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A CLASS OF RIGHT-ORDERABLE GROUPS

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1. Introduction. A group G is called *right-orderable* (or an *RO-group*) if there exists an order relation $\leqq$ on G such that $a \leqq b$ implies $ac \leqq bc$ for all a, b, c in G. This is equivalent to the existence of a subsemigroup P of G such that $P \cap P^{-1} = \{e\}$ and $P \cup P^{-1} = G$. Given the order relation $\leqq$, P can be taken to be the set of positive elements and conversely, given P, define $a \leqq b$ if and only if $ba^{-1} \in P$. A group G together with a given right-order relation on G is called *right-ordered*. A subgroup C of a right-ordered group G is called *convex* if for every g in G and x in C, $e \leqq g \leqq c$ implies $g \in G$. The set of all convex subgroups of G is ordered by inclusion and closed with respect to unions and intersections. However there is not much more one can say in general regarding this set. We shall call a right-order P on G a *C-right-order* if the set of convex subgroups form a system with torsion-free abelian factors. P. Conrad [**2**] has looked at a number of equivalent conditions for a group G to be C-right-ordered. Our main concern here is to investigate the properties of an RO-group G in which every right-order is a C-right-order. We call such a group a C_1-*group*. In Lemma 3.1 we show that a right-order P is a C-right-order if and only if it satisfies the property:

(*) For all x, y in P there exist u, v in sgr $\langle x, y\rangle$ (the semigroup generated by x and y) such that $ux \geqq vy$.

Thus in particular an RO-group G is a C_1-group if it satisfies the property:

(**) For all x, y in G there exist u, v in sgr $\langle x, y\rangle$ such that $ux = vy$.

We call G a C_3-*group* if it satisfies (**). Finally we denote by C_2 the largest subgroup closed subclass of C_1. Then $RO \supseteq C_1 \supseteq C_2 \supseteq RO \cap C_3$, and all these inclusions are proper (Corollary 3.3, Theorem 3.5).

In Section 2 we note a few properties of C_3-groups. In particular we show that locally solvable C_3-groups are locally nilpotent-by-finite (Theorem 2.6). This is not true of C_2-groups (Theorem 3.5), however orderable locally solvable C_2-groups are locally nilpotent and finitely generated orderable solvable C_1-groups are nilpotent (Theorem 3.6).

2. C_3-groups. We start by observing that the class C_3 is subgroup-closed and closed under periodic extensions; moreover a group G is in C_3 if every two-generator subgroup of G is in C_3. B. H. Neumann has shown that G is in C_3 if every two-generator subgroup of G is nilpotent.

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LEMMA 2.1. *Let H be a subgroup in the centre of a group G. If G/H is in C_3, then G is in C_3.*

Proof. Let $x, y \in G$. Then there exist $u, v \in \text{sgr}\ \langle x, y\rangle$ such that $ux = zvy$ for some $z \in H$. Thus $vyux = uxvy$ with $vyu, uxv \in \text{sgr}\ \langle x, y\rangle$.

COROLLARY 2.2. *If every two-generator subgroup of G is nilpotent-by-periodic, then G is in C_3.*

LEMMA 2.3. *A direct product of C_3-groups is in C_3.*

Proof. It is clearly enough to show that if H_1, H_2 are C_3-groups, then so is $G = H_1 \times H_2$. Take any $x = x_1x_2$, $y = y_1y_2$ in G with $x_i, y_i \in H_i$. Since $H_1 \in C_3$, there exist $a = a_1a_2$, $b = b_1b_2$ in sgr $\langle x, y\rangle$ such that $a_1x_1 = b_1y_1$. Also, since $H_2 \in C_3$, there exist $(a_1x_1)^m h_2$, $(a_1x_1)^n k_2$ in sgr $\langle ax, by\rangle$, with m, n positive integers, h_2, k_2 in H_2, such that $h_2a_2x_2 = k_2b_2y_2$. Then $(a_1x_1)^m h_2ax(a_1x_1)^m k_2by = (a_1x_1)^m k_2by(a_1x_1)^m h_2ax$, and of course $(a_1x_1)^m h_2$, ax, $(a_1x_1)^n k_2 by$, a, b are all in sgr $\langle x, y\rangle$.

LEMMA 2.4. *A polycyclic C_3-group is nilpotent-by-finite.*

Proof. Let G be a counterexample with $l(G)$ minimum where $l(G)$ is the number of infinite factors in any series of G with cyclic factors. Replacing G with a suitable normal subgroup of finite index if necessary, we may assume that it is nilpotent-by-abelian and torsion-free. Let N be the Fitting subgroup of G. By the minimality of G, N is abelian (because G/N' nilpotent-by-finite implies G nilpotent-by-finite), G/N is infinite cyclic, and the centre of G is trivial (see Lemma 2.1).

Let $G = \langle N, t\rangle$, write N additively and regard it as a module over the integral group ring $Z\ \langle t\rangle$. Let A be an indecomposable submodule of N. Then A can be identified with an additive subgroup of the complex numbers on which the action of t is that of multiplication by an algebraic integer τ whose minimal polynomial over the rationals has degree equal to $l(A)$. If all the roots of this polynomial have absolute value one, then by a theorem of Kronecker, τ is an nth root of unity for some integer n. But then t^n centralizes A, and $G_1 = \langle N, t^n\rangle$ has a non-trivial centre, so that G_1 and hence G is nilpotent-by-finite. Thus $|\tau| \neq 1$, and replacing t with a suitable power of t, if necessary, we may assume that $|\tau| < \frac{1}{4}$.

Choose any non-zero $a \in A$. By hypothesis there exist $u, v \in \text{sgr}\ \langle at, ta\rangle$ such that $uat = vta$. Then we have:

$$t^{r+1}(a\tau^{\alpha_r} + \ldots + a\tau^{\alpha_1} + a\tau) = t^{r+1}(a\tau^{\beta_r} + \ldots + a\tau^{\beta_1} + a),$$

where $a\tau = t^{-1}at$, $1 \leqq \alpha_1 \leqq \ldots \leqq a_r$, $1 \leqq \beta_1 \leqq \ldots \leqq \beta_r$, $i \leqq a_i \leqq i + 1$ and $i \leqq \beta_i \leqq i + 1$ for all i. But

$$|\tau^{\alpha_r} + \ldots + \tau^{\alpha_1} + \tau| \leqq (|\tau|^{\alpha_r} + \ldots + |\tau|^{\alpha_1}) + |\tau| < \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n + \frac{1}{4} = \frac{7}{12},$$

while

$$|\tau^{\beta_r} + \ldots + \tau^{\beta_1} + 1| \geqq 1 - (|\tau|^{\beta_r} + \ldots + |\tau|^{\beta_1}) > 1 - \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{2}{3}.$$

and we reach a contradiction.

LEMMA 2.5. *If $G = \langle A, t\rangle$ is a C_3-group and $A = \langle a_1, \ldots, a_k\rangle^G$ is abelian, then A is finitely generated and G is nilpotent-by-finite.*

Proof. The existence of u_i, v_i in sgr $\langle a_it, ta_i\rangle$ such that $u_ia_it = v_ita_i$ shows that $\langle a_i\rangle^G = \langle a_i, a_i^t, \ldots, a_i^{t^{r_i}}\rangle$ for some integer r_i. The rest follows from Lemma 2.4.

THEOREM 2.6. *If G is a locally solvable C_3-group, then G is locally nilpotent-by-finite.*

Proof. Assume, by way of induction, that the result holds for finitely generated groups of solvability length less than r, and let G be a finitely generated group of solvability length r. If A is the last non-trivial term in the derived series of G, then A is abelian and G/A is nilpotent-by-finite. Replacing G by a suitable subgroup of finite index if necessary, we may assume that G/A is nilpotent. Then $A = S^G$, where $S = \langle a_1, \ldots, a_k\rangle$ for some $a_1, \ldots, a_k$ in A. Also there exists a series $A = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ such that, for all i, $G_i \lhd G$ and $G_i = \langle G_{i-1}, t_i\rangle$ for suitable t_i in G. Repeated application of Lemma 2.5 shows that S^{G_i} is finitely generated for all $i = 1, \ldots m$. Thus G is polycyclic and the result follows from Lemma 2.4.

3. C_1 and C_2-groups.

LEMMA 3.1. *Let P be a right-order on a group G. Then the following are equivalent.*

(i) *P satisfies condition $(*)$.*

(ii) *For every x, y in $P\backslash\{e\}$, $x^ny > x$ for some $n > 0$.*

(iii) *If C and D are convex subgroups of G under P and D covers C, then C is normal in D and D/C is isomorphic to a subgroup of the additive group of the reals.*

(iv) *For all y in $P\backslash\{e\}$ the set $\{x \in G| \, |x| \ll y\}$ is a convex subgroup of G, where $|x| = x$ if $x \in P$ and x^{-1} otherwise, and $|x| \ll y$ means that $|x|^n < y$ for all n.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $x^ny \leqq x$ for all $n > 0$. By hypothesis there exist $u, v \in$ sgr $\langle xy, x\rangle$ such that $uxy \geqq vx$. Since $v > e$, $vx > x$. On the other hand $uxy = x^{\alpha_1}yx^{\alpha_2}y \ldots x^{\alpha_r}y$, where $\alpha_i \geqq 1$ for $i = 1, \ldots, r$ and $r \geqq 1$, hence $uxy \leqq x^{\alpha_2+1}y \ldots x^{\alpha_r}y \leqq \ldots \leqq x$, a contradiction.

That (ii) $\Rightarrow$ (i) is trivial. The equivalence of (ii) and (iii) was shown in [**2**] and the equivalence of (iii) and (iv) in [**1**]. We mentioned (iii) and (iv) because we will need them in the following.

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LEMMA 3.2. *Let A and B be RO-groups and G a split extension of A by B. If there exists a right-order P_A on A, invariant under conjugation by elements of B, such that not all the jumps in the chain of convex subgroups of A determined by P_A are centralized by B, then G is not a C_1-group.*

Proof. The result is obvious if P_A is not a C-order on A. Let P_B be a right-order on B and define a right-order P on G by letting $g = ab$ $(a \in A, b \in B)$ belong to P if either $e \neq a \in P_A$ or $a = e$ and $b \in P_B$. That P is indeed a right-order follows from the fact that P_A is B-invariant. We show that it is not a C-order. Let $C \prec D$ be a jump of convex subgroups of A under P_A which is not centralized by B, and choose $e < a \in D \backslash C$, $b \in B$ such that $[a, b] \notin C$.

Case 1. b normalizes D. In this case b normalizes C as well since P_A is B-invariant. Moreover D/C may be identified with a subgroup of the additive group of the reals since P_A is a C-order, and the action of b on D/C is that of multiplication by some real number $\beta > 1$ (replacing b by b^{-1} if necessary). Let $\bar{a} = Ca$ and choose $\bar{d} = Cd \in D/C$ such that

$$\bar{d} \geqq \bar{a}/(\beta - 1) > 0.$$

For instance $\bar{d}$ can be a suitable multiple of $\bar{a}$. We show that the set

$$S = \{x \in G; |x| \ll d\}$$

is not a subgroup and thus P does not satisfy Condition (iv) of Lemma 3.1. The element ab^{-1} belongs to S, for

$$(ab^{-1})^n d^{-1} = aa^b \ldots a^{b^{n-1}} d^{-b^n} b^{-n}$$

and

$$C(aa^b \ldots a^{b^{n-1}} d^{-b^n}) = \bar{a}\left(\sum_{i=0}^{n-1} \beta^i\right) - \bar{d}\,\beta^n < 0.$$

The element b also belongs to S, but $a = (ab^{-1})b$ clearly does not.

Case 2. b does not normalize D. Since P_A is B-invariant, either $D^b \supset D$ or $D \supset D^b$. Replacing b by b^{-1} if necessary, assume that $D^b \supset D$. We show that the set

$$T = \{x \in G; |x| \ll a\}$$

is not a subgroup. The element ab^{-1} is in T since

$$(ab^{-1})^n a^{-1} = aa^b \ldots a^{b^{n-1}} a^{-b^n} b^{-n} \in P^{-1}.$$

The element b also belong to T; but $a = (ab^{-1})b$ does not. This completes the proof.

COROLLARY 3.3. *Subgroups and direct products of C_1-groups need not be in C_1.*

Proof. Let Q denote the additive group of the rationals and let t be the automorphism of Q corresponding to multiplication by -2. Then $G = \langle Q, t\rangle$ is in C_1 but not in C_2. That G is not in C_2 can be seen by applying Lemma 3.2 to the subgroup $\langle Q, t^2\rangle$. To see that $G \in C_1$ let P be any right-order on G.

Without loss of generality we may assume $t \in P$. For any $x \in Q \cap P$, $x^{t-1} \in P^{-1}$, hence $x < t$ and Q is convex under P. This shows that P is a C-order.

Next consider the direct product of G with an infinite cyclic group: $H = G \times \langle z \rangle$. Every element of H can be written uniquely in the form $(t^2z)^r \times t^s$, where $x \in Q$ and r and s are integers. Let

$$R = \{(t^2z)^r \times t^s;\text{ either } s > 0, \text{ or } s = 0 \text{ and } x > 0, \\ \text{or } s = x = 0 \text{ and } r \geqq 0\}.$$

It is easy to check that R is a right-order on H and that

$$\langle e \rangle \prec \langle (t^2z) \rangle \prec \langle (t^2z), Q \rangle \prec H$$

is its convex series. But $\langle (t^2z) \rangle$ is not normal in $\langle (t^2z), Q \rangle$, hence by Lemma 3.1, R is not a C-order.

Remark. There exist also polycyclic groups which are in C_1 but not in C_2.

COROLLARY 3.4. *Let G be a finitely generated, orderable C_1-group. Then the system of convex subgroups under any order on G, is central.*

Proof. Let P be any order on G. Since G is finitely generated, there exists $J \lhd G$ such that $J \prec G$ is a convex jump under P. Thus there exists $A \geqq J$ such that $G = \langle A, x \rangle$ and G/A is infinite cyclic. By Lemma 3.2, x centralizes every convex jump in A determined by the restriction of P to A, and hence every convex jumpin G. For any a in A, $G = \{A, xa\}$ so that xa also centralizes every jump in G and hence so does a.

THEOREM 3.5. *There exist finitely generated metabelian C_2-groups which are not nilpotent-by-finite, and therefore the class C_2-contains the class $R\ O \cap C_3$ properly.*

Proof. Let $G = \langle a, t;\ a^{t^2}a^{-4t}a^5 = e, [a, a^t] = e \rangle$. Then $A = \langle a \rangle^G$ is an abelian group of rank 2 which can be identified with the subgroup of the additive group of the complex numbers generated by the numbers $(2 + i)^n$, $n \in Z$ on which t acts as multiplication by $2 + i$. Our reason for choosing $2 + i$ is that none of its powers is real.

Let H be any subgroup of G and P any order on H. If $H \leqq A$ or if $H \cap A = \langle e \rangle$, then H is abelian and P is a C-order. Otherwise $H = \langle A \cap H, u \rangle$, where $u = bt^n$ for some $b \in A$, $n \geqq 1$, and u acts on $A \cap H$ as multiplication by the non-real gaussian integer $\xi = (2 + i)^n$. Notice that a gaussian integer $h + ki$ satisfies the equation $x^2 - 2hx + h^2 + k^2 = 0$, so that by choosing $m > 0$ such that the real part of ξ^m is negative, we find a power of ξ which satisfies an equation $x^2 + rx + s = 0$ with $r > 0$ and $s > 0$. Thus for all $c \in A \cap H$, $c^{u^{2m}}\ c^{ru^m}\ c^s = e$ as well as $c\ c^{ru^{-m}}\ c^{su^{-2m}} = e$, and therefore if c is in P, either $c^{u^{-m}}$ or $c^{u^{-2m}}$ is in P^{-1}.

We now show that $A \cap H$ is convex. By changing P to P^{-1} if necessary, we may assume that $u \in P$. Suppose that $b > u^jd > e$ for some $b, d \in A \cap H$, $j \in Z$. If $j \geqq 0$ then $d^{u^{-j}} > u^{-j} > e$. If $j \geqq 0$ then $bd^{-1} > u^j \geqq e$. Thus assume that $c > u^j$ for some $c \in A \cap H$, $j \geqq 0$. Notice that $c > u^j$ implies $cu^j > u^{2j}$ and $c\, c^{u^j} > u^j c^{u^j} = c\, u^j > u^{2j}$, thus if $j \neq 0$, replacing c by another suitable element of $A \cap H$, we may assume $j \geqq 2m$. Thus we have $c > u^i > e$ and hence $cu^{-i} > e$ and $u^i c\, u^{-i} > e$ for $i = 0, 1, \ldots, 2m$. In particular c, $c^{u^{-m}}$ and $c^{u^{-2m}}$ are all in P. This is not possible, therefore $j = 0$ and $A \cap H$ is convex. This implies that P is a C-order and hence that G is a C_2-group.

It is easy to check that G is not nilpotent-by-finite and therefore by Theorem 2.6 it is not a C_3-group.

THEOREM 3.6. *Let G be a finitely generated solvable orderable C_1-group. Then G is nilpotent.*

Proof. Let G be a counterexample of smallest solvability length, and P any order on G. By Corollary 3.4, the system of convex subgroups of G is central. Moreover, as G is finitely generated, it has a descending central series

$$G = G_0 \succ\!\!- G_1 \succ\!\!- \ldots G_n \succ\!\!- G_{n+1} \succ\!\!- \ldots$$

from G to $G_\omega = \bigcap_{n=0}^{\infty} G_n$, where $G_n \succ\!\!- G_{n+1}$ is a convex jump under P. If $G_\omega = G_n$ for some n, then G is nilpotent and we have the required contradiction. If $G_\omega \neq \langle e \rangle$, observe that G/G_ω satisfies the hypotheses of the theorem since any quotient of a C_1-group is in C_1 if it is an $R\,O$-group. Thus we may replace G by G/G_ω and assume $G_\omega = \langle e \rangle$, so that G becomes a residually finitely generated torsion-free nilpotent group and hence residually F_p for all primes p, where F_p is the class of finite p-groups.

Let N be a maximal normal abelian subgroup of G containing the last non-trivial subgroup of the derived series of G. By a result of Learner (see [**5**, Lemma 6.25]), G/N is also residually F_p for all primes p, and hence orderable (see [**3**]). Also $G/N \in C_1$, and thus it is nilpotent by our choice of G. We now use the following result to complete the proof.

LEMMA 3.7. *Let G be an orderable C_1-group. If there exists $\langle e \rangle \neq A \lhd G$, A abelian and G/A finitely generated torsion-free nilpotent, then $Z(G) \cap A \neq \langle e \rangle$, where $Z(G)$ is the centre of G.*

The above lemma applies with $A = N$. Thus $Z(G) \cap N = Z_1 \neq \langle e \rangle$ and G/Z_1 is again orderable since Z_1 is an isolated subgroup in the centre of G. Since G satisfies the maximal condition on normal subgroups, repeated application of Lemma 3.7 shows that $N \leqq Z_k(G)$, the k-th centre of G, for some finite k. Thus G is nilpotent.

Proof of Lemma 3.7. Use induction on $l(G/A)$, the number of factors in any infinite cyclic series of G/A. Suppose $l(G/A) = 1$. Then $G = \langle A, c \rangle$. Take any $e \neq a$ in A and let $A_1 = \langle a \rangle^G$. Let P_1 be any G-order on A_1. Then P_1 can be extended to a G-order P on A since G is a metabelian orderable group. By Lemma 3.2, c centralizes every jump in A determined by P and hence every jump in A_1 determined by P_1. Thus if A_1 has finite rank then $A_1 \cap Z(G) \neq$

$\langle e \rangle$, as required. If A_1 has infinite rank, then it is freely generated by the elements a^{c^i}, $i \in Z$. In this case let ξ be any positive transcendental number and let P_1 consist of those elements $(a^{r_1})^{c^{n_1}} \ldots (a^{r_m})^{c^{n_m}}$ such that $\sum_{i=1}^{m} r_i \xi^{n_i} \geqq 0$. This is an archimedean G-order on A_1 and so $A_1 \leqq Z(G)$.

Now suppose that $l(G/A) = n > 1$. Then there exists $H \lhd G$ such that $A \leqq H$, $G = \langle H, d \rangle$, and $l(G/H) = 1$. Any right-order on H can be extended to a right-order on G. Thus $H \in C_1$ and by the induction hypothesis, $Z(H) \cap A = B \neq \langle e \rangle$. Now $D = \langle A, d \rangle$ is isolated in G and any right-order on D can be extended to a right-order on G since there exists a series from D to G with torsion-free abelian factors. Thus $D \in C_1$ and by the first part of the proof, for any $e \neq b \in B$, $Z(D) \cap \langle b \rangle^D \neq \langle e \rangle$. Thus $Z(G) \cap B \neq \langle e \rangle$ and hence $Z(G) \cap A \neq \langle e \rangle$.

Remark. It follows from Corollary 3.4 and Theorem 3.6 that if G is an orderable C_2-group, then the system of convex subgroups under any order on G is central and G is locally nilpotent if it is locally solvable. In the latter case every partial right-order can be extended to a total right-order (see [**4**]). In general a solvable C_2-group does not have this property as can easily be seen by considering the group $\langle a, b; b^{-1}ab = a^{-1} \rangle$.

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