mathrm{Gr}(r, n)$, $G = \prod_{p_i} \mathrm{SL}(n)$. Fix flags at each point. Let the set $\{y_j\}_{j\in J}$ consist of all the products of Schubert varieties $\Omega_{I_1} \times \cdots \times \Omega_{I_s}$ and the singular locus of each product.

Next we choose the z_i's as follows: If V is a parabolic bundle of degree zero on $\mathbb{P}^1$ with parabolic structures at $p_1, \ldots, p_s$ with the weights summing to 0 at each point, and W is a subbundle with $\mathrm{par}(W) > 0$ then it is easy to see that $\deg(W) \leqslant ns$. Now, for every $d \leqslant ns$ let $\phi_d\colon M(d, r, n) \to X$ be the product of the evaluation maps at p_i and choose a proper closed subset Z_d of $\overline{\mathrm{im}(\phi_d)}$ containing the singular locus of $\overline{\mathrm{im}(\phi_d)}$ and the complement of $\mathrm{im}(\phi_d)$ (such a Z_d exists by a theorem of Chevalley). Then let $\{z_i\}_{i\in I}$ be the collection of $\overline{\mathrm{im}(\phi_d)}$'s and Z_d's for $d \leqslant ns$.

The σ's obtained by the theorem have the property that whenever any $\sigma y_j \cap z_i$ is zero-dimensional, each point of intersection has multiplicity one (the intersection is transverse, since it takes place on the smooth parts of y_j's and z_i's, by dimension arguments). For each r we thus get an open subset U_r of G. The intersection U of all U_r's is a nonempty open subset of G such that if $\sigma \in U$ and $\sigma y_{j,r} \cap z_{i,r}$ has dimension zero, then all the the multiplicities are one.

6. The Main Theorem

THEOREM 7 (The Main Theorem). *Let $(\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_s) \in \Delta(n)^s$. Then there exist $A_1, \ldots, A_s \in \mathrm{SU}(n)$ with conjugacy class of $A_i = \bar{A}_i$ and $A_1 A_2 \cdots A_s = I$ if and only if:*

Given any $1 \leqslant r < n$ and any choice of subsets $I_1, I_2, \ldots, I_s$ of cardinality r and if $\langle \sigma_{I_1}, \ldots, \sigma_{I_s} \rangle_d = 1$ then the following inequality $\sum_{j=1}^{s} \lambda_{I_j}(\bar{A}_j) - d \leqslant 0$ is valid.

Remarks:

(1) Notice that if the Gromov–Witten number is nonzero, the following dimension equality

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$\sum_{j=1}^{s} \operatorname{codim}(\sigma_{I_j}) = nd + r(n-r)$ should hold.

This follows from zero dimensionality of intersection in the moduli space of maps $\mathbb{P}^1 \to \operatorname{Gr}(r, n)$ of degree d.

(2) More inequalities are valid for $\Gamma_n(s)$. The inequalities are valid even if the intersection numbers are greater than 1. This is a consequence of the above inequalities.

(3) The main theorem is stronger than the form previously obtained by the author and independently Agnihotri and Woodward [1] in that we are able to reduce the set of inequalities to the ones coming from intersections that are numerically one. The same improvement also holds for Klyachko's theorem.

(4) In the above, there exists an irreducible representation $\rho\colon \pi_1(\mathbb{P}^1 - \{p_1, \dots, p_s\}, u) \to \mathrm{SU}(n)$ with the given conjugacy classes above, if and only if the inequalities above are strict. Note that irreducible is the same as there being no subspace fixed by all A_i.

Proof. The proof will be in three steps

(1) By the work of Mehta and Seshadri and the modification for $\mathrm{SU}(n)$ representations (I have included a proof of the modification in the Appendix), the problem is the same as existence of parabolic vector bundle on $\mathbb{P}^1$ with the corresponding weights. By Theorem 2 we know that there exists such a bundle if and only if there is an open dense subset of points on the (product of) complete flag varieties on the fibers of $V = \mathcal{O}^n$ at the p_i, which give a semistable parabolic structure on V (with the weights coming from the $\bar{A}_i$'s).

(2) We first show the validity of the inequalities on the set $\Gamma_n(s)$. Suppose we are given $\bar{A}_1, \dots, \bar{A}_s$ which lift to $A_1, \dots, A_s$ which product to 1 (in that order). Then, we find a parabolic vector bundle with the weights given by eigenvalues of $\bar{A}_i$ as in the Mehta–Seshadri theorem. Now, suppose one of the inequalities is not valid. Say the one corresponding to $\sigma_{I_1}, \dots, \sigma_{I_s}$ of rank r and degree d. Move to generic flags at p_i using openness of stability (Theorem 2). Since the intersection number $\langle \sigma_{I_1}, \dots, \sigma_{I_s} \rangle_d$ is not zero we will find a semistable bundle with a subbundle of degree $-d$ and rank r whose fibers at p_i are in $\Omega_I(F^\bullet_{p_i})$.

Now, the left side of the inequality is less than or equal to the parabolic degree of this subbundle which should be less than or equal to zero. Hence the inequality is valid. It is clear that in any situation where we are assured of the existence of such a subbundle (not just nonempty zero-dimensional family of these) generically, we find that the corresponding inequality is valid.

(3) We now show the sufficiency of the inequalities stated in the theorem. Suppose we have the data so that the inequalities are valid but there are no matrices A_i such that the conclusion of the theorem holds. This means that the generic parabolic structure on $V = \mathcal{O}^n$ with weights dictated by the $\bar{A}_i$ is not semistable. Fix a generic parabolic structure. (Genericity means that that there are no special subbundles of any rank and degree (see the discussion on Genericity).) There is then a best candidate for a subbundle of V which contradicts semistability

(Theorem 4.3). Let W be this subbundle and let its degree be $-d$ and rank r. Let I_{p_j} be the set of cardinality r consising of i_k where i_k is the least number with $F^{\bullet}_{p_j} \cap W_{p_j}$ is k (for $k = 1, 2, \ldots, r$).

Note that the Gromov Witten number $\langle \sigma_{I_1}, \ldots, \sigma_{I_s} \rangle_d = 1$. This is because W exists and there are no other subbundle of degree $-d$ and rank r with fibers in $\Omega_I(F^{\bullet}_{p_i})$. This is so because any other M in the intersection will have degree $-d$, rank r and the parabolic part of the degree at least as much as that of W. Hence, $\mu(W) \leqslant \mu(M)$. Recalling the definition of W, we see that equality holds and we reach a contradiction to the uniqueness of W.

Since the intersection is 1, we have the corresponding inequality in the set of inequalities which our data satisfies. But $\mu(W) > 0$ since the bundle W is one that contradicts semistability of V. But $\operatorname{rank}(W)\mu(W) = \operatorname{par}(W) \leqslant 0$ is one of the inequalities on our list. So we reach a contradiction. Hence, the sufficiency is proved. □

7. Some Computations

Some computations were done in the rank 2, 3 and 4 cases with $s = 3$.

Consider roots of unity ζ_1,ζ_2 and ζ_3 of order n so that $\zeta_1\zeta_2\zeta_3 = 1$. Let B_1 be the diagonal matrix with all the diagonal entries equal to ζ_1 etc. Then $B_1B_2B_3 = I$. Hence the conjugacy classes of these elements give elements of $\Gamma_n(s)$. One can ask if the convex hull of these is $\Gamma_n(s)$.

This is true in the case $n = 2, 3$, but is false in the case $n = 4$.

The following matrix equation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

shows that the matrices on the left hand side give an element of $\Gamma_4(3)$ which we prove is not in the convex hull of the vertices considered above. The above point gives an vertex of the polyhedron $\Gamma_4(3)$ as verified by a computer check using the Porta package.

LEMMA 4. *Consider the point in* $\Gamma_4(3)$ *given by the above matrix equation. Then this is not in the convex span of the representation of the center of* SU(4).

Proof. We think of conjugacy classes in SU(4) as 4-tuples (a_1, a_2, a_3, a_4) so that $a_1 + a_2 + a_3 + a_4 = 0$ and $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant a_1 - 1$. For the purposes of this proof it is useful to think of the corresponding 4-tuple (b_1, b_2, b_3, b_4), where

$$b_1 = a_1 - a_2, \qquad b_2 = a_2 - a_3, \qquad b_3 = a_3 - a_4, \qquad b_4 = a_4 - (a_1 - 1).$$

Then the conditions on a_i are the same as $b_i \geqslant 0, b_1 + b_2 + b_3 + b_4 = 1$.

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All the three matrices on the left-hand side in the above matrix equation are conjugate and correspond to $C = (\frac{1}{2}, 0, \frac{1}{2}, 0)$. The central elements correspond to $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$, and $(0, 0, 0, 1)$ which correspond to diagonal matrices with

$$\exp\left(\frac{2\pi i}{4}\right), \quad \exp\left(\frac{4\pi i}{4}\right), \quad \exp\left(\frac{6\pi i}{4}\right), \quad \text{and} \quad \exp\left(\frac{8\pi i}{4}\right)$$

respectively. To simplify a long argument let us notice that the 2nd and 4th members of C are 0. So to write C as a convex combination of the central element we can use only $(1, 0, 0, 0)$ and $(0, 0, 1, 0)$. There is no way of finding 3 of these type multiplying to I. The two elements are ζ and ζ^3 where $\zeta = i$.

It is natural to pose the question of determining the vertices of the polyhedron $\Gamma_n(s)$. The simplest guess is wrong as the above computation shows. A knowledge of the vertices is sufficient to determine the polyhedron. □

8. Remarks on Klyachko's Theorem

Let H be a Hermitian operator on $\mathbb{C}^n$. Let

$$\lambda(H) = (\lambda^1(H) \geqslant \lambda^2(H) \geqslant \cdots \geqslant \lambda^n(H))$$

be the eigenvalues of H. Now, suppose we are given s sequences of real numbers denoted λ_i. Assume $s \geqslant 3$. Then Klyachko proved:

THEOREM 8. (Klyachko). *There are Hermitian $n \times n$ matrices $H_1, \dots, H_s$ with $\lambda(H_i) = \lambda_i$ for all i and $\sum_i H_i = c\,1$ a scalar if and only if the following conditions are satisfied (where $r \in \{1, \dots, n-1\}$ and card$(I_i) = r$)*

$$\frac{1}{r}\sum_{i=1}^{s}\sum_{j \in I_i}\lambda_i^j \leqslant \frac{1}{n}\sum_{i=1}^{s}\sum_{j=1}^{n}\lambda_i^j \quad (*)$$

whenever $\sigma_{I_1} \cdot \dots \cdot \sigma_{I_s}$ is a nonzero multiple of the class $\sigma_{[1,r]}$ (the class of a point) in the cohomology of Gr(r, n).

THEOREM 9 (*Shortened list of inequalities*). *By the Harder–Narasimhan theorem, we can shorten the set of inequalities to only those* $(*)$ *for which $\sigma_{I_1} \cdot \dots \cdot \sigma_{I_s}$ is equal to (not just a multiple of) the class $\sigma_{[1,r]}$ (the class of a point).*

Hence, the inequalities as stated by Klyachko are not independent. We do not know if the restricted set of inequalities above are independent.

To prove this, it is useful to set up the notation of a parabolic vector space:

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DEFINITION 5. A 'complete' parabolic vector space with s filtrations consists of a vector space V of rank n and the following additional data:

(1) For all $x \in \{1, 2, \dots, s\}$ a filtration: $V = V_x^n \supset V_x^{n-1} \supset \cdots \supset V_x^0 = \{0\}$ by vector subspaces with strict inclusions.
(2) Weights $a_x^1 \geqslant a_x^2 \geqslant \cdots \geqslant a_x^n$.

Also define the parabolic slope of V as $\mu(V) = \left(\frac{1}{n}\right) \sum_x \sum_{i=1}^n a_x^i$.

A subspace W of V gets an induced parabolic structure by intersecting with the fibres and looking at breaks in the sequence $W_x = W \cap V_x^n \supseteq W_x \cap V_x^{n-1} \supseteq \cdots \supseteq W_x \cap V_x^0 = \{0\}$ and assigning the highest weight possible. That is, we get from the sequence above a complete flag on W_x and define the weight $b_x^i = a_x^j$ where j is the smallest number satisfying $W_x^i = W_x \cap V_x^j$.

We want to modify the existence of Harder–Narasimhan filtrations in this setting. The proof is standard.

LEMMA 5 (Harder–Narasimhan). *If V is not a semistable vector space then let μ = maximum slope of subspaces W of V (trivially finite). Let E be a subspace of slope μ and maximum dimension possible (any subspace containing it has smaller slope). Then E is unique.*

Proof. Suppose E_1, E_2 are two such subspaces. Let S be the image of E_1 in V/E_2. We can give a parabolic structure on S via the surjective map from E_1. Call this structure S_1.

We can let π be the map from V to V/E_2. Then we can give S a parabolic structure from $\pi^{-1}(S)$ (a subspace of V). Call this S_2. We have maps:

(1) $S_1 \to S_2$ a map preserving filtrations,
(2) A sequence
$0 \longrightarrow K \longrightarrow E_1 \longrightarrow S_1 \longrightarrow 0.$
(3) A sequence
$0 \longrightarrow E_2 \longrightarrow \pi^{-1}(S) \longrightarrow S_2 \longrightarrow 0.$

From the first statement, $\mu(S_1) \leqslant \mu(S_2)$. From the second we get $\mu(S_1) \geqslant \mu(E_1)$ since the kernel has no greater slope. And from the third statement we get $\mu(S_2) < \mu(E_2)$ since the middle term strictly contains E_2 (we may assume this or the one with E_1 and E_2 interchanged). The net result is $\mu(E_2) > \mu(E_1)$ which is a contradiction to our assumption.

THEOREM 10 (Klyachko). *There are Hermitian $n \times n$ matrices $H_1, \dots, H_s$ with $\lambda(H_i) = \lambda_i$ for all i and $\sum_i H_i = c\,1$ a scalar if and only if there is a semistable parabolic vector space V with s filtrations and weights dictated by the λ_i's.*

Proof. This is just a reformulation of Klyachko's theorem as in [4]. □

We now give the proof of Theorem 5.2:

Proof. Suppose that the modified set of inequalities is wrong. Then we find s sequences of real numbers denoted λ_i as above for which the the restricted inequalities are satisfied but there are no Hermitian matrices satisfying the conditions of the theorem. This implies that the generic parabolic structure on $V = \mathbb{C}^n$ with the weights given by the λ_i is not semistable. Let $E_i^\bullet$ for $i = 1, \ldots, s$ be the filtrations (assumed general). Let W be a subspace with maximum slope and maximal dimension as in the above lemma. Let $\mathrm{rank}(W) = r$. Assume that W lies in the open cell in $\Omega_{I_t}(V)$ with respect to the filtration $E_t^\bullet$ for $t = 1, \ldots, s$. Then, W is the only element in the intersection of $\Omega_{I_t}(V)$ for $t = 1, \ldots, s$. (Note $\mathrm{card}(I_t)$= $\dim(W)$). For if Z were another element in the intersection, then $\mu(Z) \geqslant \mu(W)$ which is not possible by the lemma since Z would have the same rank as W. So we find that, since intersections are transversal with generic flags, $\sigma_{I_1} \cdot \ldots \cdot \sigma_{I_s}$ is equal to the class $\sigma_{[1,r]}$ (the class of a point) in the cohomology of $\mathrm{Gr}(r, n)$. Hence the inequality $\mu(W) \leqslant \mu(V)$ is a member of the restricted set of inequalities. But since W is a contradiction to semistability of V we should have $\mu(W) > \mu(V)$ and that leads to a contradiction. So the restricted set of inequalities suffices. □

9. Application to a Problem on Rigid Local Systems

Let $S = \{p_1, \ldots, p_s\}$ be a collection of distinct points on $\mathbb{P}^1$ and let $U = \mathbb{P}^1 - S$. A local system $\mathcal{F}$ of $\mathbb{C}$ vector spaces on U is called physically rigid (Katz [7]) if it is irreducible and given any other local system $\mathcal{G}$ on U with isomorphic local monodromies at points of S as $\mathcal{F}$ then $\mathcal{G}$ is isomorphic to $\mathcal{F}$. Recall that the local monodromy of $\mathcal{F}$ at $p \in S$ is determined by the Jordan canonical form of the monodromy transformation at p (moving around p in the counterclockwise direction).

In our earlier language if $\mathcal{F}$ arises from $\rho\colon \pi_1(\mathbb{P}^1 - S, u) \to \mathrm{GL}(n)$ and $\pi_1(\mathbb{P}^1 - S, u)$ = free group on $\gamma_1, \ldots, \gamma_s$ modulo the relation $\gamma_1 \ldots \gamma_s = I$ where γ_i are loops around each puncture (appropriately chosen and oriented in counterclockwise direction), then the local monodromies are Jordan canonical forms of $\rho(\gamma_i)$.

Katz in his book [7] posed the following problem: Given the numerical data which is known to arise from (a unique) rigid local system, can one determine if the global monodromy is finite? We show that the solution to the $\mathrm{SU}(n)$ problem leads to an algorithm for this.

The numerical data for a rigid local system on $\mathbb{P}^1 - S$ consists of Jordan canonical forms $A_{p_i} \in \mathrm{GL}(n, \mathbb{C})$ for $p_i \in S$ so that the following combinatorial condition $(*)$ is satisfied.

$$(*)\ (2 - s)n^2 + \sum_{p_i} \dim_{\mathbb{C}}(Z(A_{p_i})) = 2,$$

where $Z(A_{p_i})$ are matrices which commute with A_{p_i}.

Clearly all the A_{p_i} have to be of finite order (and hence diagonalizable) for there to exist a local system of finite global monodromy with A_{p_i} as the Jordan classes of the monodromy transform at p_i. So we assume to start with that this is the case and let $\mathbb{K}$ be the field obtained by adjoining to $\mathbb{Q}$ the eigenvalues of all the $A'_{p_i}s$ (this is a Galois extension) and let $G = \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$. Note that G is a finite group. It is a theorem of Katz that the representation (if it exists) is defined over $\mathbb{K}$, but we will not use this fact.

For each $g \in G$ one gets a new set of Jordan canonical forms $A^g_{p_i} = gA_{p_i}$ by applying g to the entries of A_{p_i}'s. If there exists a local system giving rise to A_{p_i} then there is such a representation landing in the unitary group (tensor with an nth root of the dual of the determinant representation to get to the special unitary group) and the same is true for all Galois conjugates of the original data. For rigid local systems this is sufficient as the following theorem shows (using at a crucial stage a result of Katz).

THEOREM 11. *Let A_{p_i} satisfy the numerical requirement for rigidity (*). Then there exists an irreducible rigid local system of finite global monodromy on $\mathbb{P}^1 - S$ with Jordan canonical form of the monodromy transformation at $p_i = A_{p_i}$ (counter-clockwise orientation) if and only if for every $g \in G$ there is a irreducible representation $\rho_g: \pi(\mathbb{P}^1 - S, u) \to U(n)$ with the monodromy transform at p_i being A^g_i.*

Remark. We may assume by tensoring with the the dual of a nth root of the determinant representation that the A_{p_i} are of determinant 1.

Proof. (Standard). The ‘only if’ part is clear. Let us fix for each $g \in G$ a complex vector space V_g of dimension n with a nondegenerate Hermitian form H_g and a representation ρ_g as in the statement of the theorem. Denote the fundamental group of $\mathbb{P}^1 - S$ by Γ. Among the representations V_g choose a subset $V_1, \ldots, V_r$ consisting of distinct representations of Γ. Note that the muliplicity of each V_i in the set $\{V_g: g \in G\}$ is the same. Let W be the direct sum of V_i. Γ acts on W and preserves a nondegenerate Hermitian form. The trace of each element of Γ under this representation is in $\mathbb{Q}$. This is because the trace is Galois invariant (we have summed over all Galois conjugates).

Now, let $Q: \Gamma \to \mathrm{U}(W)$. and denote the map to $\mathrm{GL}(W)$ by the same symbol. Let $A_{\mathbb{Z}}$ (resp. $A_{\mathbb{Q}}$ and $A_{\mathbb{C}}$) be the $\mathbb{Z}$- (resp. $\mathbb{Q}$ and $\mathbb{C}$) algebra generated by the image of Γ in $\mathrm{End}(W)$. Γ acts on $A_{\mathbb{Z}}$ and, hence, on $A_{\mathbb{Q}}$ and $A_{\mathbb{C}}$ on the left.

(1) The set $A_{\mathbb{Z}}$ contains a complex basis of $\oplus\mathrm{End}(V_i)$. This is because W is a faithful semisimple representation of $A_{\mathbb{C}}$ and the endomorphisms of W over $A_{\mathbb{C}}$ is $\mathbb{C}^r$ (the V_i are pairwise distinct irreducible representations). The endomorphisms of W over the ring $\mathbb{C}^r$ are precisely $\oplus\mathrm{End}(V_i)$. Thus, by Wedderburn, $A_{\mathbb{C}} = \oplus\mathrm{End}(V_i)$.
(2) The trace form on $A_{\mathbb{C}}$ (which is the sum of the trace forms on $\mathrm{End}(V_i)$) is hence nondegenerate. At this stage we do not know if $A_{\mathbb{Q}}$ is finitely generated as a $\mathbb{Q}$- vector space. Select among members of $A_{\mathbb{Z}}$ a basis $u_1, u_2, \ldots, u_l$ of $A_{\mathbb{C}}$. We have a map $A_{\mathbb{Z}} \to \mathbb{Z}^l$ by $v \mapsto (\mathrm{tr}(vu_1), \ldots, \mathrm{tr}(vu_l))$. This is injective because

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trace is nondegenerate on $A_{\mathbb{C}}$ and integral because each element of Γ has an algebraic integer as trace on each V_i by a theorem of Katz (any quasiunipotent rigid local system on $\mathbb{P}^1 - S$ is motivic [7]). Since on W all traces of elements of Γ are rational numbers (we have added all Galois conjugates), the traces are integers. Clearly $A_{\mathbb{Z}}$ is torsion free and has rank l (it has rank at least l since it contains $u_1, \dots, u_l$ and rank atmost l since it is contained in $\mathbb{Z}^l$). It follows that $A_{\mathbb{Z}}$ is a lattice in $A_{\mathbb{C}}$ (the natural map $A_{\mathbb{Z}} \bigotimes_{\mathbb{C}} \mathbb{C} \to A_{\mathbb{C}}$ is an isomorphism).

(3) Consider the left action of Γ on $A_{\mathbb{C}}$. This preserves the lattice $A_{\mathbb{Z}}$ and the representation V_1 appears with multiplicity $\dim(V_1)$ times. So it suffices to show that this representation has finite image. Also notice that Γ preserves the hermitian form $\mathrm{tr}(A\bar{B}^t)$ which is nondegenerate. Hence the image is the intersection of a compact group and a discrete group (preserves a lattice, etc.) hence finite. □

Appendix: A Variant of the Mehta–Seshadri Theorem for SU(n) Representations

The theorem of Mehta and Seshadri in its original form is one about unitary representations. A slight variant of this is required in this article. Namely we want to study SU(n) representations of the fundamental group of a punctured curve and want our weights to correspond to the the simplex corresponding to the conjugacy classes in SU(n).

The author has not found an adequate reference to the above variant. For completeness we include a proof of this fact.

First we recall the original version of the Mehta–Seshadri theorem. Let C be a smooth projective curve over $\mathbb{C}$ and $S = \{p_1, p_2, \dots, p_s\}$ be a finite set of distinct points on C.

DEFINITION 6. A parabolic vector bundle on a curve C with s marked points (the p_i's) consists of a vector bundle V of rank n on C and the following additional data:

(1) For all $x \in \{p_1, \dots, p_s\}$ a filtration of the fiber V_x: $V_x = V_x^1 \supset V_x^2 \supset \dots \supset V_x^r = \{0\}$ by vector subspaces with strict inclusions where $r = r(x)$ depends on x,

(2) Weights $0 \leqslant a_x^1 < a_x^x < \dots < a_x^r < 1$,

(3) Let $k_x^i = \dim(V_x^i / V_x^{i+1})$. Call this the multiplicity of a_x^i.

Also, define the parabolic degree of V as

$$\mathrm{Par}(V) = \deg(V) + \sum_x \sum_{i=1}^{r(x)} k_x^i a_x^i.$$

A subbundle W of V gets an induced parabolic structure by intersecting with the fibres and looking at breaks in the sequence

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$W_x = W \cap V_x^1 \supseteq W_x \cap V_x^2 \supseteq \ldots \supseteq W_x \cap V_x^{r(x)} = \{0\}$ and assigning the highest weight possible, That is we get from the sequence above a flag on W_x and the weight $b_x^i = a_x^j$ where j is the largest number satisfying $W_x^i = W_x \cap V_x^j$.

The definition of stability is made in the usual way.

Suppose we are given for each $x \in S$ weights $0 \leqslant a_x^1 < a_x^1 < \cdots < a_x^{r(x)} < 1$. Let us consider the set of parabolic bundles on C with fixed parabolic points being the points in S and fixed numerical structure of the flags (the weights and muliplicities).

THEOREM 12 (Mehta–Seshadri). *There exists an irreducible (resp. possibly reducible) representation* $\rho\colon \pi_1(\mathbb{C}^1 - \{p_1, \ldots, p_s\}, u) \to U(n)$ *with Jordan canonical forms given by the tuples* $(a_{p_j}^1, \ldots, a_{p_j}^{r(p_j)})$ *with the associated muliplicities being given and fixed if and only if there is a stable (resp. semistable) parabolic bundle of parabolic degree 0 with the weights again given by* $(a_{p_j}^1, \ldots, a_{p_j}^{r(p_j)})$ *for* $j = 1, 2, \ldots, s$.

Recall that a bundle of parabolic degree 0 is called stable (resp. semistable) if all its subbundles have negative (resp. non positive) parabolic degree.

We have to make the transition to our definitions. The differences are that in our numeration the a^i's are nonincreasing and sum to zero (at each point), the distance between any two of them (at a point) is less than or equal to 1, we are allowing them to be equal, and the flags are complete.

Let us recall our definition:

DEFINITION 7. A 'complete' parabolic vector bundle on a curve C with s marked points (the p_i's) consists of a vector bundle V of rank n on C and the following additional data:

(1) For all $x \in \{p_1, \ldots, p_s\}$ a filtration of the fiber V_x: $V_x = V_x^n \supset V_x^{n-1} \supset \cdots \supset V_x^0 = \{0\}$ by vector subspaces with strict inclusions.
(2) Weights $a_x^1 \geqslant a_x^2 \geqslant \cdots \geqslant a_x^n \geqslant a_x^1 - 1$.

The first remark is that in the Mehta–Seshadri definition we can extend the flag to a complete flag at each point of S and assign the lowest weight among the possibilities. That is, we change the Mehta–Seshadri definition by insisting on a complete flag but allowing weights to coincide. Then, in their theorem we cannot assert that irreducible unitary representations correspond uniquely to stable parabolic bundles of degree 0. The uniqueness is lost. But we can say that a stable (resp. semistable) bundle exists if and only if there is an irreducible unitary representation (resp. reducible unitary representation) with the given weights.

Next, when we are looking at SU(n) representations, the *weights at each point have to add up to an integer*. This is because all the Jordan canonical forms in this case have eigenvalues multiplying to 1. We will assume henceforth that this the case.

To prove the form of the Mehta–Seshadri theorem that we require we are going to prove:

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(1) Given a Mehta–Seshadri parabolic bundle, we will show that there is a parabolic bundle in the sense of this article. This association is not one to one.
(2) Given a parabolic vector bundle in our sense we show how to construct a Mehta–Seshadri bundle out of this.
(3) These constructions preserve stability and semistability.

We will carry out the constructions in the case of one parabolic point. The extension to more than one parabolic point presents only notational difficulties (but we will make remarks on that at the end of the constructions).

Let us start with a Mehta–Seshadri vector bundle with parabolic structure at the point P. We extend the flag at P to a complete flag and revert to our numeration (decreasing weights). Let the flag be $V_P = V_P^n \supset V_P^{n-1} \supset \cdots \supset V_P^0 = \{0\}$. Notice that stability and semi-stability of subbundles is not altered.

LEMMA 6. *Let a vector space V have a strict increasing filtration: $V^n \supset V^{n-1} \supset \cdots \supset V^0 = 0$ and $i(r) > i(r-1) > \ldots > i(1)$ be a set of numbers so that each $i(j)$ satisfies $1 \leqslant i(j) \leqslant n$. Let $a^{i(j)}$ be given for $j = 1, \ldots, r$. Let W be a vector subspace of V. Let $b^k = a^{i(j)}$ where j is the smallest number satisfying $V^{i(j)} \supseteq V^k$. Then*

$$\sum_{j=1}^{r} \dim\left(\frac{W \cap V^{i(j)}}{W \cap V^{i(j-1)}}\right) a^{i(j)} = \sum_{j=1}^{n} \dim\left(\frac{W \cap V^{j}}{W \cap V^{j-1}}\right) b^{j}.$$

Proof. It suffices to show that the coefficient of $a^{i(j)}$ on both sides is the same. This is clear since any element V^l of the flag contained in $V^{i(j)}$ and strictly containing $V^{i(j-1)}$ has been assigned the weight $a^{i(j)}$. □

We define suitable shifts of the bundles. Define V^j to be the sheaf (a bundle) which coincides with V outside of P and whose sections over an open subset containing P are the sections of V which whose fiber at P are in V_P^j. Also fix a parameter t of the curve at P. Define $V^j[i]$ to be sheaf which coincides with V outside of P, but whose stalk at P is $t^i V^j$.

We have a parabolic structure on $V^j[k]$ coming from the following inclusions of sheaves: $V^j[k] \supset V^{j-1}[k] \supset \cdots \supset V^1[k] \supset V^n[k+1] \supset \cdots \supset V^{j+1}[k+1] \supset V^j[k+1]$.

With associated weights: $a_j + k \leqslant a_{j-1} + k \leqslant \cdots \leqslant a_1 + k \leqslant a_n + k + 1 \leqslant a_{j+1} + k + 1 \leqslant a_j + k + 1$.

It is easy to see that these are strict inclusions. An easy calculation shows

(1) $\deg(V^j[k]) = \deg(V^j) - kn$
(2) $\mathrm{par}(V^j[k]) = \deg(V^j[k]) + (\sum_{i=1}^n a^i) + (n-j) + kn = \deg(V^j) + (n-j) + (\sum_{i=1}^n a^i)$
(3) $\deg(V^j) = \deg(V^n) - (n-j)$

Hence from the statements above one sees that $\mathrm{par}(V^j[k]) = \mathrm{par}(V)$. Notice also that sum of weights of $V^j[k] = \sum_{i=1}^n a^i + kn + (n-j)$. Hence for an appropriate

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choice of k and j this sum is zero (for as we decrease j by 1, the sum increases by 1, and a change in k by 1 affects the sum by n).

We have to study the effect of subbundles. We first give a natural procedure of going from subbundles of $V^j[k]$ to subbundles of some other $V^l[m]$. Let W be a subbundle of $V^j[k]$ and W_K be the generic fiber of W. Then let $\breve{W}$ be the subsheaf $W_K \cap V^l[m]$ where W_K is regarded as a constant sheaf (on the Zariski topology). We prove the following lemmas which prove finally the Mehta–Seshadri theorem in the SU(n) setting:

LEMMA 7. *$\breve{W}$ is a subbundle of $V^l[m]$.*

Proof. We have to show $V^l[m]/\breve{W}$ is torsionfree. There is no problem outside P because $V^l[m] = V$ ouside of P. Let $s \in V^l[m]$ map to torsion in $V^l[m]/\breve{W}$. Then $t^d s \in \breve{W}$ for some $d > 0$. But this implies that $s \in \breve{W}$ by our definition. So the image of s in $V^l[m]/\breve{W}$ is zero. □

LEMMA 8. *Assume W is a subbundle of V. Then* $\operatorname{par}(W) = \operatorname{par}(\breve{W})$ *where $\breve{W}$ is considered a subbundle of the parabolic bundle $V^j[k]$.*

Proof. Note that

$$(1)\quad \operatorname{par}(W)\deg(W) + \textstyle\sum_{q=1}^{n} \dim\left(\dfrac{W\cap V^q}{W\cap V^{q-1}}\right)a^q.$$

$$(2)\quad \operatorname{par}(\breve{W}) = \deg(\breve{W}) + \textstyle\sum_{q=1}^{j} \dim\left(\dfrac{\breve{W}\cap V^q[k]}{\breve{W}\cap V^{q-1}[k]}\right)(a^q + k) +$$
$$+ \textstyle\sum_{q=j+1}^{n} \dim\left(\dfrac{\breve{W}\cap V^q[k+1]}{\breve{W}\cap V^{q-1}[k+1]}\right)(a^q + k + 1).$$

We first compute $\deg(\breve{W})$. We use the exact sequence (assume for definiteness that $k < 0$)

$$0 \longrightarrow W \longrightarrow \breve{W} \longrightarrow \frac{\breve{W}}{W} \longrightarrow 0$$

where the last term is a skyscraper sheaf at P. The dimension of this space is to be computed. Call this sheaf $P(k)$. We first compute the dimension of $P(-1)$. Note that

$$P(-1) = \frac{W_K \cap t^{-1} V^j}{W^K \cap V^n},$$

which is the same as $W_K \cap V^j / t(W_K \cap V^n)$. But this is $W_P \cap V_P^j$ (the last term is the intersection of the stalk of W at P with the jth element of the flag). If $k < -1$, then $P(k)/P(k+1)$ is the same as $t^k(W_K \cap V^j)/t^{k+1}(W_K \cap V^j)$ and that has dimension $\dim(W)$. Putting these together one gets by induction that

$$\dim(P(k)) = \dim(W_P \cap V_P^j) + \dim(W)(k-1).$$

This gives

$$\deg(\check{W}) = \deg(W) - \dim(W_P \cap V_P^j) - \dim(W)(k+1).$$

We still have to calculate the parabolic degree of $\check{W}$. For this we note

$$\dim\left(\frac{W \cap V^q}{W \cap V^{q-1}}\right) = \dim\left(\frac{\check{W} \cap V^q[k]}{\check{W} \cap V^{q-1}[k]}\right)$$

for all k. Hence looking at the parabolic contributions in (1) and (2) above we see that

$$\sum_{q=1}^{n} \dim\left(\frac{W \cap V^q}{W \cap V^{q-1}}\right) a^q - \sum_{q=1}^{j} \dim\left(\frac{\check{W} \cap V^q[k]}{\check{W} \cap V^{q-1}[k]}\right)(a^q + k) -$$
$$- \sum_{q=j+1}^{n} \dim\left(\frac{\check{W} \cap V^q[k+1]}{\check{W} \cap V^{q-1}[k+1]}\right)(a^q + k + 1)$$
$$= k(\dim(W)) + \dim(W) - \dim(W_P \cap V_P^j).$$ □

The proof of the modified Mehta–Seshadri theorem (3.2) is now complete. For all the $V^j[k]$ are simultaneously stable or semistable if V has parabolic degree 0. Also one of the $V^j[k]$ will have the sum of parabolic weights at P equal to zero.

The extension to more than one point is by working at one point at a time. In the course of our constructions above we did not change anything except at P.

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