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RADICALS OF PID'S AND DEDEKIND DOMAINS

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The purpose of this paper is to characterize the radical ideals of principal ideal domains and Dedekind domains. We show that if T is a radical class and R is a PID, then $T(R)$ is an intersection of prime ideals of R. More specifically, if

$$\{0\} \subsetneq T(R) \subsetneq R,$$

then $T(R) = (p_1p_2 \dots p_k)$, where $p_1, p_2, \dots, p_k$ are distinct primes, and where $(p_1p_2 \dots p_k)$ denotes the principal ideal of R generated by $p_1p_2 \dots p_k$. We also characterize the radical ideals of commutative principal ideal rings. For radical ideals of Dedekind domains we obtain a characterization similar to the one given for PID's.

In what follows we are working in some universal (homomorphically closed and hereditary) class of associative rings. Also, if R is a ring and $x \in R$, then (x) denotes the principal ideal of R generated by the element x.

LEMMA 1. *Let R be a* PID, *and let T be a radical class. If p is a prime element of R, then $T(R) \neq (p^k)$ for $k > 1$.*

Proof. If $(p^k) \in T$, then $(p^k)/(p^{k+1}) \in T$. Now define

$$f\colon (p^{k-1})/(p^k) \to (p^k)/(p^{k+1})$$

by

$$f(xp^{k-1} + (p^k)) = xp^k + (p^{k+1}).$$

Since $xp^{k-1} - yp^{k-1} \in (p^k)$ if and only if $xp^k - yp^k \in (p^{k+1})$, f is well-defined and injective. Also, f is clearly surjective and is an additive homomorphism. Since each of $(p^{k-1})/(p^k)$ and $(p^k)/(p^{k+1})$ is a zero-ring (a ring with trivial multiplication), f is a ring homomorphism and hence a ring isomorphism. Thus

$$(p^{k-1})/(p^k) \cong (p^k)/(p^{k+1}).$$

By similar arguments we obtain

$$(p)/(p^2) \cong (p^2)/(p^3) \cong \dots \cong (p^{k-1})/(p^k) \cong (p^k)/(p^{k+1}).$$

Since $(p^k)/(p^{k+1}) \in T$ and $(p^k) \in T$, we have $(p^{k-1}) \in T$ and finally, $(p) \in T$. Thus $T(R) \supseteq (p)$.

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LEMMA 2. *Let R be a* PID, *and let T be a radical class. Let* $x \in R$ ($x \neq 0$ *and x a non-unit*), *and let*

$$x = p_1^{e_1}p_2^{e_2}\dots p_k^{e_k}$$

be the prime factorization of x into distinct primes, with $e_i > 0$ *for* $i = 1, \dots, k$. *If* $e_i > 1$ *for some i, then* $T(R) \neq (x)$.

Proof. By way of contradiction assume $(x) \in T$, and without loss of generality assume $e_k > 1$. Then, as in Lemma 1,

$$(p_1^{e_1}p_2^{e_2}\dots p_k^{e_k-1})/(p_1^{e_1}p_2^{e_2}\dots p_k^{e_k}) \cong (p_1^{e_1}p_2^{e_2}\dots p_k^{e_k})/(p_1^{e_1}p_2^{e_2}\dots p_k^{e_k+1}) \in T$$

so that $(p_1^{e_1}p_2^{e_2}\dots p_k^{e_k-1}) \in T$. Continuing in this way we arrive at $(p_1p_2\dots p_k)/(p_1^2p_2\dots p_2) \in T$ and $(p_1^2p_2\dots p_k) \in T$. Whence, $(p_1p_2\dots p_k) \in T$.

THEOREM 1. *Let R be a* PID, *and let T be a radical class. Then T(R) is a prime ideal of R, or T(R) is the intersection of a finite number of prime ideals of R. (The proper prime ideals of R are, of course, maximal ideals of R.)*

Proof. If $T(R) = R$ or $T(R) = \{0\}$, then $T(R)$ is a prime ideal of R. If $\{0\} \neq T(R) \neq R$, then $T(R) = (x)$, where $x = p_1p_2\dots p_k$, $1 \leqq k$, and where the p_i are distinct primes. This follows from Lemma 2. In this case,

$$T(R) = \bigcap_{i=1}^{k} (p_i).$$

Remark. In any case, $T(R)$ is the intersection (finite or infinite) of prime ideals of R.

COROLLARY. *Let R be a* PID, *and let T be a radical class. If* $\{0\} \neq T(R) \neq R$, *then* $R/T(R)$ *is the direct sum of a finite number of fields.*

Proof. This follows immediately from Theorem 1 and the Chinese Remainder Theorem [**3**].

For our next observations we use Yu-Lee Lee's characterization of the lower radical. Let $\mathscr{A}$ be a class of rings, and let $L(\mathscr{A})$ denote the lower radical class determined by $\mathscr{A}$. In [**4**], Yu-Lee Lee showed that $L(\mathscr{A})$ may be constructed in the following manner: Let $H(\mathscr{A})$ be the class of all homomorphic images of rings in $\mathscr{A}$. For each ring R, let $D_1(R)$ be the set of all ideals of R, and by induction define $D_{n+1}(R)$ to be the family of all rings which are ideals of rings in $D_n(R)$ and set

$$D(R) = \bigcup \{D_n(R): n = 1, 2, \dots\}.$$

A ring R is called an $L(\mathscr{A})$-ring if $D(R/I)$ contains a non-zero ring which is isomorphic to a ring in $H(\mathscr{A})$ for each ideal I of R such that $I \neq R$. The class of $L(\mathscr{A})$-rings coincides with the lower radical determined by $\mathscr{A}$. In

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[**5**], Yu-Lee Lee proved that any class $\mathscr{B}$ of rings determines an upper radical $U(\mathscr{B})$.

The following question naturally arises. Given a PID R and a prime ideal (p) of R, does there exist a radical class T for which $T(R) = (p)$? The following proposition sheds some light on this question.

PROPOSITION 1. *Let R be a* PID *and (p) a prime ideal of R. If there exists an ideal I of the ring (p) for which $(p)/I$ is ring-isomorphic to $R/(p)$, then $T(R) \neq (p)$ for each radical class T.*

Proof. If $T(R) = (p)$, and $(p)/I \cong R/(p)$, then $(p)/I$ is both T-semisimple and T-radical.

Example [S. Bronn, private communication]. Let K be a field, and let $K[x]$ denote the PID of polynomials over K in an indeterminant x. Then $(x)/(x^2 - x) \cong K[x]/(x)$.

PROPOSITION 2. *Let R be a* PID *and (p) a prime ideal of R. If $(p)/I$ is not ring-isomorphic to $R/(p)$ for each ideal I of the ring (p), then $T(R) = (p)$ for some radical class T.*

Proof. Let $\mathscr{A} = \{(p)\}$ and $\mathscr{B} = \{R/(p)\}$. If $T = L(\mathscr{A})$ or $T = U(\mathscr{B})$, then $T(R) = (p)$.

PROPOSITION 3. *Let T_1 and T_2 be radical classes in some universal class W of rings, and define $T(R) = T_1(R) \cap T_2(R)$ and set $T = \{R \in W: T(R) = R\}$. Then $T = T_1 \cap T_2$.*

Proof.
$$\begin{aligned} R \in T &\Leftrightarrow R = T(R) = T_1(R) \cap T_2(R),\\ &\Leftrightarrow R = T_1(R) = T_2(R),\\ &\Leftrightarrow R \in T_1 \cap T_2. \end{aligned}$$

COROLLARY. *Let $T_1, T_2 \ldots, T_n$ be radical classes in some universal class W of rings, and define*

$$T(R) = T_1(R) \cap T_2(R) \cap \ldots \cap T_n(R)$$

and set $T = \{R \in W: T(R) = R\}$. Then $T = T_1 \cap T_2 \cap \ldots \cap T_n$.

THEOREM 2. *Let R be a* PID, *and let $(p_1), (p_2), \ldots, (p_k)$ be distinct prime ideals of R such that there is no non-zero epimorphism of (p_i) to $R/(p_i)$ for $i = 1, 2, \ldots, k$. Then there is a radical class T for which $T(R) = (p_1p_2 \ldots p_k)$.*

Proof. Set $T = T_1 \cap T_2 \cap \ldots \cap T_k$, where $T_i = L(\{(p_i)\})$ or $T_i = U(\{R/(p_i)\})$ for $i = 1, 2, \ldots, k$.

Problem. Characterize the PID's and Dedekind domains R with the following property. If $P_1, P_2, \ldots, P_k$ are distinct prime ideals of R, then there exists a radical class T for which $T(R) = P_1P_2 \ldots P_k$.

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Definition. A principal ideal ring (PIR) is a commutative ring with identity in which every ideal is principal.

Definition. A principal ideal ring R is called *special* if it has only one prime ideal $P \neq R$ and if P is nilpotent.

THEOREM 3 [6]. *A direct sum of* PIR's *is itself a* PIR. *Every* PIR *is a finite direct sum of* PID's *and of special* PIR's.

THEOREM 4. *Let R be special* PIR *with P its unique maximal prime ideal. If T is a radical, then $T(R) = P$ or $T(R) = R$ or $T(R) = \{0\}$.*

Proof. Let m denote the index of nilpotency of $P = (p)$. By [6], the ideals of R are given in the chain

$$R \supsetneqq (p) \supsetneqq (p^2) \supsetneqq \ldots \supsetneqq (p^m) = \{0\}.$$

Moreover, each non-zero $x \in R$ has a representation as $x = ep^k$, $0 \leqq k \leqq m-1$ and e is a unit. Furthermore, the integer k in this representation is unique, and the unit e is unique modulo (p^{m-k}). Assume now that $\{0\} \neq T(R) \neq R$. Then $(p^k) \in T$ for some k, $1 \leqq k \leqq m - 1$. If $k = 1$, there is nothing to prove. Hence assume $1 < k \leqq m - 1$. Then define a mapping

$$(p^{k-1})/(p^k) \rightarrow (p^k)/(p^{k+1})$$

by

$$ep^{k-1} + (p^k) \rightarrow ep^k + (p^{k+1}).$$

Since $ep^{k-1} - up^{k-1} \in (p^k)$ implies $ep^k - up^k \in (p^{k+1})$ the mapping is a well-defined function. Our function is clearly an additive homomorphism, and thus a ring homomorphism, since both $(p^{k-1})/(p^k)$ and $(p^k)/(p^{k+1})$ are zero-rings. Also, the function is obviously surjective. To see that we have an injective function, observe that $ep^k \in (p^{k+1})$ implies that $ep^k = up^{k+l}$ for $l \geqq 1$ and u some unit. But then either $k + l \geqq m$ or (by uniqueness) $k = k + l$. If $k + l \geqq m$, then $0 = up^{k+l} = ep^k$ which is impossible, since $k \leqq m - 1$ and $ep^k = 0$ imply that

$$p^k = e^{-1}ep^k = e^{-1}0 = 0.$$

If $k = k + l$, then $l = 0$; but this contradicts $l \geqq 1$. Thus our function is injective. Hence we have an isomorphism; i.e., $(p^{k-1})/(p^k) \cong (p^k)/(p^{k+1})$. It follows as in the proof of Lemma 1 that $(p^{k-1}) \in T$. Continuing in this manner we obtain $(p) \in T$.

COROLLARY. *Let R be a* PIR, *and let T be a radical class. In order to compute $T(R)$ it suffices to compute the T-radicals of the* PID *and special* PIR *components of R.*

Proof. By Theorem 3,

$$R \cong R_1 \oplus R_2 \oplus \ldots \oplus R_n,$$

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where R_i is either a PID or a special PIR. By a theorem of Hoffman [**2**],

$$T(R_1 \oplus R_2 \oplus \ldots \oplus R_n) = T(R_1) \oplus T(R_2) \oplus \ldots \oplus T(R_n).$$

COROLLARY. *Let R be a* PIR, *and let $R_1 \oplus R_2 \oplus \ldots \oplus R_n$ be its representation as a direct sum of* PID's *and special* PIR's. *If $\{0\} \neq T(R_i) \neq R_i$ for $i = 1, 2, \ldots, n$, then $R/T(R)$ is a direct sum of fields.*

Proof. Merely observe that

$$R_1 \oplus R_2 \oplus \ldots \oplus R_n/T(R_1) \oplus T(R_2) \oplus \ldots \oplus T(R_n) \cong R_1/T(R_1) \oplus R_2/T(R_2) \oplus \ldots \oplus R_n/T(R_n).$$

Definition. A ring R is said to be a Dedekind domain if it is an integral domain and if every ideal in R is a product of prime ideals.

We need the following results concerning commutative rings.

PROPOSITION 4 [**6**]. *In a Dedekind domain R, every proper prime ideal is inventible and maximal.*

PROPOSITION 5 [**6**]. *Let R be a ring with identity, and let D and P be ideals in R such that*
(a) $D \subseteq P$,
(b) *if $b \in P$, then $b^m \in D$ for some m,*
(c) *P is a maximal ideal.*
Then D is primary and P is its radical.

PROPOSITION 6 [**6**]. *Let D and P be ideals in a ring R. Then D is primary and P is its radical if and only if the following conditions are satisfied:*
(a′) $D \subseteq P$.
(b′) *If $b \in P$, then $b^m \in D$ for some m (m may depend on b).*
(c′) *If $ab \in D$ and $a \notin D$, then $b \in P$.*

PROPOSITION 7 [**6**]. *A residue class ring R/A of a Dedekind domain R by a proper ideal A is a* PIR.

PROPOSITION 8 [**1**]. *If $P_1, P_2, \ldots, P_k$ are proper prime ideals of a Dedekind domain and $n_1, n_2, \ldots, n_k$ are positive integers, then*

$$P_1^{n_1}P_2^{n_2}\ldots P_k^{n_k} = P_1^{n_1} \cap P_2^{n_2} \cap \ldots \cap P_k^{n_k}.$$

PROPOSITION 9 [**1**]. *Let R be a Dedekind domain and A a proper ideal of R with factorization $P_1^{n_1}P_2^{n_2}\ldots P_k^{n_k}$. Then R/A is isomorphic to*

$$R/P_1^{n_1} \oplus R/P_2^{n_2} \oplus \ldots \oplus R/P_k^{n_k}.$$

LEMMA 3. *Let R be a Dedekind domain, and let P be a proper prime ideal of R. If T is a radical class, then $T(R) \neq P^k$ for $k > 1$.*

Proof. By Proposition 2, P is a maximal ideal of R. By Proposition 5, P^n is primary for each positive integer n. By Proposition 7, R/P^n is a PIR for each

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positive integer n. Since R/P^{k+1} and R/P^k are PIR's, let $P^k/P^{k+1} = (a + P^{k+1})$ and $P^{k-1}/P^k = (b + P^k)$, where $a \in P^k$, $a \notin P^{k+1}$ and $b \in P^{k-1}$, $b \notin P^k$. Now suppose that $P^k \in T$. Then $P^k/P^{k+1} \in T$. Define a mapping

$$P^k/P^{k+1} \rightarrow P^{k-1}/P^k$$

by

$$ar + P^{k+1} \rightarrow br + P^k.$$

Now by Proposition 6, $a(r - s) = ar - as \in P^{k+1}$ implies $r - s \in P$; and this implies that $b(r - s) \in P^k$. Thus the mapping is a well-defined function. The mapping is clearly an additive homomorphism and is surjective. Since both P^k/P^{k+1} and P^{k-1}/P^k are zero-rings, the function is a ring homomorphism. Thus $P^{k-1}/P^k \in T$. This together with $P^k \in T$ implies $P^{k-1} \in T$. Continuing in this way we obtain $P \in T$.

LEMMA 4. *Let R be a Dedekind domain, and let T be a radical class. Let $P_1, P_2, \ldots, P_k$ be distinct proper prime ideals of R, and let $e_1, e_2, \ldots, e_k$ be positive integers. If $e_i > 1$ for some i, then*

$$T(R) \neq P_1^{e_1}P_2^{e_2} \ldots P_k^{e_k}.$$

Proof. Assume that $P_1^{e_1}P_2^{e_2} \ldots P_k^{e_k} \in T$ and, without loss of generality, assume $e_k > 1$. Using the technique of Lemma 3, define a mapping

$$P_1^{e_1}P_2^{e_2} \ldots P_k^{e_k}/P_1^{e_1}P_2^{e_2} \ldots P_k^{e_k+1} \rightarrow P_1^{e_1}P_2^{e_2} \ldots P_k^{e_k-1}/P_1^{e_1}P_2^{e_2} \ldots P_k^{e_k}$$

by

$$ar + P_1^{e_1}P_2^{e_2} \ldots P_k^{e_k+1} \rightarrow br + P_1^{e_1}P_2^{e_2} \ldots P_k^{e_k}.$$

We use Proposition 6 to prove that the mapping is a well-defined function. For this,

$$a(r - s) = ar - as \in P_1^{e_1}P_2^{e_2} \ldots P_k^{e_k} = P_1^{e_1} \cap P_2^{e_2} \cap \ldots \cap P_k^{e_k+1}$$

and $a \notin P_k^{e_k+1}$ imply $r - s \in P_k$. This in turn implies that

$$b(r - s) \in P_1^{e_1}P_2^{e_2} \ldots P_k^{e_k}.$$

Hence the mapping is a well-defined function and is clearly surjective. Also it is easy to see that the function is an additive homomorphism. Since both rings are zero-rings, the function is a ring homomorphism. Thus, as in the proof of Lemma 3, $P_1^{e_1}P_2^{e_2} \ldots P_k^{e_k-1} \in T$. Continuing in this way we obtain $P_1P_2 \ldots P_k \in T$.

THEOREM 5. *Let R be a Dedekind domain, and let T be a radical class. If $\{0\} \neq T(R) \neq R$, then $T(R)$ is the product of a finite number of distinct prime ideals.*

Proof. This is an immediate consequence of Lemma 4.

COROLLARY. *If the radical $T(R)$ of the Dedekind domain R is a proper ideal of R, then $R/T(R)$ is the direct sum of a finite number of fields.*

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Proof. By Theorem 5, $T(R) = P_1P_2 \ldots P_k$, where the P_i are distinct prime (hence maximal) ideals of R. By Proposition 9,

$$R/T(R) \cong R/P_1 \oplus R/P_2 \oplus \ldots \oplus R/P_k.$$

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