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NONLINEARLY CONSTRAINED OPTIMAL CONTROL PROBLEMS INVOLVING PIECEWISE SMOOTH CONTROLS

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Abstract

In this paper, we consider a class of optimal control problems involving inequality continuous-state constraints in which the control is piecewise smooth. The requirement for this type of control is more stringent than that for the control considered in standard optimal control problems in which the controls are usually taken as bounded measurable functions. In this paper, we shall show that this class of optimal control problems can easily be transformed into an equivalent class of combined optimal parameter selection and optimal control problems. We shall then use the control parametrisation technique to devise a computational algorithm for solving this equivalent dynamic optimisation problem. Furthermore, convergence analysis will be given to support this numerical approach. For illustration, two nontrivial optimal control problems involving transferring cargo via a container crane will be solved using the proposed approach.

1. Introduction

In [2] and [16], the concept of control parametrisation is used to devise a computational method for solving a general class of optimal control problems subject to canonical constraints, both in equality as well as inequality form. By the concept of control parametrisation, we mean that the control is approximated by a piecewise constant function with possible discontinuities at the preassigned switching points.

For the inequality continuous state constraints, a simple constraint transcription is used in [2] to convert these inequality continuous state constraints into equivalent equality constraints in canonical form. However, there are

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many nonstandard optimal control problems which cannot be solved using the algorithm of [2]. Examples include optimal parameter selection problems; free terminal time optimal control problems, including minimum time problems; minimax optimal control problems (problems with Chebyschev cost functional); and boundary value control problems, including problems with periodic boundary conditions and inter-related boundary conditions. Thus, a more general class of dynamic optimisation problems known as combined optimal parameter selection and optimal control problems is considered in [16]. It is then shown that all the above-mentioned nonstandard optimal control problems can be converted into special cases of this general class of dynamic optimisation problems.

Note that the constraint transcription used in [2] to deal with the inequality continuous state constraints is very easy to apply. However, it has the disadvantage that the equality constraints so obtained do not satisfy the usual constraint qualification, and hence convergence is not guaranteed and some oscillation may result in numerical computation. According to [17], the algorithm rarely converges for nontrivial optimal control problems involving inequality continuous state constraints, although it does give good approximate results. Again, according to [17] it is impossible to overcome numerically the violations of the continuous state constraints, using the constraint transcription presented in [2]. Nonetheless, a general purpose optimal control software known as MISER (cf. [3]) was developed using the results of [2] and [16].

In [17], the idea of [5] together with the concept of control parametrisation is used to devise a computational algorithm for solving a general class of optimal control problems involving inequality continuous state constraints. The main contribution of the paper is to overcome the two disadvantages existing in the constraint transcription introduced in [2]. Numerical experience conducted in [17] has demonstrated that the new algorithm is a much more stable one. The idea of this constraint transcription can be easily incorporated in MISER for dealing with inequality continuous state constraints. We refer the reader to [1] for many interesting theoretical results for the class of optimal control problems considered in [17].

In this paper, we are concerned with a class of optimal control problems involving inequality continuous state constraints in which the control is piecewise smooth (i.e., the control is continuous and its derivative is piecewise smooth). Since the requirement for this type of control is more stringent than that for the control in [17], this class of optimal control problems cannot be solved directly using the results of [17]. Furthermore, the optimal control software MISER cannot be used directly. However, in view of the idea presented in Section 6 of [16], we can transform this class of optimal control problems into an equivalent class of combined optimal parameter selection and optimal control problems. The equivalent problems can be solved numerically by several optimal control algorithms such as gradient-restoration algorithms due to Miele (cf. [4], [6–11]), multiplier methods (cf. [12] and [13]), and control parametrisation algorithms (cf. [16]). In this paper, we shall use the control parametrisation technique to handle this equivalent dynamical optimisation problem. Furthermore, vigorous convergence analysis will be given to support this numerical approach. For illustration, two non-trivial optimal control problems involving transferring cargo containers via a container crane will be solved using the proposed approach.

2. Problem statement

Consider a process described by the following state differential equations defined on the fixed time interval $(0, T]$:

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{2.1a}$$

where

$$x = [x_1, \dots, x_n]^{\mathrm{T}} \in \mathbb{R}^n, \qquad u = [u_1, \dots, u_r]^{\mathrm{T}} \in \mathbb{R}^r$$

are, respectively, state and control vectors; $f = [f_1, \dots, f_n]^{\mathrm{T}} \in \mathbb{R}^n$ is a given real valued function; and the superscript $^{\mathrm{T}}$ denotes the transpose.

The initial condition for the differential equation (3.2.1a) is:

$$x(0) = x^0 \tag{2.1b}$$

where $x^0 = [x_1^0, \dots, x_n^0]^{\mathrm{T}} \in \mathbb{R}^n$ is a given vector. Define

$$U = \{v = [v_1, \dots, v_r]^{\mathrm{T}} \in \mathbb{R}^r : \alpha_i \le v_i \le \beta_i, \ i = 1, \dots, r\} \tag{2.2}$$

where α_i, $i = 1, \dots, r$, and β_i, $i = 1, \dots, r$, are given real numbers. Note that U is a compact and convex subset of $\mathbb{R}^r$.

DEFINITION 2.1. A function $u : [0, T] \to \mathbb{R}^r$ is said to be piecewise smooth if it is continuous and its derivative is piecewise continuous.

Let u be a piecewise smooth function defined on $[0, T]$ with values in U, and let $\dot{u}$ denote the derivative of u. If

$$c_i \le \dot{u}_i(t) \le d_i, \quad \forall t \in [0, T], \ i = 1, \dots, r, \tag{2.3}$$

where c_i, $i = 1, \dots, r$, and d_i, $i = 1, \dots, r$, are given real numbers, then the u is called an admissible control. Let $\mathscr{U}$ be the class of all such admissible controls. Furthermore, let $\mathring{\mathscr{U}}$ be a subset of the set $\mathscr{U}$ defined by

$$\mathring{\mathscr{U}} = \{u \in \mathscr{U} : \alpha_i < u_i(t) < \beta_i, \ \forall t \in [0, T], \ i = 1, \dots, r\} \tag{2.4}$$

Note that the class $\mathscr{U}$ of admissible controls considered in this section is more restrictive than that considered in [17], where bounded measurable functions are taken as admissible controls.

For each $u \in \mathscr{U}$, let $x(\cdot \mid u)$ be an absolutely continuous function defined on $[0, T]$ which satisfies the differential equation (2.1a) almost everywhere in $(0, T]$ and the initial condition (2.1b). This function is called the solution of the system (2.1) corresponding to the control $u \in \mathscr{U}$.

The inequality terminal state constraints and inequality continuous state constraints are specified as follows:

$$\Phi_i(x(T \mid u)) \geq 0, \qquad i = 1, \ldots, N_{\mathrm{T}} \tag{2.5}$$

where Φ_i, $i = 1, \ldots, N_{\mathrm{T}}$, are given real valued functions defined on $\mathbb{R}^n$, and

$$h_i(t, x(t \mid u), u(t)) \geq 0, \quad \forall t \in [0, T], \; i = 1, \ldots, N, \tag{2.6}$$

where h_i, $i = 1, \ldots, N$, are given real valued functions defined on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^r$.

Note that our admissible controls are required to be piecewise smooth. Thus, they are allowed to appear in the inequality continuous state constraints (2.6). This is a slight generalisation of that considered in [17].

If $u \in \mathscr{U}$ satisfies the constraints (2.5) and (2.6), then it is called a feasible control. Let $\mathscr{F}$ be the class of all feasible controls.

We may now state our optimal control problem as follows:

PROBLEM (P). Given the system (2.1), find a control $u \in \mathscr{F}$ such that the cost functional

$$g_0(u) = \Phi_0(x(T \mid u)) + \int_0^{\mathrm{T}} \mathscr{L}_0(t, x(t \mid u), u(t))\, dt \tag{2.7}$$

is minimised over $\mathscr{F}$, where Φ_0 and $\mathscr{L}_0$ are given real valued functions, and T is the terminal time of the problem.

The following conditions are assumed throughout:

(A1) $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ is piecewise continuous on $[0, T]$ for each $(x, u) \in \mathbb{R}^n \times \mathbb{R}^r$, and continuously differentiable with respect to each of the components of x and u for each $t \in [0, T]$; and furthermore, for any given compact subset $C \subset \mathbb{R}^r$, there exists a constant $K > 0$ such that

$$|f(t, x, u)| \leq K(1 + |x|)$$

for all $(t, x, u) \in [0, T] \times C \times \mathbb{R}^n$, where $|\cdot|$ denotes the usual Euclidean norm;

(A2) For each $i = 1, \ldots, N_{\mathrm{T}}$, $\Phi_i : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable;

(A3) For each $i = 1, \ldots, N$, $h_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ is continuously differentiable;

(A4) $\Phi_0 : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable;

(A5) $\mathscr{L}_0 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ is piecewise continuous on $[0, T]$ for each $(x, u) \in \mathbb{R}^n \times \mathbb{R}^r$, and continuously differentiable with respect to each of the components of x and u for each $t \in [0, T]$.

Define

$$\Theta = \{u \in \mathscr{U} : \Phi_i(x(T \mid u)) \geq 0, \quad i = 1, \ldots, N_{\mathrm{T}}\} \tag{2.8}$$

and

$$\mathscr{F} = \{u \in \Theta : h_i(t, x(t \mid u), u(t)) \geq 0, \quad \forall t \in [0, T], \quad i = 1, \ldots, N\}. \tag{2.9}$$

Let $\mathring{\Theta}$ and $\mathring{\mathscr{F}}$ be, respectively, subsets of the sets Θ and $\mathscr{F}$ defined by

$$\mathring{\Theta} = \{u \in \mathring{\mathscr{U}} : \Phi_i(x(T \mid u)) > 0, \quad i = 1, \ldots, N_{\mathrm{T}}\} \tag{2.10}$$

and

$$\mathring{\mathscr{F}} = \{u \in \mathring{\Theta} : h_i(t, x(t \mid u), u(t)) > 0, \quad \forall t \in [0, T], \quad i = 1, \ldots, N\} \tag{2.11}$$

To continue, we assume that the following condition is satisfied.

(A6) For any $u \in \mathscr{F}$, there exists a $\bar{u} \in \mathring{\mathscr{F}}$ such that

$$\alpha\bar{u} + (1 - \alpha)u \in \mathring{\mathscr{F}} \quad \forall \alpha \in (0, 1].$$

3. Model transformation

In this section, our aim is to convert the problem (P) into a form solvable by the optimal control software MISER. To begin, we follow the idea of Section 6 of [16] to introduce an extra set of differential equations for the control u:

$$\dot{u}(t) = v(t) \tag{3.1a}$$

with the initial conditions:

$$u(0) = \xi \tag{3.1b}$$

where $v = [v_1, \ldots, v_r]^{\mathrm{T}} \in \mathbb{R}^r$; and $\xi = [\xi_1, \ldots, \xi_r]^{\mathrm{T}}$.

In view of (3.1a), we see that u is now a state function rather than the control function. It is determined by the new control function v and the initial vector ξ. For convenience, let ξ be referred to as the system parameter. Clearly, for a given system parameter vector ξ, if v is approximated by a piecewise constant function, then u will be a piecewise linear function.

Define

$$\tilde{x} = [x_1, \dots, x_n, x_{n+1}, \dots, x_{n+r}]^{\mathrm{T}}, \tag{3.2}$$

where

$$x_{n+i} = u_i, \qquad i = 1, \dots, r. \tag{3.3}$$

Thus, by appending (3.1) to (2.1), we have

$$\dot{\tilde{x}}(t) = \tilde{f}(t, \tilde{x}(t), v(t)) \tag{3.4a}$$

$$\tilde{x}(0) = \tilde{x}^0(\xi) \tag{3.4b}$$

where

$$\tilde{f} = [f_1, \dots, f_n, v_1, \dots, v_r]^{\mathrm{T}} \tag{3.4c}$$

and

$$\tilde{x}^0(\xi) = [x_1^0, \dots, x_n^0, \xi_1, \dots, \xi_r]. \tag{3.4d}$$

In view of the definition of $\mathscr{U}$, the following constraints must be satisfied.

$$\alpha_i \le \xi_i \le \beta_i, \qquad i = 1, \dots, r, \tag{3.5}$$

$$\alpha_i \le x_{n+i}(t) \le \beta_i, \quad \forall t \in [0, T], \; i = 1, \dots, r, \tag{3.6}$$

and

$$c_i \le v_i(t) \le d_i, \quad \forall t \in [0, T], \; i = 1, \dots, r. \tag{3.7}$$

Let $\mathscr{Z}$ be the set containing all vectors ξ such that the constraints (3.5) are satisfied. Furthermore, let $\mathscr{V}$ be the set containing all functions v such that the constraints (3.7) are satisfied.

We shall call elements from $\mathscr{V}$ admissible controls and $\mathscr{V}$ the class of admissible controls. Furthermore, we shall call elements from $\mathscr{Z}$ system parameter vectors and $\mathscr{Z}$ the set of system parameter vectors.

To continue, let $\mathring{\mathscr{V}}$ be a subset of the set $\mathscr{V}$ such that the following constraints are satisfied.

$$\alpha_i < x_{n+i}(t) < \beta_i, \quad \forall t \in [0, T], \; i = 1, \dots, r.$$

For each $(\xi, v) \in \mathscr{Z} \times \mathscr{V}$, let $\tilde{x}(\cdot \mid \xi, v)$ be the corresponding solution of the system (3.4). The constraints (3.6) can be written as:

$$\tilde{h}_{i+N}(t, \tilde{x}(t \mid \xi, v)) = \beta_i - x_{n+i}(t \mid \xi, v) \ge 0, \quad \forall t \in [0, T], \; i = 1, \dots, r \tag{3.8a}$$

and

$$\tilde{h}_{i+N+r}(t, \tilde{x}(t \mid \xi, v)) = x_{n+i}(t \mid \xi, v) - \alpha_i \ge 0, \quad \forall t \in [0, T], \; i = 1, \dots, r. \tag{3.8b}$$

For the inequality terminal state constraints (2.5) and the inequality continuous state constraints (2.6), they are written as:

$$\tilde{\Phi}_i(\tilde{x}(T \mid \xi, v)) = \Phi_i(x(T \mid U)) \ge 0, \qquad i = 1, \dots, N_{\mathrm{T}} \tag{3.9}$$

and

$$\tilde{h}_i(t, \tilde{x}(t \mid \xi, v)) = h_i(t, x(t \mid u), u(t)) \geq 0, \quad \forall t \in [0, T], \; i = 1, \dots, N. \tag{3.10}$$

Define

$$\tilde{\Theta} = \{(\xi, v) \in \mathscr{Z} \times \mathscr{V} : \tilde{\Phi}_i(\tilde{x}(T \mid \xi, v)) \geq 0, \; i = 1, \dots, N_{\mathrm{T}}\} \tag{3.11}$$

and

$$\widetilde{\mathscr{F}} = \{(\xi, v) \in \tilde{\Theta} : \tilde{h}_i(t, \tilde{x}(t \mid \xi, v)) \geq 0, \; \forall t \in [0, T], \; i = 1, \dots, N + 2r\}. \tag{3.12}$$

Let $\mathring{\tilde{\Theta}}$ and $\mathring{\widetilde{\mathscr{F}}}$ be, respectively, the subsets of the sets $\tilde{\Theta}$ and $\widetilde{\mathscr{F}}$ defined by

$$\mathring{\tilde{\Theta}} = \left\{(\xi, v) \in \mathscr{Z} \times \mathring{\mathscr{V}} : \tilde{\Phi}_i(\tilde{x}(T \mid \xi, v)) > 0, \; i = 1, \dots, N_{\mathrm{T}}\right\} \tag{3.13}$$

and

$$\mathring{\widetilde{\mathscr{F}}} = \left\{(\xi, v) \in \mathring{\tilde{\Theta}} : \tilde{h}_i(t, \tilde{x}(t \mid \xi, v)) > 0, \; \forall t \in [0, T], \; i = 1, \dots, N + 2r\right\}. \tag{3.14}$$

We now consider the following combined optimal parameter selection and optimal control problem:

Given the dynamical system (3.4), find a combined parameter vector and control $(\xi, v) \in \widetilde{\mathscr{F}}$ such that the cost functional

$$\tilde{g}_0(\xi, v) = \tilde{\Phi}_0(\tilde{x}(T \mid \xi, v)) + \int_0^{\mathrm{T}} \widetilde{\mathscr{L}}_0(t, \tilde{x}(t \mid \xi, v), v(t))\, dt \tag{3.15}$$

is minimised over $\widetilde{\mathscr{F}}$, where

$$\tilde{\Phi}_0(\tilde{x}(T \mid \xi, v)) = \Phi_0(x(T \mid u)) \tag{3.16}$$

and

$$\widetilde{\mathscr{L}}_0(t, \tilde{x}(t \mid \xi, v), v(t)) = \mathscr{L}_0(t, x(t \mid u), u(t)) \tag{3.17}$$

while $\Phi_0(x(T \mid u))$ and $\mathscr{L}_0(t, x(t \mid u), u(t))$ are given in (2.6)—the cost functional of the problem (P).

For convenience, let this combined optimal parameter selection and optimal control problem be referred to as the problem (Q). In fact, it is obvious that the problem (P) is equivalent to the problem (Q). Nonetheless, this trivial result is presented in the following as a theorem for convenience in future reference.

Theorem 3.1. *The problem* (P) *is equivalent to the problem* (Q) *in the sense that for each* $u \in \mathscr{F}$ *there corresponds uniquely a* $(\xi, v) \in \widetilde{\mathscr{F}}$ *such that* $g_0(u) = \tilde{g}_0(\xi, v)$ *and vice versa.*

4. Control parametrisation

In view of Theorem 3.1, it suffices to solve the problem (Q). Since the problem (Q) is in the form considered in [17], the constraint transcription introduced in [5] can be used together with the concept of control parametrisation to solve the problem (Q). To begin, we use the problem (Q) to construct a sequence of problems such that the solution of each of these approximate problems is a suboptimal solution to the problem (Q). This is achieved through the discretisation of the control space by approximating each control with a piecewise constant function with possible discontinuities at preassigned switching points. Since the problem (P) is equivalent to the problem (Q), each of these suboptimal solutions generates readily a suboptimal solution to the problem (P). The details of the approximation of control space are given as follows:

Consider a monotonically nondecreasing sequence $\{S^p\}_{p=1}^{\infty}$ of finite subsets of $[0, T]$. For each p, let n_p+1 points of S^p be denoted by $t_0^p, t_1^p, \dots, t_{n_p}^p$. These points are chosen such that $t_0^p = 0$, $t_{n_p}^p = T$, and $t_{k-1}^p < t_k^p$, $k = 1, 2, \dots, n_p$. Thus, associated with each S^p there is the obvious partition $\mathscr{I}^p$ of $[0, T)$ defined by $\mathscr{I}^p \equiv \{I_k : k = 1, \dots, n_p\}$, where $I_k^p = [t_{k-1}^p, t_k^p)$.

We choose S^p such that the following two properties are satisfied:

(C1) S^{p+1} is a refinement of S^p; and

(C2) $\lim_{p\to\infty} S^p$ is dense in $[0, T]$. This is equivalent to requiring that $\lim_{p\to\infty} \max_{k=1,\dots,n_p} |I_k^p| = 0$

where $|I_k^p| = t_k^p - t_{k-1}^p$, the length of the kth interval.

Let $\mathscr{V}^p$ consist of all those elements from $\mathscr{V}$ which are piecewise constant and consistent with the partition $\mathscr{I}^p$. It is clear that each $v \in \mathscr{V}^p$ can be written as:

$$v^p(t) = \sum_{k=1}^{n_p} \sigma^{p,k} \chi_{I_k^p}(t), \tag{4.1}$$

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where χ_I denotes the indicator function of I defined by

$$\chi_I(t) = \begin{cases} 1 & t \in I \\ 0 & \text{elsewhere} \end{cases} \tag{4.2}$$

$\sigma^{p,k} \in V$, and

$$V = \{v \in \mathbb{R}^r : c_i \leq v_i \leq d_i,\ \ i = 1, \dots, r\}. \tag{4.3}$$

Let $\sigma^p = [\sigma^{p,1^{\mathsf{T}}}, \dots, \sigma^{p,n_p^{\mathsf{T}}}]^{\mathsf{T}}$ where $\sigma^{p,k^{\mathsf{T}}} = [\sigma_1^{p,k}, \dots, \sigma_r^{p,k}]$. Restricting to $\mathscr{V}^p$, the control constraints defined in (4.3) become:

$$c_i \leq \sigma_i^{p,k} \leq d_i, \qquad i = 1, \dots, r;\ k = 1, \dots, n_p. \tag{4.4}$$

Let Ξ^p be the set of all those σ^p vectors which satisfy the constraints (4.4).

Clearly, for each control $v^p \in \mathscr{V}^p$ there exists a unique control parameter vector $\sigma^p \in \Xi^p$ such that (4.1) is satisfied. Conversely, there also exists a unique control $v^p \in \mathscr{V}^p$ corresponding to each control parameter vector $\sigma^p \in \Xi^p$.

With $v \in \mathscr{V}^p$, the differential equation (3.4) takes the form:

$$\dot{\tilde{x}}(t) = \tilde{\tilde{f}}(t, \tilde{x}(t), \sigma^p), \tag{4.5a}$$

where

$$\tilde{\tilde{f}}(t, \tilde{x}(t), \sigma^p) = \tilde{f}\left(t, \tilde{x}(t), \sum_{k=1}^{n_p} \sigma^{p,k} \chi_{I_k^p}(t)\right)$$

and $\sigma^p = [\sigma^{p,1^{\mathsf{T}}}, \dots, \sigma^{p,n_p^{\mathsf{T}}}]^{\mathsf{T}}$. The initial condition remains the same:

$$\tilde{x}(0) = \tilde{x}^0(\xi). \tag{4.5b}$$

Define $\theta^p = (\xi, \sigma^p)$. For convenience, let θ^p be referred to as the combined vector. Let $\tilde{x}(\cdot \mid \theta^p)$ be the solution of the system (4.5) corresponding to the combined vector $\theta^p \in \mathscr{Z} \times \Xi^p$.

Similarly, by restricting v in $\mathscr{V}^p$, the constraints in (3.11) and (3.12) are reduced, respectively, to

$$\tilde{\Phi}_i(\tilde{x}(T \mid \theta^p)) \geq 0, \qquad i = 1, \dots, N_{\mathsf{T}} \tag{4.6}$$

and

$$\tilde{h}_i(t, \tilde{x}(t \mid \theta^p)) \geq 0, \qquad i = 1, \dots, N + 2r \tag{4.7}$$

Define

$$\Lambda^p = \{\theta^p \in \mathscr{Z} \times \Xi^p : \tilde{\Phi}_i(\tilde{x}(T \mid \theta^p)) \geq 0,\ i = 1, \dots, N_{\mathsf{T}}\} \tag{4.8}$$

and

$$\Omega^p = \{\theta^p \in \Lambda^p : \tilde{h}_i(t, \tilde{x}(t \mid \theta^p)) \geq 0,\ i = 1, \dots, N + 2r\}. \tag{4.9}$$

Furthermore, let $\mathring{\Lambda}^p$ and $\mathring{\Omega}^p$ be, respectively, subsets of the sets Λ^p and Ω^p defined by

$$\mathring{\Lambda}^p = \{\theta^p \in \mathscr{Z} \times \Xi^p : \tilde{\Phi}_i(\tilde{x}(T \mid \theta^p)) > 0, \ \ i = 1, \dots, N_{\mathrm{T}}\} \tag{4.10}$$

and

$$\mathring{\Omega}^p = \{\theta^p \in \mathring{\Lambda}^p : \tilde{h}_i(T, \tilde{x}(t \mid \theta^p)) > 0, \ \ i = 1, \dots, N + 2r\}. \tag{4.11}$$

We may now specify the approximate problem $(Q(p))$ as follows:

PROBLEM $(Q(p))$. Find a control parameter vector $\theta^p \in \Omega^p$ such that the cost functional

$$\hat{\tilde{g}}_0(\theta^p) = \tilde{\Phi}_0(\tilde{x}(T \mid \theta^p)) + \int_0^{\mathrm{T}} \widehat{\widetilde{\mathscr{L}}}_0(t, \tilde{x}(t \mid \theta^p), \theta^p)\, dt \tag{4.12}$$

is minimised over Ω^p, where $\widehat{\widetilde{\mathscr{L}}}_0$ is obtained from $\widetilde{\mathscr{L}}_0$ in an obvious manner.

In the rest of this section, some preliminary results are summarised in two lemmas and two remarks.

LEMMA 4.1. *Let $\{\theta^{p,k}\}_{k=1}^{\infty}$ be a sequence of combined vectors in $\mathscr{Z} \times \Xi^p$ such that* $\lim_{k\to\infty} |\theta^{p,k} - \bar{\theta}^p| = 0$. *Then,*

(i) $\bar{\theta}^p \in \mathscr{Z} \times \Xi^p$;

(ii) $\{x(\cdot \mid \theta^{p,k})\}_{k=1}^{\infty} \subset X$, *where X is a bounded subset of* $\mathbb{R}^n$;

(iii) $\lim_{k\to\infty} \|\tilde{x}(\cdot \mid \theta^{p,k}) - \tilde{x}(\cdot \mid \bar{\theta}^p)\|_{\infty} = 0$;

(iv) *for each* $t \in [0, T]$, $\lim_{k\to\infty} |\tilde{x}(t \mid \theta^{p,k}) - \tilde{x}((t \mid \bar{\theta}^p)| = 0$; *and*

(v) $\lim_{k\to\infty} \hat{\tilde{g}}_0(\theta^{p,k}) = \hat{\tilde{g}}(\bar{\theta}^p)$.

PROOF. (i) follows from the fact that the set $\mathscr{Z} \times \Xi^p$ is compact. The results presented in (ii)–(v) can be established by techniques similar to those given in the relevant parts of the proofs of Lemmas 4.2 to 4.4 of [19].

REMARK 4.1. For each $v \in \mathscr{V}$ and each p, let $v^p(t)$ be constructed from v according to (4.1) with

$$\sigma^{p,k} = \frac{1}{|I_k^p|} \int_{I_k^p} v(s)\, ds \tag{4.13a}$$

and

$$|I_k^p| = |t_k^p - t_{k-1}^p|. \tag{4.13b}$$

Then, it follows from Lemma 4.1 and its proof of [18] that

$$v^p \to v \tag{4.14a}$$

almost everywhere in $[0, T]$, as $p \to \infty$; and furthermore,

$$\lim_{p\to\infty} \int_0^T |v^p(t) - v(t)|\, dt = 0. \tag{4.14b}$$

Again, by using techniques similar to those given in the relevant parts of the proofs of Lemmas 4.2 and 4.4 of [19], we have the results presented in the following lemma.

LEMMA 4.1. *Let* $\{(\xi^p, v^p)\}_{p=1}^{\infty}$ *be a bounded sequence of elements in* $\mathbb{R}^r \times L_\infty^r$ *such that*

$$\lim_{p\to\infty} |v^p(t) - v(t)| = 0, \quad \textit{a.e. on } [0, 1],$$

and $\lim_{p\to\infty} |\xi^p - \xi| = 0$. *Then*

(i) $\{x(\cdot \mid \xi^p, v^p)\}_{p=1}^{\infty} \subset X$, *where* X *is a bounded subset of* $\mathbb{R}^r \times L_\infty^r$;

(ii) $\lim_{p\to\infty} \|\tilde{x}(\cdot \mid \xi^p, v^p) - \tilde{x}(\cdot \mid \xi, v)\|_\infty = 0$;

(iii) *for each* $t \in [0, T]$, $\lim_{p\to\infty} |\tilde{x}(t \mid \xi^p, v^p) - \tilde{x}(t \mid \xi, v)| = 0$;

(iv) $\lim_{p\to\infty} \tilde{g}_0(\xi^p, v^p) = \tilde{g}_0(\xi, v)$.

REMARK 4.2. In view of Lemma 4.2(ii)–(iii) and (3.3), it follows that $\lim_{p\to\infty} \|u^p - u\|_\infty = 0$, and for each $t \in [0, T]$, $\lim_{p\to\infty} |u^p(t) - u(t)| = 0$, where u^p and u are, respectively, related to (ξ^p, v^p) and (ξ, v) according to (3.1). Furthermore, by virtue of the argument similar to that given for Lemma 4.4 of [19], we have $\lim_{p\to\infty} g_0(u^p) = g_0(u)$.

5. Constraints approximation

In view of Theorem 3.1, we recall that the problem (P) is equivalent to the problem (Q). For problem (Q), it is to be solved via the control parametrisation technique. In other words, instead of solving the problem (Q), we need only to solve a sequence of finite dimensional optimisation problems $(Q(p))$. However, the problem $(Q(p))$ cannot be solved directly as such by the software MISER. In this section, we shall convert the problem $(Q(p))$ into a form solvable by MISER.

For each $i = 1, \dots, N+2r$, the corresponding inequality continuous state constraint in (3.10) is equivalent to

$$\hat{\tilde{g}}_i(\theta^p) = \int_0^T \min\{\tilde{h}_i(t, \tilde{x}(t \mid \theta^p)), 0\}\, dt = 0. \tag{5.1}$$

However, it should be noted that the equality constraints (5.1) are non-differentiable with respect to those θ^p at which $\tilde{h}_i = 0$.

For convenience, let $(Q(p))$ with (4.7) replaced by (5.1) be again denoted by $(Q(p))$. Clearly, the set Ω^p can also be written as:

$$\Omega^p = \{\theta^p \in \Lambda^p : \hat{\tilde{g}}_i(\theta^p) = 0, \ \ i = 1, \dots, N + 2r\}. \tag{5.2}$$

Since the equality constraints (5.1) are nondifferentiable, the smoothing technique of [5] will be used, that is, replace $\min\{\tilde{h}_i(t, \tilde{x}(t \mid \theta^p)), 0\}$ by $\widetilde{\mathscr{L}}_{i,\varepsilon}(t, \tilde{x}(t \mid \theta^p))$, where

$$\widetilde{\mathscr{L}}_{i,\varepsilon}(t, \tilde{x}(t \mid \theta^p)) = \begin{cases} \tilde{h}_i(t, \tilde{x}(t \mid \theta^p)), & \text{if } \tilde{h}_i(t, \tilde{x}(t \mid \theta^p)) < -\varepsilon \\ -(\tilde{h}_i(t, \tilde{x}(t \mid \theta^p)) - \varepsilon)^2/4\varepsilon, & \text{if } -\varepsilon \leq \tilde{h}_i(t, \tilde{x}(t \mid \theta^p)) \leq \varepsilon \\ 0, & \text{if } \tilde{h}_i(t, \tilde{x}(t \mid \theta^p)) > \varepsilon \end{cases} \tag{5.3}$$

This function is obtained by smoothing out the sharp corner of the function $\min\{\tilde{h}_i(t, \tilde{x}(t \mid \theta^p)), 0\}$.

For each $i = 1, \dots, N + 2r$, define

$$\hat{\tilde{g}}_{i,\varepsilon}(\theta^p) = \int_0^T \widetilde{\mathscr{L}}_{i,\varepsilon}(t, \tilde{x}(t \mid \theta^p))\, dt. \tag{5.4}$$

We now define two related approximate problems which will be referred to as $(Q_\varepsilon(p))$ and $(Q_{\varepsilon,\gamma}(p))$. The first approximate problem is:

Problem $(Q_\varepsilon(p))$. The problem (Q) with the inequality continuous state constraints (4.7) replaced by

$$\hat{\tilde{g}}_{i,\varepsilon}(\theta^p) = 0, \qquad i = 1, \dots, N + 2r. \tag{5.5}$$

Let Ω^p_ε be the feasible region of $(Q_\varepsilon(p))$ defined by

$$\Omega^p_\varepsilon = \{\theta^p \in \Lambda^p : \hat{\tilde{g}}_{i,\varepsilon}(\theta^p) = 0, \ \ i = 1, \dots, N\} \tag{5.6}$$

Then, for each $\varepsilon > 0$, $\Omega^p_\varepsilon \subset \Omega^p$.

Note that the equality constraints (5.5) fail to satisfy the usual constraint qualification. Thus, we may encounter numerical difficulty if they are used in their present form. This situation is similar to that in Remark 5.3 of [2].

To overcome this difficulty, we consider our second approximate problem as follows:

Problem $(Q_{\varepsilon,\gamma}(p))$. The problem (Q) with (4.7) replaced by

$$\gamma + \tilde{g}_{i,\varepsilon}(\theta^p) \geq 0, \qquad i = 1, \dots, N + 2r. \tag{5.7}$$

To continue, we assume that the following condition is satisfied.

(A7) For any $\theta^p \in \Omega^p$, there exists a $\bar{\theta}^p \in \mathring{\Omega}^p$ such that

$$\alpha\bar{\theta}^p + (1 - \alpha)\theta^p \in \mathring{\Omega}^p \quad \forall \alpha \in (0, 1].$$

LEMMA 5.1. *Let* $\{\theta_\varepsilon^{p,*}\}$ *be a sequence in* ε *of the optimal combined vectors of the problems* $(Q_\varepsilon(p))$. *Then,*

$$\lim_{\varepsilon\to 0}\hat{\hat{g}}_0(\theta_\varepsilon^{p,*}) = \hat{\hat{g}}_0(\theta^{p,*}), \tag{5.8}$$

where $\theta^{p,*}$ *is an optimal combined vector of the problem* $(Q(p))$.

PROOF. The proof is similar to that given for Lemma 3.1 of [17].

LEMMA 5.2. *Let* $\theta^{p,*}$ *and* $\theta_\varepsilon^{p,*}$ *be as defined in Lemma* 5.1. *If there exists a combined vector* $\bar{\theta}^p \in \mathscr{Z}\times\Xi^p$ *such that* $\lim_{\varepsilon\to 0}|\theta_\varepsilon^{p,*} - \bar{\theta}^p| = 0$, *then* $\bar{\theta}^p$ *is also an optimal combined vector of* $(Q(p))$.

PROOF. The proof is similar to that given for Lemma 3.2 of [17]. The main alterations are to replace Lemmas 4.2 and 4.3 of [19], Lemma 2.1 of [17] and Lemma 3.1 of [19] by Lemma 3.4.1(iv), (iii), (ii) and Lemma 3.5.1, respectively.

LEMMA 5.3. *There exists a* $\gamma(\varepsilon) > 0$ *such that for all* γ, $0 < \gamma < \gamma(\varepsilon)$, *any feasible combined vector* $\theta_{\varepsilon,\gamma}^p$ *of the problem* $(Q_{\varepsilon,\gamma}(p))$, *i.e.,*

$$\gamma + \hat{\hat{g}}_{i,\varepsilon}(\theta_{\varepsilon,\gamma}^p) \geq 0, \qquad i = 1, \dots, N,$$

is also a feasible combined vector of the problem $(Q(p))$.

PROOF. The proof is similar to that given for Lemma 3.3 of [17].

At this stage, the algorithm presented in Section 4 of [17] can be used to generate a solution of the problem $(Q(p))$. For convenience, this algorithm is recalled as follows:

ALGORITHM 5.1.

Data. $\varepsilon > 0$, $\gamma > 0$. (In particular, we may choose $\varepsilon = 10^{-1}$ and $\gamma = T\varepsilon/16$).

Step 1. Solve $(Q_{\varepsilon,\gamma}(p))$ to give $\theta_{\varepsilon,\gamma}^{p,*}$.

Step 2. Check feasibility of $\tilde{h}_i(t, \tilde{x}(t \mid \theta_{\varepsilon,\gamma}^{p,*})) \geq 0$ for all $t \in [0, T]$ and for all $i = 1, \dots, N + 2r$.

Step 3. If $\theta_{\varepsilon,\gamma}^{p,*}$ is feasible, go to Step 5.

Step 4. Set $\gamma = \gamma/2$ and go to Step 1.

Step 5. Set $\varepsilon = \varepsilon/10$, $\gamma = \gamma/10$. Go to Step 1.

REMARK 5.1. From Lemma 5.3, we see that the halving process of γ in Step 4 of the algorithm needs only to be carried out a finite number of times. Thus, the algorithm produces a sequence of suboptimal parameter vectors to the problem $(Q(p))$, where each of them is in the feasible region of $(Q(p))$.

Note that the other two remarks given for the algorithm in Section 4 of [17] are obviously valid, also.

THEOREM 5.1. *Let* $\{\theta^{p,*}_{\varepsilon,\gamma}\}$ *be a sequence in* ε *of the suboptimal combined vectors produced by the algorithm* 5.1. *Then* $\hat{\tilde{g}}_0(\theta^{p,*}_{\varepsilon,\gamma}) \to \hat{\tilde{g}}_0(\theta^{p,*})$, *as* $\varepsilon \to 0$, *where* $\theta^{p,*}$ *is an optimal combined vector of* $(Q(p))$. *Furthermore, any accumulation point of* $\{\theta^{p,*}_{\varepsilon,\gamma}\}$ *is a solution of* $(Q(p))$.

PROOF. Clearly,

$$\hat{\tilde{g}}_0(\theta^{p,*}) \leq \hat{\tilde{g}}_0(\theta^{p,*}_{\varepsilon,\gamma}) \leq \hat{\tilde{g}}_0(\theta^{p,*}_{\varepsilon}), \tag{5.23}$$

where $\theta^{p,*}_{\varepsilon}$ is as defined for Lemma 5.1. Thus, by the same lemma, we have $\hat{\tilde{g}}_0(\theta^{p,*}_{\varepsilon,\gamma}) \to \hat{\tilde{g}}_0(\theta^{p,*})$.

To prove the second part of the theorem, we note that the sequence $\{\theta^{p,*}_{\varepsilon,\gamma}\}$ in ε is in $\mathscr{Z} \times \Xi^p$ which is a compact subset of $\mathbb{R}^{s+rN_p}$. Thus, the existence of an accumulation point is ensured. On this basis, the proof of the second part of the theorem follows easily from an argument similar to that given for Lemma 5.2.

REMARK 5.4. Let $\theta^{p,*}$ be an optimal combined vector of the approximate problem $(Q(p))$. Then, we know that $\theta^{p,*}$ defines a unique combined parameter vector and control $(\xi^{p,*}, v^{p,*})$ in $\mathscr{F}$. Thus, we shall refer to $(\xi^{p,*}, v^{p,*})$ as an optimal combined parameter vector and control of the approximate problem $(Q(p))$. Furthermore, it follows from Theorem 3.1 that there corresponds uniquely a piecewise smooth control $u^{p,*}$ of the problem $(P(p))$.

6. Some convergence results

In this section, we shall discuss some convergence properties of the sequence of approximate optimal controls. To be more precise, for each $p = 1, 2, \ldots$, let $\theta^{p,*} = (\xi^p, \sigma^p)$ be an optimal combined vector to the finite dimensional optimization problem $Q(p)$; furthermore, let $\{(\xi^{p,*}, v^{p,*})\}_{p=1}^{\infty}$ be the corresponding sequence of elements in $\widetilde{\mathscr{F}}$. In view of (C1) given in Section 3, we see that each of these elements is a suboptimal one to the problem (Q), and is such that $\tilde{g}_0(\xi^{p+1,*}, v^{p+1,*}) \leq \tilde{g}_0(\xi^{p,*}, v^{p,*})$ for all $p = 1, 2, \ldots$.

In view of Theorem 3.1, we note that $\{(\xi^{p,*}, v^{p,*})\}_{p=1}^{\infty}$ generates a unique sequence $\{u^{p,*}\}_{p=1}^{\infty}$ of elements in $\mathscr{F}$ such that $g_0(u^{p,*}) = \tilde{g}_0(\xi^{p,*}, v^{p,*})$ where $g_0(u^{p,*})$, which is the cost functional of the problem (P), is defined by (2.6). Furthermore, it is easy to see that $g_0(u^{p+1,*}) \le g_0(u^{p,*})$ for all $p = 1, 2, \ldots$.

THEOREM 6.1. *Let* $(\xi^{p,*}, v^{p,*})$ *be an optimal combined parameter vector and control of the approximate problem* $(Q(p))$. *Suppose that the original problem* (P) *has an optimal control* u^*. *Then there exists a unique element* $u^{p,*} \in \mathscr{F}$ *such that* $\lim_{p\to\infty} g_0(u^{p,*}) = g_0(u^*)$.

PROOF. From (A6), there exists $\bar{u} \in \mathring{\mathscr{F}}$ such that

$$u_\alpha \equiv \alpha\bar{u} + (1-\alpha)u^* = u^* + \alpha(\bar{u} - u^*) \in \mathring{\mathscr{F}}, \quad \forall \alpha \in (0, 1].$$

For any $\delta_1 > 0$, $\exists\, \alpha_1 \in (0, 1)$ such that

$$g_0(u^*) \le g_0(u_\alpha) \le g_0(u^*) + \delta_1, \quad \forall \alpha \in (0, \alpha_1). \tag{6.1}$$

Choose $\alpha_2 = \alpha_1/2$. Then it is clear that $u_{\alpha_2} \in \mathring{\mathscr{F}}$. Thus there exists a $\delta_2 > 0$ such that

$$\Phi_i(x(T \mid u_{\alpha_2})) > \delta_2, \qquad i = 1, \ldots, N_T, \tag{6.2}$$

and, for all $t \in [0, T]$,

$$h_i(t, x(t \mid u_{\alpha_2})) > \delta_2, \qquad i = 1, \ldots, N, \tag{6.3a}$$

$$\beta_i - u_{\alpha_2, i}(t) > \delta_2, \qquad i = 1, \ldots, r, \tag{6.3.b}$$

and

$$u_{\alpha_2, i}(t) - \alpha_i > \delta_2, \qquad i = 1, \ldots, r, \tag{6.3c}$$

where $u_{\alpha_2, i}$ denotes the ith component of the control u_{α_2}.

In view of Theorem 3.1, we see that u^* (respectively, u_{α_2}) gives rise uniquely to an element (ξ^*, v^*) (respectively, $(\xi_{\alpha_2}, v_{\alpha_2})$) in $\widetilde{\mathscr{F}}$. Furthermore, it can be easily verified that $(\xi_{\alpha_2}, v_{\alpha_2})$ satisfies:

$$\widetilde{\Phi}_i(\tilde{x}(T \mid \xi_{\alpha_2}, v_{\alpha_2})) > \delta_2, \qquad i = 1, \ldots, N_T. \tag{6.4}$$

and, for all $t \in [0, T]$,

$$\tilde{h}_i(t, \tilde{x}(t \mid \xi_{\alpha_2}, v_{\alpha_2})) > \delta_2, \qquad i = 1, \ldots, N + 2r. \tag{6.5}$$

Let $v^p_{\alpha_2}$ be the control defined from v_{α_2} according to (4.1). Then, by virtue of Remark 4.1, Lemma 4.2(iv), (A1) and (A3) there exists a p_0 such that $(\xi_{\alpha_2}, v^p_{\alpha_2}) \in \widetilde{\mathscr{F}}$ for all $p > p_0$.

From (4.1), $(\xi_{\alpha_2}, v^p_{\alpha_2})$ gives rise uniquely to a combined vector $\theta^p_{\alpha_2} \in \Omega^p$. Clearly,

$$\hat{\tilde{g}}_0(\theta^{p,*}) \leq \hat{\tilde{g}}_0(\theta^p_{\alpha_2}) \tag{6.6}$$

for all $p \geq p_0$, where $\theta^{p,*}$ is an optimal combined vector of the problem $(Q(p))$.

Let $(\xi^{p,*}, v^{p,*})$ be the corresponding element in $\widetilde{\mathscr{F}}$. Furthermore, let $u^{p,*}$ be the element in $\mathscr{F}$ corresponding to $(\xi^{p,*}, v^{p,*})$. Then, for all $p \geq p_0$, we have

$$\tilde{g}_0(\xi^{p,*}, v^{p,*}) \leq \tilde{g}_0(\xi_{\alpha_2}, v^p_{\alpha_2}) \tag{6.7}$$

and

$$g_0(u^{p,*}) \leq g_0(u^p_{\alpha_2}). \tag{6.8}$$

Next, by virtue of Remark 4.1 and the second part of Remark 4.2, we have

$$\lim_{p\to\infty} g_0(u^p_{\alpha_2}) = g_0(u_{\alpha_2}). \tag{6.9}$$

Combining (6.1), (6.8) and (6.9), we obtain

$$g_0(u^*) \leq \lim_{p\to\infty} g_0(u^{p,*}) \leq g_0(u_{\alpha_2}) < g_0(u^*) + \delta_1. \tag{6.10}$$

Since $\delta_1 > 0$ is arbitrary and u^* is an optimal control, we conclude that $\lim_{p\to\infty} g_0(u^{p,*}) = g_0(u^*)$. This completes the proof.

The next theorem presents a convergence result for $\{u^{p,*}\}$. This result is stronger than that presented in Theorem 5.2 of [17]. More precisely, in this theorem, the sequence $\{u^{p,*}\}$ is shown, rather than being assumed as in Theorem 5.2 of [3], to possess an accumulation point in the uniform topology. Any such accumulation point is then shown to be an optimal control of the original optimal control problem (P).

THEOREM 6.2. *Let $(\xi^{p,*}, v^{p,*})$ be an optimal solution of the approximate problem $(Q(p))$, and let $\{u^{p,*}\}$ be the corresponding sequence in $\mathscr{F}$. Then, the sequence $\{u^{p,*}\}$ possesses an accumulation point in the uniform topology in $[0, T]$. Furthermore, any such accumulation point is an optimal control of the original optimal control problem* (P).

PROOF. In view of (2.3), it follows from the Ascoli-Arzela theorem that the sequence $\{u^{p,*}\}$ has a subsequence, again denoted by the original sequence, such that

$$u^{p,*} \to \bar{u} \tag{6.11}$$

uniformly in $[0, T]$. Next, we shall show that $\bar{u}$ satisfies the constraints specified in (2.3). For this, we note that, for each $i = 1, \dots, r$, and for any

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$\Delta > 0$,

$$\begin{aligned}&[(\bar{u}_i(t+\Delta) - \bar{u}_i(t))/\Delta] \\ &\quad = -[(u_i^{p,*}(t+\Delta) - \bar{u}_i(t+\Delta))/\Delta] + [(u_i^{p,*}(t) - \bar{u}_i(t))/\Delta] \\ &\qquad + [(u_i^{p,*}(t+\Delta) - u_i^{p,*}(t))/\Delta].\end{aligned}$$

By the definition of $\{u^{p,*}\}$, we recall that it is in $\mathscr{U}$. Hence,

$$c_i \leq (u_i^{p,*}(t+\Delta) - u_i^{p,*}(t))/\Delta \leq d_i.$$

Thus,

$$\begin{aligned}&c_i - [(u_i^{p,*}(t+\Delta) - \bar{u}_i(t+\Delta))/\Delta] + [(u_i^{p,*}(t) - \bar{u}_i(t))/\Delta] \\ &\quad \leq [(\bar{u}_i(t+\Delta) - \bar{u}_i(t))/\Delta] \\ &\quad \leq d_i - [(u_i^{p,*}(t+\Delta) - \bar{u}_i(t+\Delta))/\Delta] + [(u_i^{p,*}(t) - \bar{u}_i(t))/\Delta]. \qquad (6.12)\end{aligned}$$

Therefore, it follows from (6.11) and (6.12) that $\bar{u}$ is also in $\mathscr{U}$.

Next, by virtue of the second part of Remark 4.2, we have

$$\lim_{p\to\infty} g_0(u^{p,*}) = g_0(\bar{u}). \qquad (6.13)$$

An argument similar to that given in the proof of Lemma 3.2 of [17] shows that $\bar{u}$ is a feasible point of the problem (P). Let u^* be an optimal control of (P). Then, by a similar argument as that used to obtain (6.1), there exists, for each $\delta_1 > 0$, an $\alpha_2 \in (0, 1)$ such that

$$g_0(u^*) \leq g_0(u_{\alpha_2}) < g_0(u^*) + \delta_1, \qquad (6.14)$$

where $u_{\alpha_2} \in \mathring{\mathscr{F}}$. Thus, there exists a $\delta_2 > 0$ such that

$$\Phi_i(x(T \mid u_{\alpha_2})) > \delta_2, \qquad i = 1, \dots, N_{\mathsf{T}}. \qquad (6.15)$$

and, for all $t \in [0, T]$,

$$h_i(t, x(t \mid u_{\alpha_2})) > \delta_2, \qquad i = 1, \dots, N. \qquad (6.16a)$$

$$\beta_i - u_{\alpha_2, i}(t) > \delta_2, \qquad i = 1, \dots, r \qquad (6.17b)$$

and

$$u_{\alpha_2, i}(t) - \alpha_i > \delta_2, \qquad i = 1, \dots, r \qquad (6.17c)$$

In view of Theorem 3.1, we see that u^* (respectively, u_{α_2}) gives rise uniquely to an element (ξ^*, v^*) (respectively, $(\xi_{\alpha_2}, v_{\alpha_2})$) in $\widetilde{\mathscr{F}}$. Furthermore, it can be easily verified that $(\xi_{\alpha_2}, v_{\alpha_2})$ satisfies:

$$\widetilde{\Phi}_i(\tilde{x}(T \mid \xi_{\alpha_2}, v_{\alpha_2})) > \delta_2, \qquad i = 1, \dots, N_{\mathsf{T}}. \qquad (6.18)$$

and, for all $t \in [0, T]$,

$$\tilde{h}_i(t, \tilde{x}(t \mid \xi_{\alpha_2}, v_{\alpha_2})) > \delta_2, \qquad i = 1, \ldots, N + 2r. \tag{6.19}$$

Let $v^p_{\alpha_2}$ be the control defined from v_{α_2} according to (4.1). Then, by virtue of Remark 4.1, Lemma 4.2(iv), (A2) and (A3) there exists a p_0 such that $(\xi_{\alpha_2}, v^p_{\alpha_2}) \in \widetilde{\mathscr{F}}$ for all $p > p_0$.

From (4.1), $(\xi_{\alpha_2}, v^p_{\alpha_2})$ gives rise uniquely to a combined vector $\theta^p_{\alpha_2} \in \Omega^p$. Clearly,

$$\hat{\tilde{g}}_0(\theta^{p,*}) \leq \hat{\tilde{g}}_0(\theta^p_{\alpha_2}) \tag{6.20}$$

for all $p \geq p_0$, where $\theta^{p,*}$ is an optimal combined vector of the problem $(Q(p))$.

Let $(\xi^{p,*}, v^{p,*})$ be the corresponding element in $\widetilde{\mathscr{F}}$. Furthermore, let u^p be the element in $\mathscr{F}$ corresponding to $(\xi^{p,*}, v^{p,*})$. Then, for all $p \geq p_0$, we have

$$\tilde{g}_0(\xi^{p,*}, v^{p,*}) \leq \tilde{g}_0(\xi_{\alpha_2}, v^p_{\alpha_2}) \tag{6.21}$$

and

$$g_0(u^{p,*}) \leq g_0(u^p_{\alpha_2}). \tag{6.22}$$

Next, by virtue of Remark 4.1 and the second part of Remark 4.2, we have

$$\lim_{p \to \infty} g_0(u^{p,*}) = g_0(\hat{u}) \leq \lim_{p \to \infty} g_0(u^p_{\alpha_2}) = g_0(u_{\alpha_2}). \tag{6.23}$$

Combining (6.14) and (6.23), we have $g_0(\bar{u}) < g_0(u^*) + \delta_1$. Since $\delta_1 > 0$ is arbitrary and u^* is an optimal control, it is clear that $\bar{u}$ is also an optimal control of the problem (P). This completes the proof.

7. Examples

To illustrate the applicability of the proposed numerical procedure, we consider two optimal control problems involving cargo transfer via container crane.

EXAMPLE 7.1. (minimum swing). In [14], a realistic and complex problem of transferring containers from a ship to a cargo truck at the port of Kobe was considered. The container crane is driven by a hoist motor and a trolley drive motor. For safety reasons, the objective is to minimise the swing during and at the end of the transfer.

This problem was solved in [14], using the algorithm of [12]. It was resolved in [2] and [17], where piecewise constant functions were used to approximate the controls in the control parametrisation procedure. This is feasible theoretically, but in practice, the controls are difficult to realise. Essentially, piecewise constant controls in this crane problem corresponds to the stepping of the current in the motors to provide the necessary driving torque. This creates electrical noise in the system which might affect the performance of other electronic circuitries, in particular, sensing equipments. The time constants of the current drives will also mean that the piecewise constant controls cannot be realised exactly. Furthermore, piecewise constant controls entail infinite jerk at the switching points, thus resulting in undesirable jerky forces on the load. In addition, it may also excite a large bandwidth of vibration modes and induce structural vibrations. Hence smooth control is much preferred for smoother operations of the motors and the problem is best solved by the proposed method. In this example, we solve the same problem with the controls approximated by piecewise linear functions.

Recall that in the crane problem in [14], the controls are given by $u_1(t)$ and $u_2(t)$. Since the controls are required to be smooth, we introduce an extra set of differential equations for the controls as follows:

$$\dot{u}_1(t) = v_1(t) \tag{7.1a}$$

$$\dot{u}_2(t) = v_2(t) \tag{7.1b}$$

with the initial conditions

$$u_1(0) = \xi_1 \tag{7.2a}$$

$$u_2(0) = \xi_2 \tag{7.2b}$$

where the ξ_1 and ξ_2 are the system parameters to be determined.

In view of (7.1), $u_1(t)$ and $u_2(t)$ are now state functions instead of control functions. They are determined by the new control functions $v_1(t)$, $v_2(t)$ and the system parameters ξ_1, ξ_2. Let us denote $u_1(t)$ and $u_2(t)$ respectively by $x_7(t)$ and $x_8(t)$. To prevent the original control variables u from changing too rapidly with time, we need to impose bounds for the new control functions v. Here, we assume that $|v(t)| \le 10$, $\forall t$.

The final formulation of the problem after appropriate normalisation is:

$$\underset{v(\cdot)}{\text{minimise}}\, 4.5 \int_0^1 [x_3^2(t) + x_6^2(t)]\, dt \tag{7.3}$$

subject to the dynamical equations

$$\dot{x}_1 = 9x_4, \tag{7.4a}$$

$$\dot{x}_2 = 9x_5, \tag{7.4b}$$

$$\dot{x}_3 = 9x_6, \tag{7.4c}$$

$$\dot{x}_4 = 9(x_7 + 17.2656x_3), \tag{7.4d}$$

$$\dot{x}_5 = 9x_8, \tag{7.4e}$$

$$\dot{x}_6 = -9[x_7 + 27.0756x_3 + 2x_5x_6]/x_2, \tag{7.4f}$$

$$\dot{x}_7 = 9v_1, \tag{7.4g}$$

$$\dot{x}_8 = 9v_2, \tag{7.4h}$$

where

$$x(0) = [0, 22, 0, 0, -1, 0, \xi_1, \xi_2]^{\mathsf{T}}, \tag{7.5}$$

$$x(1) = [10, 14, 0, 2.5, 0, 0]^{\mathsf{T}}, \tag{7.6}$$

with $x_7(1)$ and $x_8(1)$ being unspecified; and control constraints

$$|v_1(t)| \leq 10, \quad \forall t \in [0, 1], \tag{7.7a}$$

$$|v_2(t)| \leq 10, \quad \forall t \in [0, 1], \tag{7.7b}$$

together with state inequality constraints

$$|x_4(t)| \leq 2.5, \quad \forall t \in [0, 1], \tag{7.8a}$$

$$|x_5(t)| \leq 1.0, \quad \forall t \in [0, 1], \tag{7.8b}$$

$$|x_7(t)| \leq 2.83374, \quad \forall t \in [0, 1], \tag{7.8c}$$

$$-0.80865 \leq x_8(t) \leq 0.71265, \quad \forall t \in [0, 1]. \tag{7.8d}$$

TABLE 1. Results for piecewise linear controls: Example 7.1.

ε	γ	$g_0(v)$
10^{-3}	10^{-4}	0.5493×10^{-2}
10^{-5}	4×10^{-8}	0.5441×10^{-2}
10^{-6}	4×10^{-9}	0.5412×10^{-2}

These results were obtained using the general optimal control software MISER. The value of the cost functional is marginally higher than that obtained in [17] using the same algorithm. This is probably a tradeoff for using a smoother control.

Figures 1(a)–1(e) present the states and controls of the crane problem and should be compared to the graphs given in [17]. The horizontal axis has been

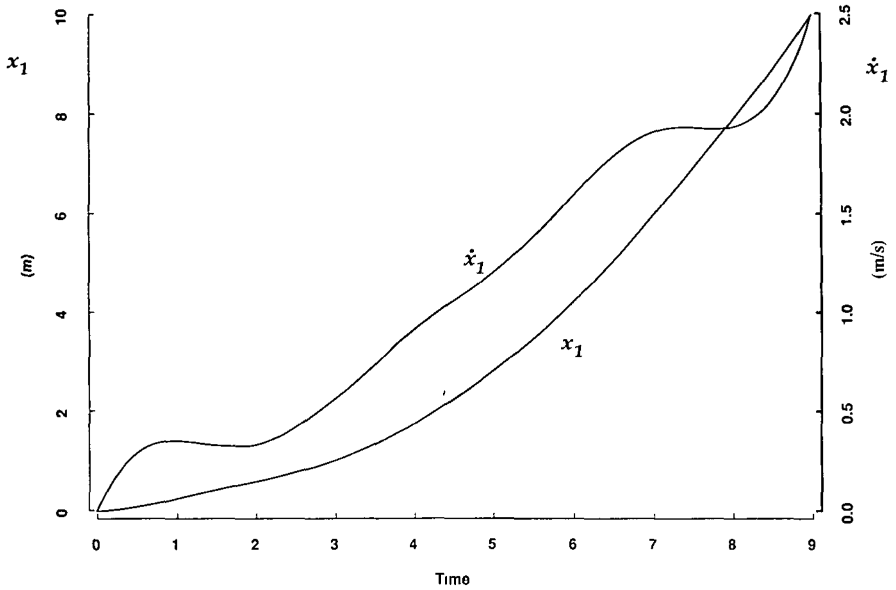

FIGURE 1a. History of state variable x_1, Example 7.1.

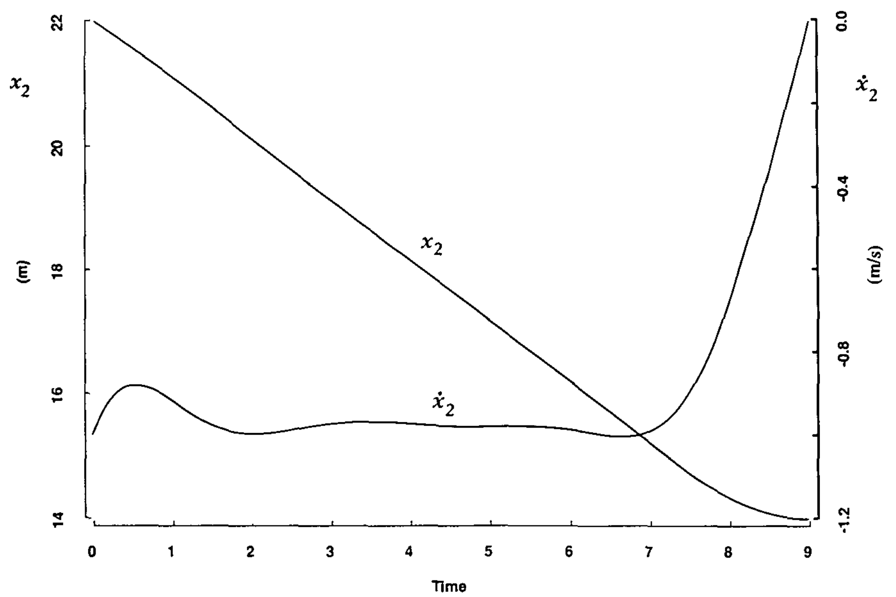

FIGURE 1b. History of state variable x_2, Example 7.1.

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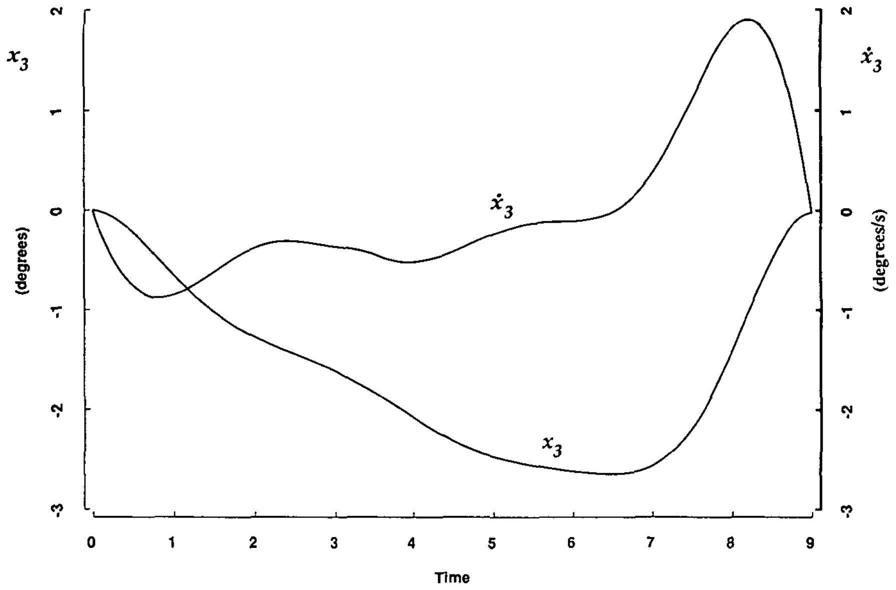

FIGURE 1c. History of state variable x_3, Example 7.1.

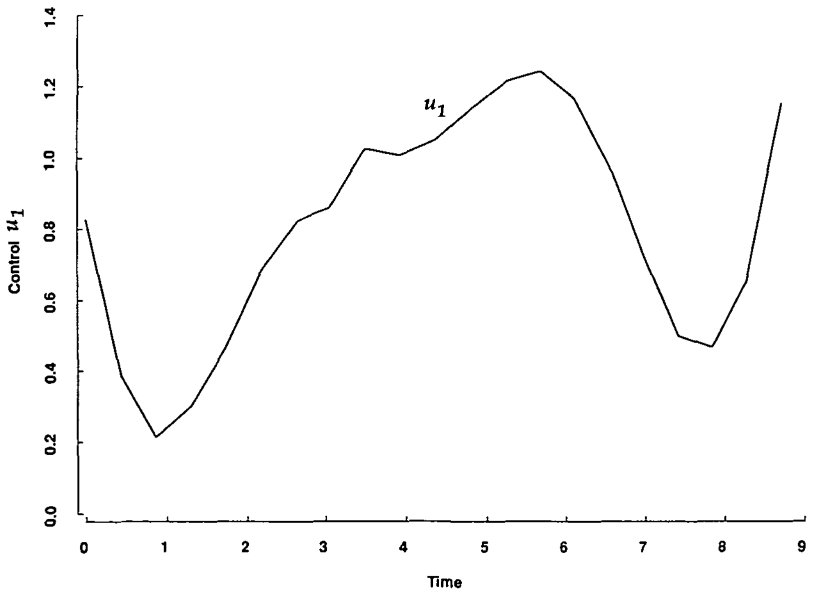

FIGURE 1d. History of optimal control u_1, Example 7.1.

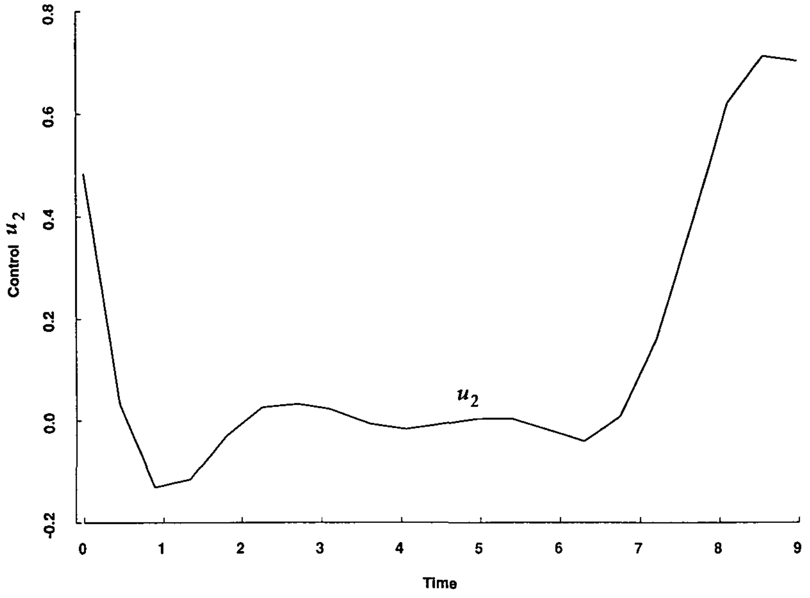

FIGURE 1e. History of optimal control u_2, Example 7.1.

rescaled so that the graphs shown are in real time rather than normalised time (cf. [17]). All the trajectories of the various state variables and controls are observed to satisfy the given boundary conditions. Using piecewise linear functions as approximations to the controls, we see that the rate of change of the swing, given by $\dot{x}_3$ in Figure 1(c) is much more gradual as compared to that of Figure 2(c) in [17]. Figures 1(d) and 1(e) shows the corresponding piecewise linear controls used in this crane problem.

EXAMPLE 7.2. (minimum time). This is a follow up to Example 7.1 above. Since the main goal in any industry is profit making, as thousands of cargo transfer operations are carried out at the port each day, or anywhere for that matter, it is obvious that one would like to minimise the transfer time for greater efficiency if safety in the working environment is not compromised. Hence in this example, we will consider the corresponding minimum time problem where the swing is subjected to lie within the acceptable bounds as obtained from Example 7.1. Similarly, piecewise linear controls are used as the approximations to provide smooth operations of the motors of the container crane.

The problem is to seek a control such that the cost functional

$$g_0(u) = t_f = \int_0^{t_f} 1\,dt$$

is minimised, subjecting to the same constraints and boundary conditions as in Example 7.1 together with the following additional constraints:

$$x_{3\,\min} \le x_3(t) \le x_{3\,\max}, \qquad t \in [0, t_f]$$

$$x_{6\,\min} \le x_6(t) \le x_{6\,\max}, \qquad t \in [0, t_f]$$

where

t_f is an unknown parameter to be determined;

$x_{3\,\min}$–lower bound of the swing angle obtained from the minimum swing problem;

$x_{3\,\max}$–the corresponding upper bound;

$x_{6\,\min}$–lower bound of the velocity of the swing obtained from the minimum swing problem; and

$x_{6\,\max}$–the corresponding upper bound.

We shall represent t_f by ξ_3 in conjunction with the notations for system parameters as used in Example 7.1.

After appropriate normalisation, the minimum time problem becomes:

$$\text{minimise}_{v(\cdot)}\, \xi_3$$

subject to the dynamical equations

$$\dot{x}_1 = \xi_3 x_4, \quad \dot{x}_2 = \xi_3 x_5, \quad \dot{x}_3 = \xi_3 x_6, \quad \dot{x}_4 = \xi_3(x_7 + 17.2656 x_3),$$

$$\dot{x}_5 = \xi_3 x_8, \quad \dot{x}_6 = -\frac{\xi_3}{x_2}(x_7 + 27.0756 x_3 + 2 x_5 x_6), \quad \dot{x}_7 = \xi_3 V_1, \quad \dot{x} = \xi_3 V_2,$$

where

$$x(0) = [0, 22, 0, 0, -1, 0, \xi_1, \xi_2]^{\mathrm{T}},$$

$$x(\xi_3) = [10, 14, 0, 2.5, 0, 0]^{\mathrm{T}},$$

with $x_7(\xi_3)$ and $x_8(\xi_3)$ being unspecified; and control constraints

$$|v_1(t)| \le 10, \quad |v_2(t)| \le 10, \quad \forall t \in [0, 1],$$

together with state inequality constraints

$$|x_4(t)| \le 2.5, \qquad |x_5(t)| \le 1.0, \qquad |x_7(t)| \le 2.83374,$$

$$-0.80865 \le x_8(t) \le 0.71265,$$

$$x_{3\,\min} \le x_3(t) \le x_{3\,\max},$$

$$x_{6\,\min} \le x_6(t) \le x_{6\,\max}, \qquad \forall t \in [0, 1].$$

TABLE 2. Results for piecewise linear controls: Example 7.2.

ε	γ	$g_0(v)$
10^{-3}	10^{-4}	8.7115
10^{-4}	2×10^{-6}	8.7120
10^{-5}	4×10^{-8}	8.7167
10^{-6}	4×10^{-9}	8.7166

The results are again obtained using the MISER program. It is observed that in solving the time optimal control problem, the program is sensitive to the lower bounds of the cost functional ξ_3 (recall that ξ_3 represents the terminal time to be determined). That is, for whatever initial guess of ξ_3 (this cannot be negative for obvious reasons), the program will start iterating from the specified lower bound, ignoring the hard constraints and then work its way until the hard constraints are satisfied. This is due to the optimisation routine NLPQL (cf. Ref. 15) employed in MISER.

From numerical experiences, it was noted that if the lower bound is too small, then convergence fails. The problem may, however, be solved as follows:

An artificial lower bound for ξ_3 was initially imposed so that a local optimal solution can be found. The lower bound is then decreased gradually to obtain subsequent local optimal solutions. For each run of the program with ξ_3 reduced, the control inputs were taken from the outputs of the previous run. This process is repreated until the required accuracy for the minimum time is achieved.

The states and controls of this minimum time crane problem are as shown in Figures 2(a)–2(e). As in the previous example, the graphs are given in real time. Comparing with Figures 1(a)–1(e), there is little difference between the two sets of graphs except that in this minimum time problem, the terminal time for the transfer is 8.7166 seconds as compared to 9 seconds in the minimum swing problem. Note also that the second control in Figure 2(e) is much smoother than that given in Figure 1(e).

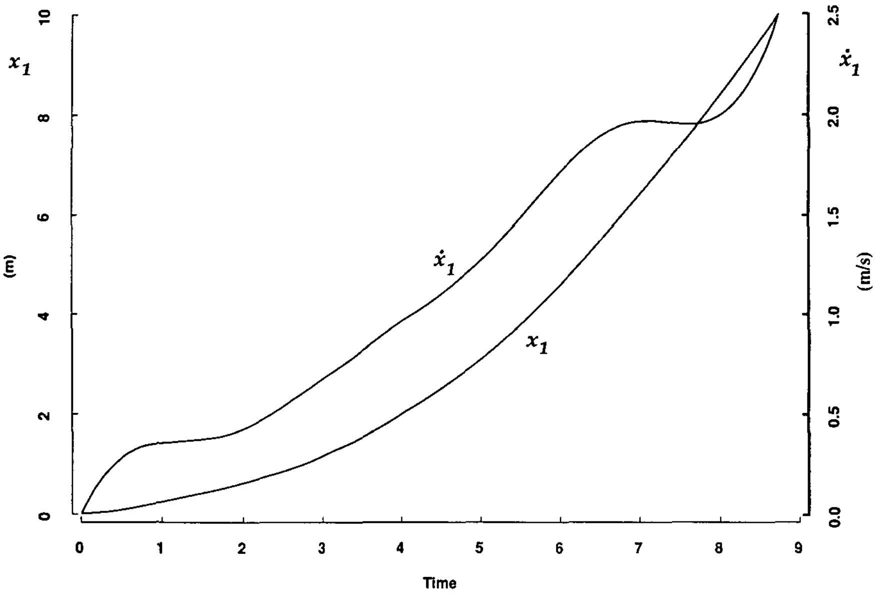

FIGURE 2a. History of state variable x_1, Example 7.2.

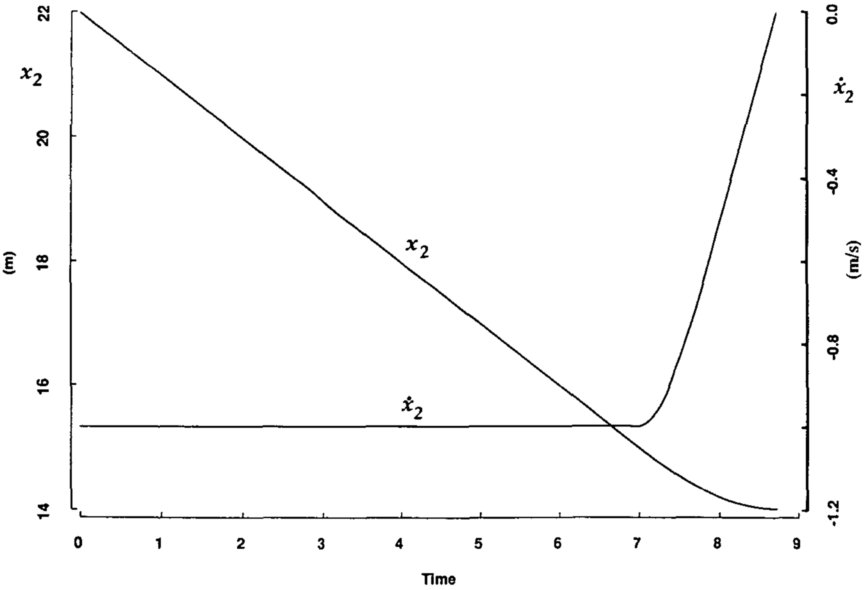

FIGURE 2b. History of state variable x_2, Example 7.2.

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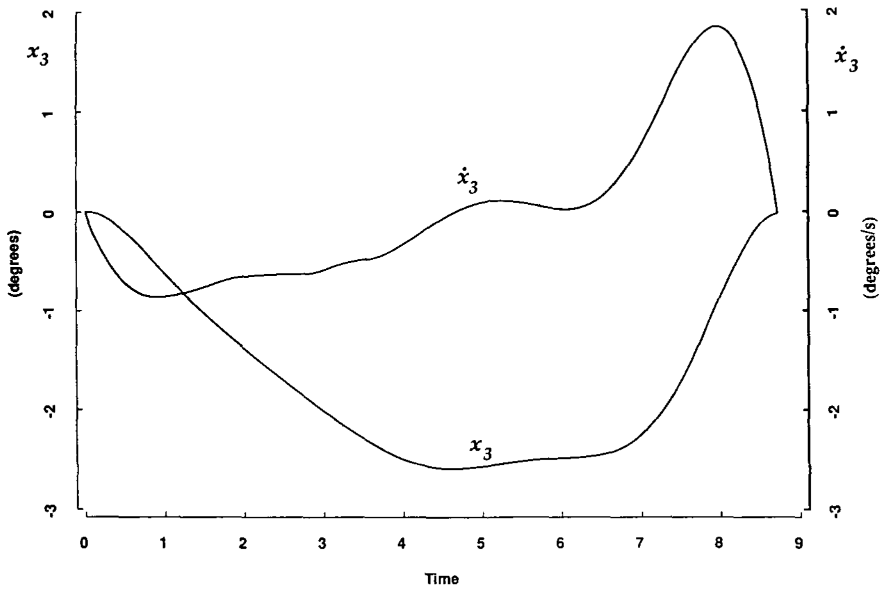

FIGURE 2c. History of state variable x_3, Example 7.2.

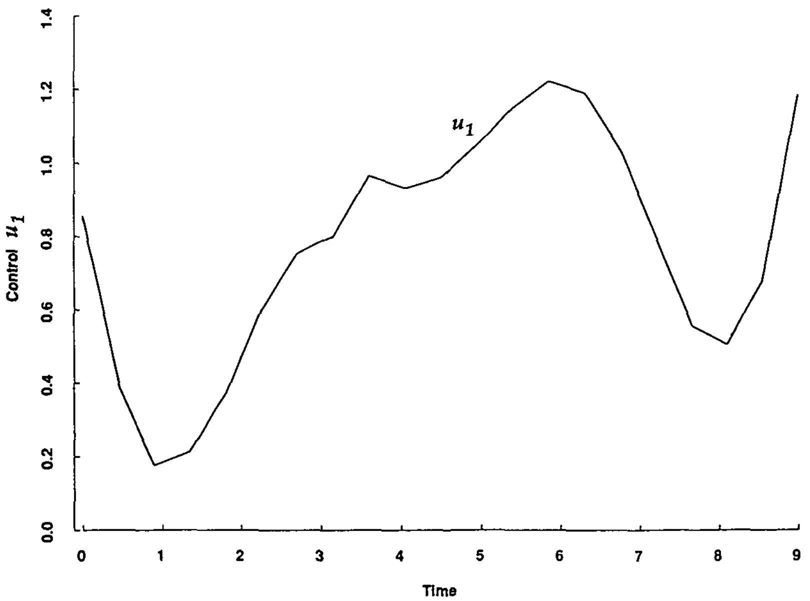

FIGURE 2d. History of optimal control u_1, Example 7.2.

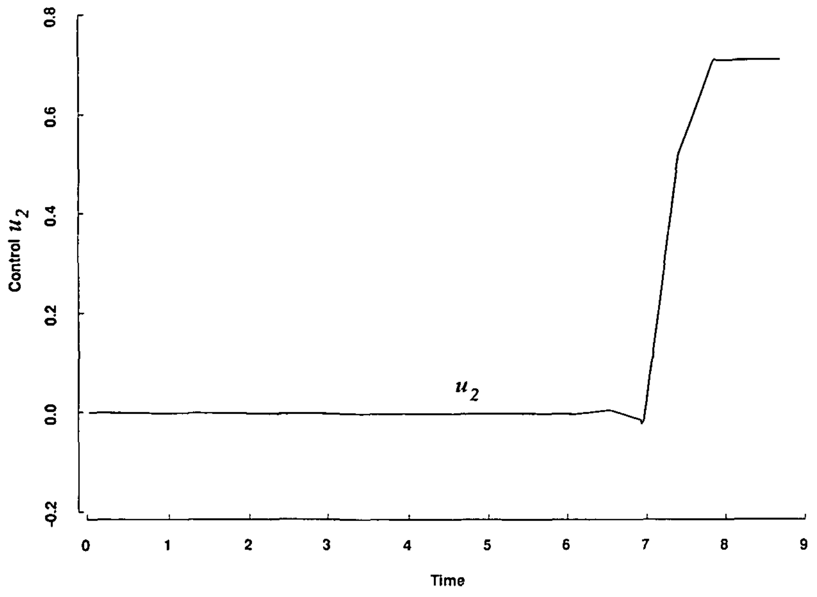

FIGURE 2e. History of optimal control u_2, Example 7.2.

Recall that this transfer time is only for the diagonal motion in the total transferring path (see [14]). In this example, we have reduced this transfer time by 3.1 percent. This may not look much, but bearing in mind that there are thousands of cargo transfer operations being carried out each day, a substantial amount of time can be saved, thus resulting in greater productivity and ultimately increased profits.

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