Canad. Math. Bull.Vol. **35** (1), 1992 pp. 21–28

KRASNOSELSKI-MANN ITERATIONS IN NORMED SPACES

JONATHAN BORWEIN[1], SIMEON REICH[2] AND ITAI SHAFRIR

ABSTRACT. We provide general results on the behaviour of the Krasnoselski-Mann iteration process for nonexpansive mappings in a variety of normed settings.

1. **Krasnoselski-Mann iterations.** Let C be a closed convex subset of a normed space $(X, \|\ \|)$ and consider a nonexpansive mapping $p: C \longrightarrow C$. Let $\{t_n\}$ be an arbitrary sequence of real numbers in $[0,1]$ and consider the sequence of iterates $\{x_n\}$ in C generated by: $x_0 \in C$

$$x_{n+1} := (1 - t_n)x_n + t_n p(x_n). \tag{1}$$

This iteration is often said to be a *segmenting Mann iteration* [13], [5], [6] or to be of *Krasnoselski-type* [12], [2], [7]. In this note we use an iterative inequality due to Goebel and Kirk [9], [4], [10] that unifies the basic work of Ishikawa [8] and Edelstein and O'Brien [2]. By refining this inequality—given as Proposition 2 below—we are able to significantly extend certain results on the behaviour of iteration (1). More precisely, we are able to allow for any sequence $\{t_n\}$ with divergent sum and with $\limsup t_n < 1$, while in [17], [18] $\{t_n\}$ was required to be bounded away from 0. Thus, we cover the case of Cesaro and other summability methods [1], [5], [13].

Throughout, we let $y_n := p(x_n)$, and write $d(x,y) := \|x - y\|$. We let $\{x_n^*\}$ denote iteration (1) commencing at x^*.

LEMMA 1. *The following inequalities hold for n and i in N:*

(a) $d(x_{n+1}, y_{n+1}) \le d(x_n, y_n)$;

(b) $d(x_{n+1}, x_{n+1}^*) \le d(x_n, x_n^*)$;

In particular, if $t_n \equiv \alpha$ *then*

(c) $d(x_{n+i+1}, x_{n+1}) \le d(x_{n+i}, x_n)$.

PROOF. (a) is standard and in fact this holds in any hyperbolic space in the sense of [4]. We write

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &= \|(1 - t_n)(x_n - y_n) + y_n - y_{n+1}\| \\ &\le (1 - t_n)\|x_n - y_n\| + \|y_n - y_{n+1}\| \\ &\le (1 - t_n)\|x_n - y_n\| + t_n\|x_n - y_n\|. \end{aligned}$$

[1]Research supported in part by NSERC and completed while the author was visiting the Technion.

[2]Research supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund.

Received by the editors October 9, 1990.

AMS subject classification: Primary: 47H09; secondary: 47H10, 47H17.

The next two inequalities rely more on the linear structure, but do hold in any hyperbolic space in the more restrictive sense [18].

(b) $d(x_{n+1}, x^*_{n+1}) = \|(1-t_n)(x_n - x^*_n) + t_n(y_n - y^*_n)\| \leq (1-t_n)\|x_n - x^*_n\| + t_n\|y_n - y^*_n\| \leq \|x_n - x^*_n\|$, since $\|x_n - x^*_n\| \geq \|y_n - y^*_n\|$.

(c) follows similarly or from applying (b) with $x^*_n = x_{n+i}$. ■

We now list the key elementary inequality from [4], [10].

PROPOSITION 2. *The following inequalities hold for n and i in N:*

$$\Big(1 + \sum_{s=i}^{n+i-1} t_s\Big) d(x_i, y_i) \leq d(x_i, y_{i+n}) + \Big[\prod_{s=i}^{n+i-1} \frac{1}{1-t_s}\Big]\{d(x_i, y_i) - d(x_{i+n}, y_{i+n})\}. \quad ■$$

In particular

$$\Big(\sum_{s=i}^{n+i-1} t_s\Big) d(x_i, y_i) \leq d(x_i, x_{i+n}) + \Big[\prod_{s=i}^{n+i-1} \frac{1}{1-t_s}\Big]\{d(x_i, y_i) - d(x_{i+n}, y_{i+n})\} \tag{2}$$

(on using Lemma 1(a)). This is the form we will work with.

It enables us to prove the following very general result, which with the additional restrictive hypothesis that $\{t_n\}$ is bounded away from zero, may be found in Theorem 6.6 in [18].

2. **The main results.**

We suppose X is complete and we define

$$r_C(p) := \inf\{d\big(x, p(x)\big) : x \in C\}.$$

THEOREM 3. *Suppose that $\{t_n\}$ is divergent in sum with* $\limsup\{t_n : n \in N\} < 1$. *For any initial value x, in the Krasnoselski-Mann iteration (1), $d(x_n, y_n)$ converges to $r_C(p)$.*

PROOF. We first show that $r(x) := \lim d(x_n, y_n)$ (which exists by Lemma 1(a)) is independent of the initial value. Suppose not. Then select points x and x^* such that $r(x) > r(x^*)$. Set $\varepsilon > 0$ with $r(x) > r(x^*) + \varepsilon$ and now fix i with $d(x_j, y_j) < r(x) + \varepsilon$, $d(x^*_j, y^*_j) < r(x^*) + \varepsilon$ for $j \geq i$. We abbreviate

$$S_n := \sum_{s=i}^{n+i-1} t_s$$

and observe (as in [8], [4]) that since $\limsup\{t_n : n \in N\} < 1$, the product term in (2) is bounded above by $\exp(KS_n)$ for some constant K. Thus (2) yields for all n

$$\begin{aligned} S_n r(x) &\leq d(x_i, x_{n+i}) + \varepsilon \exp(KS_n) \\ &\leq d(x^*_i, x^*_{n+i}) + d(x_i, x^*_i) + d(x_{n+i}, x^*_{n+i}) + \varepsilon \exp(KS_n). \end{aligned}$$

By using Lemma 1(b) we have

$$S_n r(x) \leq d(x^*_i, x^*_{i+n}) + 2d(x, x^*) + \varepsilon \exp(KS_n).$$

https://doi.org/10.4153/CMB-1992-003-0 Published online by Cambridge University Press

Now write

$$d(x_i^*, x_{i+n}^*) \le \sum_{s=i}^{n+i-1} d(x_s^*, x_{s+1}^*) = \sum_{s=i}^{n+i-1} t_s d(x_s^*, y_s^*) \le \sum_{s=i}^{n+i-1} t_s d(x_i^*, y_i^*)$$
$$= S_n d(x_i^*, y_i^*),$$

(using Lemma 1(a) with $\{x_n^*\}$). Hence we have

$$S_n[r(x) - r(x^*) - \varepsilon] \le 2d(x, x^*) + \varepsilon \exp(KS_n). \tag{3}$$

Now, much as in [8] or [4], select n such that

$$M \le S_n \le M + 1 \text{ for } M := \big(1 + 2d(x, x^*)\big) / \big(r(x) - r(x^*)\big).$$

Then (3) yields

$$\big(1 + 2d(x, x^*)\big)(r(x) - r(x^*) - \varepsilon / \big(r(x) - r(x^*)\big) \le 2d(x, x^*) + \varepsilon \exp\big(K(M + 1)\big)$$

so that since x, x^*, K, M, $r(x)$, $r(x^*)$ are independent of ε, we may let ε go to 0 to deduce that $1 + 2d(x, x^*) \le 2d(x, x^*)$. This contradiction shows that $r(x)$ is independent of x. Finally, since $r(x) \le d\big(x, p(x)\big)$ for each x, we reach the desired conclusion. ■

Let us recall that p is said to have an approximate fixed point or to be *approximately fixed* in C precisely when $r_C(p) = 0$. Also, p will be said to be *asymptotically regular* in C if, in iteration (1), $d\big(x_n, p(x_n)\big) \longrightarrow 0$, for each x in C.

COROLLARY 4. *If p is approximately fixed then p is asymptotically regular.*

PROOF. This is immediate from Theorem 3. ■

LEMMA 5. *A nonexpansive mapping $p: C \longrightarrow C$ is approximately fixed if any of the following hold:*

(i) p has a fixed point;
(ii) some (each) sequence of iterates $\{x_n\}$ is bounded;
(iii) C is closed and convex and norm bounded;
(iv) X is reflexive and C is closed and convex with no non-zero recession directions;
(v) C is directionally bounded (in the sense of [19]).

PROOF. (i) is obvious.

(ii) Lemma 1(b) shows that if one sequence is bounded, all are. Proposition 2 in [3] shows that p is asymptotically regular (the argument uses (2) and is an easier variant of Theorem 3). This also will be established in Corollary 9 below.

(iii) is standard and follows from the Banach contraction principle applied to $(1 - t)p + tc$ for c in C and t in $(0, 1]$.

(iv) is due to Reich [16]. We repeat the relevant argument for completeness. In [11] it is shown by direct methods that there is a continuous linear functional f^x (depending on x) in the unit dual ball such that

$$f^x\Big(\frac{x - p^{(n)}(x)}{n}\Big) \ge r_C(p) \tag{4}$$

https://doi.org/10.4153/CMB-1992-003-0 Published online by Cambridge University Press

for all positive n in N. Now $\|[x-p^{(n)}(x)]/n\| \leq \|x-p(x)\|$ and so $p^{(n)}(x)/n$ has a weak cluster point d^*. For all $t>0$ and $c \in C$, $(1-t/n)c+tp^{(n)}(x)/n$ lies in C for n large and so $c+td^*$ lies in C. Thus if C has no non-zero recession directions, $d^*=0$ and (4) shows $r_C(p)=0$.

(v) See [19, Theorem 2.4]. ■

The next two results are known, see Fujihara [3] and Reich and Shafrir [17], [18]. Again we sketch simple proofs for the sake of completeness.

LEMMA 6. *Let $\{x_n\}$ be given by (1). Then, for all $n \geq 1$,*

$$d(x_0,x_n) \geq \sum_{i=0}^{n-1} t_i r_C(p).$$

PROOF. We begin with an elementary inequality from [11] for nonexpansive p leaving C invariant, where we assume $0 \in C$ (with no loss of generality): if $x \in C$, $s>0$, and $p(x^*)=(1+s)x^*$ (as exists by Banach's principle) then

$$\|x-x^*\|-\|p(x)-x^*\| \geq \|x^*-p(x^*)\|-2s\|p(x)\|. \tag{5}$$

Now $p_n := (1-t_n)I+t_np$ is nonexpansive and leaves C invariant, and $p_n(x^*)=(1+st_n)x^*$. Thus we may apply (5) for each n. Since $p_n(x_n)=x_{n+1}$ and $\|x^*-p_n(x^*)\| \geq t_n r_C(p)$, we deduce that

$$\|x_n-x^*\|-\|x_{n+1}-x^*\| \geq t_n r_C(p)-2st_n\|x_{n+1}\|,$$

and on summing

$$\|x_0-x_n\| \geq \|x_0-x^*\|-\|x_n-x^*\| \geq S_n r_C(p)-O(s),$$

where S_n is the sum of the first n scalars t_n.

Letting s go to 0 gives the claimed inequality. ■

A little more work with subgradients shows that for each x_0 there is a functional f of norm not exceeding 1 such that for all n

$$f(x_0-x_n) \geq \sum_{i=0}^{n-1} t_i r_C(p).$$

PROPOSITION 7. *If t_n lies in [0,1] and is divergent in sum then*

$$\lim_{n\to\infty} \frac{d(x_0,x_n)}{\sum_{i=0}^{n-1} t_i} = r_C(p).$$

PROOF. By Lemma 6 it is clear that

$$\liminf_{n\to\infty} \frac{d(x_0,x_n)}{\sum_{i=0}^{n-1} t_i} \geq r_C(p).$$

Since $\|x_n-x_n^*\| \leq \|x-x^*\|$, $\limsup d(x_n,x_0)/S_n$ is independent of the initial value x_0 [again S_n is the sum of the first n scalars t_n]. Since $\limsup d(x_n,x_0)/S_n \leq d\big(x_0,p(x_0)\big)$, it follows as required that $\limsup d(x_n,x_0)/S_n \leq r_C(p)$. ■

From now on we assume throughout that $\sum t_n$ diverges with $0<t_n<1-\delta$ for some $\delta>0$.

https://doi.org/10.4153/CMB-1992-003-0 Published online by Cambridge University Press

THEOREM 8. *For any initial value in iteration (1) we have,*

$$\lim_{n\to\infty} d(x_n, x_{n+1})\big/t_n = \lim_{n\to\infty} d(x_n, x_{n+k})\Big/\sum_{j=n}^{n+k-1} t_j = \lim_{n\to\infty} d(x_0, x_n)\Big/\sum_{j=0}^{n-1} t_j \tag{6}$$
$$= r_C(p)$$

for all $k \geq 1$.

PROOF. In view of Theorem 3 and Proposition 7, we need only show that

$$\lim_{n\to\infty} d(x_n, x_{n+k})\Big/\sum_{j=n}^{n+k-1} t_j = r_C(p).$$

Now

$$r_C(p) \leq d(x_n, x_{n+k})\Big/\sum_{j=n}^{n+k-1} t_j \leq d\big(x_n, p(x_n)\big)$$

where the first inequality is a consequence of Lemma 6 with initial value x_n. The second follows from convexity of the norm and Lemma 1(a). Now Theorem 3 completes the argument. ∎

COROLLARY 9. *Let* $\{x_n\}$ *be defined by (1).*

(a) Either $\{x_n\}$ *is bounded and* p *is asymptotically regular or* $\{\|x_n\|\}$ *tends to infinity.*

(b) If all closed bounded convex subsets of C *have the fixed point property for nonexpansive mappings then either* p *has a fixed point or* $\{\|x_n\|\}$ *tends to infinity.*

PROOF. (a) It follows from Theorem 3 and Lemma 6 that, when $\{x_n\}$ is bounded, p is asymptotically regular.

Suppose $\liminf \|x_n\| < \infty$. Pick a bounded subsequence $\{x_{n(k)}\}$ and define $R := \limsup \|x_{n(k)} - x_0\|$. Consider

$$D := \{y \in C : \limsup_{k\to\infty} d(y, x_{n(k)}) \leq R\}.$$

Clearly, D is closed convex bounded and nonempty. For any y in C we have

$$d\big(p(y), x_{n(k)}\big) \leq d(y, x_{n(k)}) + d\big(x_{n(k)}, p(x_{n(k)})\big)$$

since p is nonexpansive. As p is asymptotically regular, $d\big(x_{n(k)}, p(x_{n(k)})\big) \longrightarrow 0$. It follows that D is left invariant by p. Since x_0 lies in D, the entire sequence does and (a) is proven.

(b) By hypothesis, p has a fixed point and (b) follows. ∎

COROLLARY 10. *Let* $\{x_n\}$ *be defined by (1). If* $\{x_n\}$ *has a norm cluster-point* x^* *then* $\{x_n\}$ *converges to* $x^* = p(x^*)$.

PROOF. It follows from Theorem 3 that $d\big(x^*, p(x^*)\big) = r_C(p)$. By Lemma 6, $r_C(p) = 0$. Since $\{d(x^*, x_n)\}$ is a decreasing sequence, we are finished. ∎

Goebel and Kirk [9], [4, Theorem 3] show that Corollary 10 holds in any of their hyperbolic metrics when t_n is constant, say α.

COROLLARY 11. *If C is boundedly relatively compact, in particular if X is finite dimensional, then either $\{x_n\}$ converges to $x^* = p(x^*)$ or $\{\|x_n\|\}$ tends to infinity.*

PROOF. This is immediate from Corollary 10. ∎

REMARK 12. Theorem 3 and Theorem 8 continue to hold with essentially the same proof in any hyperbolic space (in the sense of Reich and Shafrir). In consequence the condition that $\{t_n\}$ be bounded away from 0 may be dropped from the appropriate results in both [17] and [18].

3. **Two geometric corollaries.** We finish by rederiving two essentially known conclusions that exploit the convexity structure of the norm on the space. The first result is the counterpart to Lemma 6, and the comment following it.

THEOREM 13. *If the dual norm on X^* is strictly convex then there is a functional f in the dual unit ball such that, independent of x,*

$$f(x - x_n) \geq \sum_{s=0}^{n-1} t_s r_C(p) \tag{7}$$

and so, as before

$$\|x - x_n\| \geq \sum_{s=0}^{n-1} t_s r_C(p) \tag{8}$$

and, in particular, either p is asymptotically regular or $\|x_n\|$ tends to infinity.

PROOF. We may assume that $r_C(p) > 0$. Reich [14, p. 140] has observed that if the dual norm on X^* is strictly convex then the norm-one functional in (4) can be supposed independent of x. Indeed for x and y in C $\{p^{(n)}(x)/n\}$ and $\{p^{(n)}(y)/n\}$ are bounded and have a common cluster-point, say $-F$, in X^{**} [as $\|p^{(n)}(x) - p^{(n)}(y)\|/n \longrightarrow 0$]. Since we have $\|F\|_{**} \leq r_C(p)$, because $\|[x - p^{(n)}(x)]/n\| \leq \|x - p(x)\|$, (4) yields

$$F(f^x) = F(f^y) = r_C(p)$$

which shows that $F([f^x + f^y]/2) = r_C(p)$ and that $\|[f^x + f^y]/2\| = 1$. The assumption of strict convexity now shows $f^x = f^y$. We have

$$x - x_n = \sum_{s=0}^{n-1} t_s\big(x_s - p(x_s)\big)$$

and so applying (4) within $n := 1$ and $f := f^x$ gives

$$f(x - x_n) = \sum_{s=0}^{n-1} t_s f\big(x_s - p(x_s)\big) \geq \sum_{s=0}^{n-1} t_s r_C(p)$$

so that since $\|f\| = 1$ the result follows. ∎

Note that (7) is actually a stronger inequality than (8).

https://doi.org/10.4153/CMB-1992-003-0 Published online by Cambridge University Press

COROLLARY 14. *If X has a uniformly convex and Fréchet differentiable norm then either $\{x_n\}$ converges weakly to $x^* = p(x^*)$ or $\{\|x_n\|\}$ tends to infinity.*

PROOF. If $r_C(p) > 0$ then Theorem 8 or 13 applies (since X is reflexive, the norm on X^* is strictly convex when the norm on X is Gateaux differentiable). Otherwise, $r_C(p) = 0$ and by Corollary 4, $d\big(x_n, p(x_n)\big) \longrightarrow 0$. Now either $\{\|x_n\|\} \longrightarrow \infty$ or $\{x_n\}$ has a weak cluster point x^*. In the latter case, as is well known, $x^* = p(x^*)$ since $d\big(x_n, p(x_n)\big) \longrightarrow 0$ [15]. Thus Theorem 2 in [15] applies and shows that $\{x_n\}$ converges weakly to x^*. ■

EXAMPLE 15. (a) The next example attributed to J. F. Mertens in [11] illustrates the need for strict convexity in (7) of Theorem 13. See also Theorem 15 in [17] and the remark following it. Let $p: R^2 \longrightarrow R^2$ be defined by

$$p(x,y) := \big(x\text{-signum}\,(x), y\big) \quad \text{if } |x| \geq 1$$

and

$$p(x,y) := (0, 1 + y - |x|) \quad \text{if } |x| < 1.$$

Then p is nonexpansive in the l_1 norm. Moreover $r_C(p) = 1$. But, 0 lies in the convex hull of the range of $I - p = \{(x, 1 - |x|) : |x| \leq 1\}$ and a fortiori one can not pick f independent of x.

(b) We let X be any of the traditional sequence spaces and consider $C := \{x \in X : |x_n| \leq 1 \text{ for } n \in N\}$ with $p: C \longrightarrow C$ given via $p\big((x_1, \ldots, x_n, \ldots)\big) := (1, x_1, \ldots, x_n, \ldots)$. Thus p is an isometry in $l_q(N)$, $1 \leq q \leq \infty$, and C is linearly bounded. Despite this, for $q := 1$, $r_C(p) = 1$. For $q > 1$, Lemma 4 applies and shows that p is asymptotically regular. For $1 < q < \infty$, Corollary 14 implies that $\{\|x_n\|\} \longrightarrow \infty$, because p is fixed point free. For $q = \infty$, we observe that p is w^*-continuous and so that $\{x_n\}$ converges w^* to the unique fixed point $(1, 1, \ldots 1, \ldots)$. In particular, if x lies in c_0 it follows from Corollary 10 that $\{x_n\}$ has no norm cluster-points. ■

ACKNOWLEDGEMENT. The first author would like to thank Jonathan Spingarn for several very useful discussions on the topic.

REFERENCES

1. W. G. Dotson, *On the Mann iterative process*, Trans. Amer. Math. Soc. **149**(1970), 655–73.
2. M. Edelstein and R. C. O'Brien *Nonexpansive mappings, asymptotic regularity and successive approximations*, J. London Math. Soc. **17**(1978), 547–554.
3. T. Fujihara, *Asymptotic behavior of nonexpansive mappings in Banach spaces*, Tokyo J. Math. **7**(1984), 119–128.
4. K. Goebel, and W. A. Kirk, *Iteration processes for nonexpansive mappings*, Contemporary Mathematics **21**(1983), 115–123.
5. C.W. Groetsch, *A note on segmenting Mann iterates*, J. Math. Anal. and Appl. **40**(1972), 369–372.
6. T. L. Hicks and J. D. Kubicek, *On the Mann iteration process in Hilbert space*, J. Math. Anal. and Appl. **59**(1977), 498–504.
7. B. P. Hillam, *A generalization of Krasnoselski's theorem on the real line*, Mathematics Magazine **48**(1975), 167–168.
8. S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. **59**(1976), 65–71.

9. W. A. Kirk, *Krasnoselskii's iteration process in hyperbolic space*, Numer. Funct. Anal. Optimiz. **4**(1982), 371–381.

10. ———, *Fixed point theory for nonexpansive mappings, I and II*, (I) appears in Lecture Notes in Mathematics **886** (Springer-Verlag, 1981), pp. 484–505; (II) appears in: Fixed Points and Nonexpansive Mappings (R. Sine, ed.), Contemporary Math. **18**, Amer. Math. Soc., Providence RI, (1983), pp. 121–140.

11. E. Kohlberg, and A. Neyman, *Asymptotic behaviour of nonexpansive mappings in normed linear spaces*, Israel J. Math. **38**(1981), 269–275.

12. M. A. Krasnoselski, *Two observations about the method of successive approximations*, Usp. Math. Nauk, **10**(1955), 123–127.

13. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4**(1953), 506–510.

14. S. Reich, *On the asymptotic behaviour of nonlinear semigroups and the range of accretive operators II*, J. Math. Anal. Appl. **87**(1982), 134–146.

15. ———, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67**(1979), 274–276.

16. ———*The almost fixed point property for nonexpansive mappings*, Proc. Amer. Math. Soc. **88**(1983), 44–45.

17. S. Reich and I. Shafrir, *On the method of successive approximations for nonexpansive mappings*, Nonlinear and Convex Analysis, Marcel Dekker, New York, (1987), 193–201.

18. ———, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Analysis **15**(1990), 537-558.

19. I. Shafrir, *The approximate fixed point property in Banach and hyperbolic spaces*, Israel J. Math, **71**(1990), 211–223.

Department of Mathematics
Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia
B3H 3J5

Present Address:
Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario
N2L 3G1

Department of Mathematics
University of Southern California
Los Angeles, California
U.S.A. 90089
and
Department of Mathematics
Technion—Israel Institute of Technology
32000 Haifa
Israel

Department of Mathematics
Technion—Israel Institute of Technology
32000 Haifa
Israel

Present Address:
Analyse Numerique
Université de Paris VI
75252 Paris
France