2 Contest Success Functions

As discussed in Chapter 1, Contest Theory heavily relies on Game Theory, for the time being on Nash equilibrium, to make predictions. In determining the properties of the equilibrium and ultimately the expected outcomes of a contest, the form of the contest success function (CSF) plays a crucial role. As a result, it is imperative to be cautious in selecting an appropriate CSF to model a contest.

This chapter presents a comprehensive review of the primary CSFs utilized in the literature and investigates their fundamental properties. Additionally, we provide a preliminary analysis of the existence and properties of the Nash equilibrium with various CSFs in a contest game. A more thorough examination of the existence of equilibrium in technically demanding cases is reserved for Chapter 3.

The origin of the functional forms defining the CSFs is not always clear, as tractability is not the sole consideration. In Chapter 4, we aim to address the question of understanding the microfoundations of the CSFs used in the literature. We will see that these microfoundations, sometimes, suggest new CSFs!

Throughout this chapter and most of the book, we assume that $c_i(g_i) = g_i$, and thus the payoff for each contestant i is given by

$$\pi_i(\mathbf{g}) = p_i(\mathbf{g})V_i - g_i.$$

2.1 All Pay Auction (Hillman and Riley, 1989)

We begin by assuming that the prize is awarded through an auction in which the highest bid wins. In other words, the contestant who exerts the greatest effort wins the prize with a probability one. In the event of a tie between the largest efforts, a lottery is held to determine the winner. Formally, the following rules for allocating the prize apply. Let K be the set of agents with maximal effort, i.e., if $i, j \in K$, $g_i = g_j$ and $g_i > g_l$ for all $l \notin K$. Then

(i) If $g_i \in K$, $p_i = 1/\#K$, where $\#K$ denotes the cardinality of the set K.
(ii) If $g_l \notin K$, $p_l = 0$.

The All Pay Auction yields a contest in which the prize is awarded through an auction process, but unlike traditional auctions, all participants are required to pay regardless of whether they win or not, hence the term "all pay." This type of auction

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represents a reverse form of Bertrand competition, where instead of the firm offering the lowest price winning the entire demand, the prize is awarded to the participant with the highest bid.

All Pay Auction models are characterized by intense competition, where even a small increase in effort can make a significant difference. Examples of the latter are the final sprint in an athletic competition or the Tour de France, where the time difference between the winner and the 10th place finisher may be as little as 0.4%.[1]

The All Pay Auction has no equilibrium in pure strategies. Suppose that **g** is such that g_i is the unique largest effort. Then $g_i - \varepsilon$ with ε sufficiently small also wins the prize and reduces costs, so there is a profitable deviation and no equilibrium in pure strategies with this characteristic is possible. Suppose now that **g** is such that there are, at least, two contestants i and j with the largest effort. Clearly $g_i < V_i$. By playing $g_i + \epsilon$ she wins the prize for sure with, almost, the same cost. So, again, we have a profitable deviation.

When there is no equilibrium in pure strategies, Game Theory recommends looking for an equilibrium in mixed strategies.[2] Here we just prove that such equilibrium exists in the case of two contestants.

PROPOSITION 2.1 *Suppose an All Pay Auction with $n = 2$ and $V_1 \geq V_2$. An equilibrium in mixed strategies exists where both contestants randomize on* $[0, V_2]$ *according to the following distribution functions:*

$$F_1(g_1) = \frac{g_1}{V_2}, \tag{2.1}$$

$$F_2(g_2) = \frac{g_2}{V_1} + 1 - \frac{V_2}{V_1}. \tag{2.2}$$

Proof First note that no contestant would expend more than the value of the prize. Consequently, there is no gain for contestant 1 to expend more than V_2. For each contestant, any g_i in the support has to be optimal. If g_i is in the support of contestant i, the probability of winning, given the mixed strategy of contestant j, is the probability that $g_j < g_i$, i.e., $F_j(g_i)$. The payoff for contestant i is $F_j(g_i)V_i - g_i$. So, g_i is optimal if $V_i dF_j(g_i)/dg_i = 1$. Integrating both sides of this equation, we obtain $F_j(g_i) = g_i/V_i + K_j$, where K_j is a constant. For contestant 1, $F_1(V_2) = 1$, thus $K_1 = 0$. For contestant 2, $F_2(V_2) = 1 = V_2/V_1 + K_2$, thus $K_2 = 1 - V_2/V_1$. □

In equilibrium, the expected effort for contestants are

$$Eg_1 = \int_0^{V_2} g_1 \frac{1}{V_2} dg_1 = \left[\frac{g_1^2}{2V_2}\right]_0^{V_2} = \frac{V_2^2}{2V_2} = \frac{V_2}{2},$$

1 In the 2021 Tour de France, the winner Tadej Pogačar spent a total of 83 hours on the road. The difference between his time and that of the runner-up was 5 minutes, representing a difference of 0.1% of the winner's time. Meanwhile, the difference between his time and that of the 10th place finisher was approximately 18 minutes, equating to a difference of 0.36%.

2 For a discussion on this equilibrium concept, the reader may refer to textbooks on Game Theory, such as Fudenberg and Tirole (1991) or Binmore (1991).

$$Eg_2 = \int_0^{V_2} g_2 \frac{1}{V_1} dg_2 = \left[\frac{g_2^2}{2V_1} \right]_0^{V_2} = \frac{V_2^2}{2V_1}.$$

Each contestant expends a fraction $V_2/2V_1$ of their valuation. The ranking of efforts is the same as the ranking of valuations. Moreover, the ratio between expected efforts matches exactly the ratio of valuations. Effort for contestant 2 is increasing in her valuation (the more you like the prize, the harder you fight for it) and decreasing in the valuation of contestant 1. However, the effort of contestant 1 depends only on the valuation of contestant 2 and it is increasing. This is because contestant 1 expects a tougher fight with contestants whose valuations are closer to hers and adjusts her effort accordingly. Total expected effort is given by

$$Eg_1 + Eg_2 = \frac{V_2}{2}\left(1 + \frac{V_2}{V_1}\right).$$

Note that when both contestants have identical valuations, the sum of expected expenses equals the value of the prize. In the parlance of Tullock (1980), rents are completely dissipated.

In equilibrium, all efforts in the support of the mixed strategy should give the same payoff. Hence, given $g_1^* \in [0, V_2]$, payoff for contestant 1 is

$$\pi_1(g_1^*) = F_2(g_1^*)V_1 - g_1^* = \left(\frac{g_1^*}{V_1} + 1 - \frac{V_2}{V_1}\right)V_1 - g_1^* = V_1 - V_2.$$

And similarly for contestant 2, given $g_2^* \in [0, V_2]$,

$$\pi_2(g_2^*) = F_1(g_2^*)V_2 - g_2^* = \frac{g_2^*}{V_2}V_2 - g_2^* = 0.$$

Note that contestant 1 can obtain the prize for sure by making an effort infinitesimally above V_2. Thus, the payoff for contestant 1 in equilibrium is $\pi_1(g^*) = V_1 - V_2$. Given this, contestant 2 cannot expect to obtain more than zero. Continuing the comparison between this model and Bertrand's, it becomes evident that in both models, the weaker competitor (in this case, the contestant with a lower valuation, and in Bertrand's model, the firm with higher costs) earns no profit. If both competitors are identical, both earn zero profit. This intense competition, where even a small difference can have a significant impact, undermines the profits of underdogs.

The All Pay Auction with more than two contestants is analyzed in Chapter 3, Section 3.5.

2.2 Difference CSF (Hirshleifer, 1989)

In the Difference CSF, the probability of winning is based on the difference between efforts. As stated by its originator, "this CSF captures the tremendous advantage of being slightly stronger than one's opponent" (Hirshleifer, 1991, p. 131).

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Assuming $n = 2$, the difference CSF is written as

$$p_1(\mathbf{g}) = F_1(g_1 - g_2), \;\; p_2(\mathbf{g}) = 1 - F_1(g_1 - g_2), \tag{2.3}$$

where F_1 is a strictly increasing function with a range in $[0, 1]$. Notice that this CSF is akin to the All Pay Auction. In fact, when $n = 2$, it can be regarded as a generalization of the All Pay Auction where $F_i(a) = 1$ if $a > 0$, $F_i(a) = 0$ if $a < 0$, and $F_i(a) = 0.5$ if $a = 0$.

In contests with more than two contestants, the previous definition is easily generalized. Let $M(\mathbf{g}_{-i})$ be a measure of the average of the efforts made by all contestants but i. This measure can be the arithmetic average (i.e., the sum of efforts divided by $n - 1$), the geometric average (the $n - 1$ root of the multiplication of efforts), the maximum effort made by competitors, etc. Then

$$p_i(\mathbf{g}) = F_i(g_i - M(\mathbf{g}_{-i})), \; i \in \{1, 2, \ldots, n\}.$$

When $n = 2$ and $M(g_j) = g_j$, we have (2.3).

Variations of this CSF were proposed by Baik (1998) and Che and Gale (2000). The first author assumes the following:

$$p_1(\mathbf{g}) = F_1(g_1 - \sigma g_2) \text{ and } p_2(\mathbf{g}) = 1 - p_1(\mathbf{g}), \text{ with } \; \sigma > 0,$$

which, for $\sigma = 1$, collapses (2.3).

Che and Gale (2000) proposed a special non-smooth form of (2.3):

$$p_1(\mathbf{g}) = \max\left\{\min\left\{\frac{1}{2} + s(g_1 - g_2), 1\right\}, 0\right\} \text{ and } p_2(\mathbf{g}) = 1 - p_1(\mathbf{g}). \tag{2.4}$$

The number s measures how responsive the probability of winning is to the difference of efforts. When $s = 0$, this CSF is a pure lottery. When $s \to \infty$, the difference of efforts is paramount and the CSF becomes the All Pay Auction in which the highest bidder takes it all.

A defining feature of the Difference CSF, when F is differentiable, is that, in any equilibrium in pure strategies, only the contestant with the highest valuation exerts a positive effort. To see this in a very simple way, let $F'(g_1 - g_2)$ be the derivative of $F(g_1 - g_2)$. Let $V_1 > V_2$. Suppose both contestants exert positive effort. Since $p_2(\mathbf{g}) = 1 - p_1(\mathbf{g})$, and the first-order conditions of payoff maximization hold with equality for both contestants, we have that

$$F_1'(g_1 - g_2)V_1 - 1 = 0 \text{ and } -F_1'(g_1 - g_2)V_2 - 1 = 0,$$

which is a contradiction.

In other cases, such as with the CSF outlined in (2.4), a pure strategy equilibrium does not exist, and instead, the equilibrium is in mixed strategies (see Che and Gale, 2000).

A drawback of the Difference CSF is that it is not unit-independent. As a result, in the Che and Gale CSF, the parameter s must be adjusted to the units used to measure the difference $g_1 - g_2$, to keep the impact of the difference constant.

A comprehensive analysis with heterogeneous contestants for the following alternative difference CSF

$$p_i(\mathbf{g}) = \frac{1}{\sum_{j=1}^{n} \exp(\alpha(g_j - g_i))} = \frac{\exp(\alpha g_i)}{\sum_{j=1}^{n} \exp(\alpha g_j)}$$

can be found in Ewerhart and Sun (2022). They found that when α is small (a high level of noise) two types of equilibria in pure strategies exist, one in which all contestants exert zero effort, and another with at most one active contestant. In all other cases, there are mixed-strategy equilibria.

2.3 Ratio Form CSF (Tullock, 1980)

In the Ratio CSF, the probability that a contestant wins equals the ratio of her effort to the aggregate effort, namely

$$p_i(\mathbf{g}) = \frac{g_i}{\sum_{j=i}^{n} g_j}. \tag{2.5}$$

The Ratio Form CSF can be interpreted as saying that the probability that contestant i obtains the prize when she holds g_i tickets and there are $\sum_{j=i}^{n} g_j$ tickets is just the proportion of tickets in i's hand. Hence, this CSF is also known as the "Tullock lottery."

It's noteworthy that the probability in Equation (2.5) is a homogeneous function of degree zero, meaning it does not change with the units used to measure expenses. This makes the Ratio Form CSF unit invariant.

To fully define the CSF, we need to explain how the prize is allocated when no contestant makes an effort. The standard convention is to assume that when $\mathbf{g} = \mathbf{0}$, $p_i(\mathbf{g}) = 1/n$, meaning that the prize is awarded through a fair lottery. Note that this CSF is discontinuous at $\mathbf{g} = \mathbf{0}$. This means that when all contestants minus, say i, make zero effort, an infinitesimal effort by i yields the full prize. But, an even smaller effort yields the prize too, so the best reply of i when $\mathbf{g}_{-i} = \mathbf{0}$ does not exist. Fortunately, this does not cause much trouble from the point of view of the existence of equilibrium. It just implies that the standard procedure of showing the existence of a Nash equilibrium cannot be applied here because payoffs are not continuous.

The next proposition illustrates how to handle existence issues in the specific scenario where $V_1 = V_2 = \cdots = V_n = V$.

PROPOSITION 2.2 *There is a unique Nash equilibrium of the contest game in which the CSF is the Ratio Form and all valuations are identical.*

Proof The strategy of the proof is that, first, we look for a candidate equilibrium. Once we have identified it, we show that this candidate is indeed an equilibrium.
How to find the candidate? The equilibrium candidate must fulfill the first-order condition of payoff maximization. Thus for each i,

$$\frac{\sum_{j=i}^{n} g_j - g_i}{(\sum_{j=i}^{n} g_j)^2} V \le 1, i \in \{1, 2, \ldots, n\} \tag{2.6}$$

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with strict inequality if $g_i = 0$. We first see that if an inequality is strict, say for contestant j, it must be strict for all contestants. Suppose not, so there are two contestants, say i and k such that

$$\frac{\sum_{j=i}^{n} g_j - g_i}{(\sum_{j=i}^{n} g_j)^2} V < 1 \; ; \; \frac{\sum_{j=i}^{n} g_j - g_k}{(\sum_{j=i}^{n} g_j)^2} V = 1.$$

In this case $g_i = 0$. But this implies that $g_k < g_i$ which is a contradiction. And $\mathbf{g} = \mathbf{0}$ is not an equilibrium because if a contestant makes an infinitesimal effort, he will obtain the full prize. So we are left with the case in which, for all contestants, (2.6) holds with equality. Next, note that any solution to this system of equations must be symmetric, i.e., $g_i = g_j$ for all i, j since the term $\sum_{j=i}^{n} g_j$ appears in the same way in all of them. We have found a candidate, namely

$$g_i^* = \frac{n-1}{n^2} V. \tag{2.7}$$

It is only left to show that this candidate is indeed a Nash equilibrium. Given $\mathbf{g}_{-i}^*$, payoff for i looks like

$$\pi_i(g_i, \mathbf{g}_{-i}^*) = \frac{g_i}{g_i + (\frac{n-1}{n})^2 V} V - g_i.$$

Clearly, (2.7) is a solution to the first-order conditions of payoff maximization, and, given that payoffs are strictly concave in g_i, it is the only solution, so the candidate is indeed the unique Nash equilibrium. □

From (2.7), probabilities of winning, total effort, and payoffs in equilibrium are given by

$$p_i(\mathbf{g}^*) = \frac{1}{n}, \quad \sum_{j=i}^{n} g_j^* = \frac{n-1}{n} V, \quad \pi_i(\mathbf{g}^*) = \frac{V}{n^2}. \tag{2.8}$$

The symmetry of equilibrium actions translates into equal probabilities (shares) of obtaining the prize. But this could have been achieved by making no effort at all! In this sense, symmetric contests can be seen as a rat race in which participants invest substantial effort in vain.

Other comparative statics results are as follows:

(a) Individual and aggregate equilibrium efforts increase with V. Individual effort decreases with n, but aggregate effort increases with n.
Contestants respond to incentives, and when expected reward decreases – either because the prize is less valuable or there is more competition – individual efforts decrease as well.[3]

[3] The impact of prizes motivating effort is well-documented. For example, Ehrenberg and Bognanno (1990a,b) studied the relationship between the magnitude of prizes and the performance of participants in major golf tournaments. Their findings revealed a negative association between larger prizes and lower scores, which was interpreted as an indicator of effort. These results can be refined by including the composition of the contestants (higher prizes attract better athletes) and gender (men and women seem to react differently to prizes); see Matthews, Sommers, and Peschiera (2007).

(b) When $n \to \infty$, aggregate efforts tend to V, and individual payoff tends to 0. This result was named by Tullock as *Rent Dissipation* because the value of efforts (almost) equals the value of the rent in contests with a large number of contestants.[4] In very competitive contests, contestants should expect very little reward. In other words, the candy (rent) dissipates into too many mouths. We recall that similar results can be found in oligopolist markets with a large number of firms.[5] In fact, a contest with the Ratio Form CSF is formally identical to a Cournot model with inverse demand $V/\sum_{j=i}^{n} g_j$ (so the elasticity of demand is one) and marginal costs are constant and equal to one (in this interpretation g_i stands for the output of firm i).

The last observation leads us to inquire if the methods developed in Industrial Organization can be applied to our field. Despite the similarity noted above, there is a serious problem. In Industrial Organization, at least since the work of Bulow, Geanakoplos, and Klemperer (1985), the models are divided into two subfields: Those in which strategies are strategic substitutes, i.e., the best reply of any player is decreasing in the actions of the competitors (typically the Cournot model), and those in which strategies are strategic complements, i.e., the best reply of any player is increasing in the actions of competitors (typically Bertrand models). These two cases use different methods of analysis.[6] Thus, we are led to ask what the best replies look like in a contest in which the CSF is the Ratio. The answer is that they do not fall into the category of strategic substitutes or complements. Let us see why.[7]

From the first-order conditions of payoff maximization and letting $g_{-i} = \sum_{j \neq i} g_j$ (note that now g_{-i} is a number and not a vector as when we write $\mathbf{g}_{-i}$), the best reply function for contestant i with valuation V_i is given by

$$BR_i(g_{-i}) = \sqrt{V_i g_{-i}} - g_{-i}.$$

We see that

$$\frac{dg_i}{dg_{-i}} = \frac{\sqrt{V_i}}{2\sqrt{g_{-i}}} - 1.$$

For $g_{-i} = V_i/4$, it is zero. For $g_{-i} < V_i/4$, it is positive. For $g_{-i} > V_i/4$, it is negative. Thus, strategies are neither strategic substitutes nor strategic complements. In Figure 2.1, the best replies for two contestants in a contest with $V_1 = V_2 = 1$ are depicted.

Note that equilibrium in Figure 2.1 occurs at $g_i^* = 0.25$, exactly where $dg_i/dg_{-i} = 0$. With more contestants, since equilibrium is symmetric, it is located in the intersection of the best reply and a straight line of slope $1/(n-1)$. In Figure 2.2, the case for $n = 7$ and $V_i = 1$ for all i is plotted.

4 We have seen another example of Rent Dissipation in the section when the CSF is the All Pay Auction.
5 An exposition of these results can be found in Corchón (2001), propositions 2.2, 2.3, and 3.5.
6 For an exposition of these methods, see the books by Corchón (2001) and Vives (2001).
7 The careful reader will have noticed that the Cournot model with an isoelastic demand curve mentioned in the previous paragraph yields a best reply that is neither a strategic substitute nor a complement. This is why we said that “typically” Cournot models yield strategic substitution.

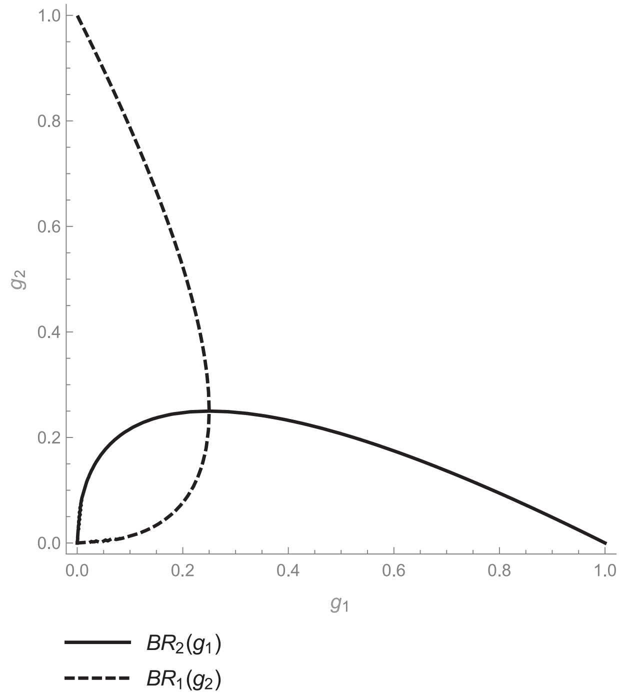

Figure 2.1 Best replies in a two-contestant case with a Ratio Form CSF and $V_1 = V_2 = 1$

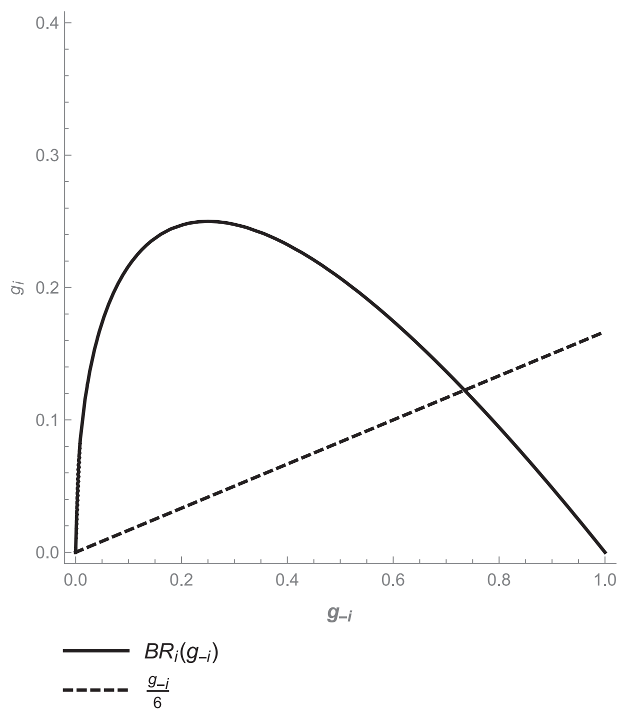

Figure 2.2 Equilibrium in a contest with $n = 7$, $V_i = 1$ for all i, and a Ratio Form CSF

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When $n > 2$, equilibrium occurs in the decreasing part of best replies, so, locally, strategies are strategic substitutes. When n is very large, the straight line representing symmetry is really close to the horizontal axis so g_i^* is close to 0 and total expenses tend to 1, which is the value of the prize (Rent Dissipation).

The study of the heterogeneous contestants brings a new issue, namely, that some contestants may opt to spend no effort in equilibrium. The analysis of this scenario is deferred to Chapter 3.

2.4 Extensions of Ratio: Logit CSF (Dixit, 1987)

In the Logit CSF, the linear terms g_i in the Ratio Form CSF are substituted by a more general function ϕ of g_i. When $\mathbf{g} \neq \mathbf{0}$,

$$p_i(\mathbf{g}) = \frac{\phi(g_i)}{\sum_{j=1}^{n} \phi(g_j)}.$$

The function ϕ measures the impact (or the merit) of g_i in the contest, and $\phi(g_i)/\sum_{j=1}^{n} \phi(g_j)$ measures the relative impact (merit) of i. The probability (share) of contestant i winning the contest equals her relative merit. For instance,the *generalized Ratio Form* is $\phi(g_i) = g_i^{\epsilon}$, as originally written by Tullock (1980). When $\epsilon = 1$, this form yields our old friend (2.5). When $\epsilon = 0$, it yields a pure lottery. You can check in Exercise 2.1 that if $\epsilon \in [0, 1]$, the CSF is concave in g_i and the unique symmetric Nash equilibrium is

$$p_i(\mathbf{g}^*) = \frac{1}{n}, \quad g_i^* = \frac{\epsilon(n-1)V}{n^2}, \text{ and } \pi_i^*(\mathbf{g}^*) = \frac{V(n - \epsilon(n-1))}{n^2}.$$

These expressions generalize those in (2.8). But now there is no rent dissipation when n is very large, which should serve as a warning about the generality of this result. See Exercise 2.11 for further properties of this CSF.

Another example is given by $\phi(g_i) = g_i + k$ (Amegashie, 2006). The parameter k can be interpreted as noise or as a head start (Kirkegaard, 2012). See Exercise 2.2 for the calculation of the unique Nash equilibrium in this case. Another special case of $\phi(g_i)$ in which this function is not everywhere increasing is presented in Exercise 2.19. See Cornes and Hartley (2005) for a general discussion of the importance of the form of ϕ.

Finally, note that if ϕ is strictly increasing, defining G_i as $G_i = \phi(g_i)$ payoffs can be written as

$$\pi_i(\mathbf{G}) = \frac{G_i}{\sum_{j=1}^{n} G_j} V - \phi^{-1}(G_i).$$

In this case, the contest with a Logit CSF is strategically equivalent to a contest with a Ratio Form CSF in which costs are not necessarily linear.

We will see that the main insights obtained with the Ratio CSF hold when the CSF is Logit. We assume that ϕ is differentiable, strictly increasing and concave to make sure that the first-order conditions of payoff maximization yield a maximum. Indeed, the first-order conditions are

$$\frac{\phi'(g_i)\sum_{j\neq i}\phi(g_j)}{(\sum_{j=1}^{n}\phi(g_j))^2}V-1=0,\ i\in\{1,\ldots,n\}, \tag{2.9}$$

where ϕ' is the derivative of ϕ. Since ϕ is concave, ϕ' is nonincreasing in g_i, and so the numerator inherits this property. And the denominator is increasing in g_i. Thus, since all these magnitudes are nonnegative, the ratio is decreasing in g_i. In other words, the second derivative of the payoff is nonpositive. Therefore, the first-order conditions are also sufficient conditions for payoff maximization.

To derive the best reply, we write the left-hand side of (2.9) as $F(g_i,\mathbf{g}_{-i})$. Totally differentiating this function with respect to g_i and g_j, we obtain

$$\frac{dg_i}{dg_j}=\frac{\frac{\partial F(g_i,\mathbf{g}_{-i})}{\partial g_j}}{-\frac{\partial F(g_i,\mathbf{g}_{-i})}{\partial g_i}}.$$

Note that in the Ratio case, payoffs are aggregative, i.e., depend on the sum of efforts.[8] Assuming that in the maximum, the second-order conditions of payoff maximization hold with strict inequality, we see that $sign(dg_i/dg_j)=sign(\partial F(g_i,\mathbf{g}_{-i})/\partial g_j)$. In our case

$$sign\frac{\partial F(g_i,\mathbf{g}_{-i})}{\partial g_j}=sign\left(\phi(g_i)-\sum_{j\neq i}\phi(g_j)\right).$$

As in the Ratio Form case, given the efforts of all other contestants, there is a unique effort of i at which $dg_i/dg_j=0$, namely $\phi(g_i)=\sum_{j\neq i}\phi(g_j)$, which for $n=2$ implies $g_1=g_2$. If for $\sum_{j\neq i}\phi(g_j)$ large enough the best reply of i is 0 (as it happens when $\phi(g_k)=g_k^{\epsilon}$ and $k\in\{1,2,\ldots,n\}$), the best reply in this case has the same form as in the Ratio CSF, namely hump-shaped. It can be proved that an increase of the prize increases efforts but an increase in the number of contestants decreases efforts; see Exercise 2.11.

2.5 Extensions of Ratio: Ratio Plus Luck and Relative Difference CSFs

The Ratio CSF does not account for the influence of pure luck on the contest result. As mentioned in Section 2.4, Amegashie proposed a CSF that specifically considers this factor. Nitzan (1991a) introduced another CSF to model the impact of luck, specifically

[8] See Corchón (2021) for a survey on aggregative games.

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$$p_i(\mathbf{g}) = a\frac{1}{n} + (1-a)\frac{g_i}{\sum_{j=1}^{n} g_j},\ a \in [0,1]. \tag{2.10}$$

Here, success is driven by two factors: merit (as in the Ratio CSF) and luck (as in the Che and Gale CSF), which is assumed to be evenly distributed among all contestants. If the CSF is imposed by a planner, it reflects her preference for equality. In Nitzan's CSF, luck does not affect merit directly (as in Amegashie), but it does impact the outcome. In the context of elections, this fixed effect means that some voters are firmly committed to one political party (e.g., Democrats or Republicans) and will never change their vote. As a result, even if a party makes minimal effort in an election, it will still receive a significant number of votes and have a nonnegligible chance of winning. In Amegashie's CSF, merit is perceived with a degree of noise.

Note that the sum of probabilities in (2.10) is one, as it should be. It is left to the reader to show that, in the unique Nash equilibrium,

$$g_i^* = \frac{(n-1)(1-a)V}{n^2}$$

(see Exercise 2.3). When the contest is decided by pure luck (i.e., $a = 1$), equilibrium efforts are zero, and when it is decided by relative merit (i.e., $a = 1$), they equal those obtained with the Ratio Form CSF.

Beviá and Corchón (2015) proposed an extension which adds, to the luck factor, the relative difference between contestants. This extension considers that a difference of, say, one battalion in a battle (or a million dollars in an R&D contest) is very different when the absolute number of soldiers is in the tens (or the R&D expenditure is in a few million dollars) than when it is in the thousands (or when R&D expenditure is in the billion dollars). Differences count, but they should be weighted by the expenses incurred by contestants. Sun Tzu's *The Art of War* counsels that how to arrange an army depends on ratios: "It is the rule in war, if our forces are ten to the enemy's one, to surround him; if five to one, to attack him; if twice as numerous, to divide our army into two" (p. 9).

The formal definition of this idea is not straightforward. A preliminary concept is needed. A *notional* CSF, denoted by f, is defined ignoring the requirement that probabilities are nonnegative and must sum up to one.

With two contestants and $\sum_{k=1}^{2} g_k \neq 0$,

$$f(g_i, g_j) = \alpha + \beta\frac{g_i - s g_j}{\sum_{k=1}^{2} g_k},\ i,j \in \{1,2\},\ i \neq j \tag{2.11}$$

with α, β, and s being nonnegative numbers. On top of the luck term α, the relative difference between expenses, where the competitor's expense is weighted by a factor s, is introduced. This is made to recover, as a special case, the Ratio Form CSF ($s = 0$ and $\alpha = 0$). The parameter s can be interpreted as how the probability of winning reacts to differences in efforts and β as how this probability reacts to relative differential efforts. Both parameters reflect how competitive a contest is.

To convert a notional CSF into a CSF, first we have to guarantee that the functions f defined in (2.11) add up to one, which is the case if and only if $2\alpha + \beta(1-s) = 1$.

So from the three parameters of this CSF, only two are truly independent. If $s = 0$ and $\alpha = 0.5a$, $a + \beta = 1$, and the Nitzan's CSF defined in (2.10) is recovered. Finally, we have to make sure that the values coming from the notional CSF are between zero and one. With all these ingredients in hand, the probability of winning is defined using the notional CSF,

$$p_i(\mathbf{g}) = \min(\max(f_i(g_i, g_j), 0), 1),\ i, j \in \{1,2\}, i \neq j.$$

This is the Relative Difference CSF. To find a Nash equilibrium in this setting, we disregard the nonnegativity constraint and go to the first-order condition of payoff maximization of the notional CSF. In the case of identical valuations, the first-order condition for contestant 1 is

$$\beta V \frac{(1+s)g_2}{(g_1+g_2)^2} - 1 = 0.$$

By the same reasonings made in Proposition 2.2, a candidate equilibrium is symmetric and is given by

$$g_1^* = g_2^* = \beta V \frac{1+s}{4},$$

which coincides with the Ratio case when $\beta = 1$ and $s = 0$. Note that the probability of obtaining the prize, if contestants exercise these efforts, is $\alpha + \beta(1-s)/2 = 0.5$.

Now consider the maximization of payoffs of contestant 1 when $g_2 = g_2^*$, namely

$$\min\left(\max\left(\alpha + \beta \frac{g_1 - s g_2^*}{g_1 + g_2^*}, 0\right), 1\right) V - g_1.$$

Since $g_1 \in [0, V]$ and payoffs are continuous in g_1 by a theorem by Weierstrass the maximum exists. It could be only located either in the extremes of $[0, V]$ or when the first-order conditions are met. Clearly, $g_1 = V$ is worse than $g_1 = 0$ because in the former the best outcome would be to obtain the prize with a probability 1, payoffs are going to be nonpositive. Therefore, if we show that in the candidate equilibrium profits are nonnegative, this candidate is indeed an equilibrium. Indeed,

$$\pi_i(\mathbf{g}^*) = \frac{V}{2} - \beta V \frac{1+s}{4} = \frac{V}{4}(2 - \beta(1+s)) \geq 0 \quad \text{if and only if } 2 \geq \beta(1+s).$$

What to take home from this argument is that when using the relative difference CSF, you can disregard the constraints imposed by nonnegativity and use the notional CSF (2.11). And the first-order conditions of payoff maximization when these constraints are ignored yield indeed the Nash equilibrium we are looking for, provided that $2 \geq \beta(1+s)$. The latter indicates that the contest cannot be very competitive (recall the interpretation of s and β given above) so we stay away from the All Pay Auction.[9]

The same procedure can be applied when $n > 2$. In this case, the notional CSF can be written as

[9] Note that the inequality $2 \geq \beta(1+s)$ always holds for the Ratio Form CSF since $s = 0$ and $\beta = 1$.

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$$f(g_i, \mathbf{g}_{-i}) = \alpha + \beta \frac{\left(g_i - \frac{s}{n-1} \sum_{j \neq i} g_j\right)}{\sum_{k=1}^{n} g_k}.$$

In this case, the condition for the first-order condition to be an equilibrium is $n \geq \beta(n - 1 + s)$, and equilibrium effort for each contestant i is

$$g_i^* = \beta V \frac{n - 1 + s}{n^2}.$$

See Exercise 2.4 for the calculations leading to the above formula.[10]

2.6 Additive Separable CSF

In the Additive Separable CSF, success in a contest is represented by

$$p_i(\mathbf{g}) = \frac{1}{n} + \varphi(g_i) - \frac{1}{n-1} \sum_{j \neq i} \varphi(g_j) \tag{2.12}$$

with $\varphi(0) = 0$ and φ increasing. Clearly, the sum of the terms in (2.12) is 1. To make each term nonnegative, we add that $\varphi(V) \leq 1/n$, and thus

$$p_i(\mathbf{g}) = \frac{1}{n} + \varphi(g_i) - \frac{1}{n-1} \sum_{j \neq i} \varphi(g_j) \geq \frac{1}{n} - \frac{1}{n} = 0$$

for the relevant range of efforts, namely for those in $[0, V]$. The success in a contest depends on luck ($1/n$) and the difference between your impact on the contest ($\varphi(g_i)$) and the average impact of the others, $\sum_{j \neq i} \varphi(g_j)/(n-1)$. The difference with the Che and Gale CSF is that we do not need the max operator to guarantee that p_i's are nonnegative. In this case, the trick is provided by the boundedness assumption $\varphi(V) \leq 1/n$. The reader can check that when $n = 2$, (2.12) looks similar to what is inside the minmax operator in (2.4) when $s = 1$. But this would hold if $\varphi(g_i) = g_i$ and this form violates the boundedness assumption.

An advantage of this CSF is that Nash equilibrium strategies are dominant strategies, i.e., they are not only the best reply to what others do, but they are the best reply to anything the others might do! Contestants do not have to wonder about what the others are going to do. To prove this, we see that the terms that do not depend on g_i enter additively in payoffs. Thus, when other contestant actions increase, payoffs are just smaller, but the relationship between expected revenue ($p_i V$) and costs is the same. In other words, payoffs are transformed monotonically by the strategies of other contestants and, as consumer theory teaches us, this transformation does not affect the maximum. Mathematically this is seen by looking at the first-order condition of payoff maximization, $\varphi'(g_i)V = 1$, and realize that it only depends on g_i.

[10] Hwang (2012) presented another generalization of the Difference and the Ratio Form CSF; see Section 3.6.

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Particular forms of this family of CSF were proposed by Skaperdas and Vaidya (2012) and Polishchuk and Tonis (2013) and axiomatized by Cubel and Sanchez-Pages (2016).

There are other CSF that yield dominant strategies. Consider a CSF like the following one: Given a cutoff $\underline{g} \in [0, V/n)$, define

$$p_i(\mathbf{g}) = \frac{1}{\#C(\boldsymbol{g})} \text{ for all } i \in C(\boldsymbol{g}),$$
$$p_i(\mathbf{g}) = 0 \text{ for all } i \notin C(\boldsymbol{g}), \text{ and}$$
$$p_i(\mathbf{g}) = \frac{1}{n} \text{ for all } i \text{ if } C(\boldsymbol{g}) = \varnothing,$$
$$\text{where } C(\boldsymbol{g}) = \{i \in \{1, \dots, n\} \mid g_i \geq \underline{g}\}.$$

In this CSF, contestants are requested to make a minimum effort $\underline{g}$ to qualify for a pure lottery of the prize unless no one makes this minimum effort in which case all qualify for this lottery. It is easy to see that $g_i = \underline{g}$ is a dominant strategy for all contestants. But, this CSF is not additively separable. If this CSF were additively separable, it should satisfy the following property: For all $(g_i, \mathbf{g}_{-i})$, $(g_i', \mathbf{g}_{-i}')$, $p_i(g_i, \mathbf{g}_{-i}) + p_i(g_i', \mathbf{g}_{-i}') = p_i(g_i, \mathbf{g}_{-i}') + p_i(g_i', \mathbf{g}_{-i})$. Suppose that $g_i \geq \underline{g}$, $\mathbf{g}_{-i}$ is such that $C(\boldsymbol{g})\backslash\{i\} = k$, $g_i' < \underline{g}$ and $\mathbf{g}_{-i}'$ is such that $C(\boldsymbol{g})\backslash\{i\} = m$. Then, $p_i(g_i, \mathbf{g}_{-i}) + p_i(g_i', \mathbf{g}_{-i}') = 1/(k+1)$ but $p_i(g_i, \mathbf{g}_{-i}') + p_i(g_i', \mathbf{g}_{-i}) = 1/(m+1)$.

However, it is possible to construct an additive separable CSF that gives the same equilibrium efforts, probabilities, and payoffs than the contest described above. Let $p_i(\mathbf{g})$ be such that

$$p_i(\mathbf{g}) = \frac{1}{n} + \varphi(g_i) - \frac{1}{n-1}\sum_{j \neq i} \varphi(g_j), \text{ with}$$
$$\varphi(g_i) = 1/n \text{ if } g_i \geq \underline{g} \text{ and } \varphi(g_i) = 0 \text{ otherwise.}$$

This CSF is additively separable with $g_i = \underline{g}$ being a dominant strategy for all contestants. Beviá and Corchón (2022) prove that given any CSF yielding a contest with dominant strategies, there is an additively separable CSF yielding a contest with the same efforts and probabilities in equilibrium. This implies that additively separable CSFs, somehow, characterize those CSFs yielding dominant strategies.

2.7 Advanced Material: The Existence of a Symmetric Equilibrium with General CSF

At this point, the reader may wonder that given the variety of CSFs presented so far, we would need idiosyncratic arguments to show the existence of a Nash equilibrium in symmetric contests. Recall a general theorem showing the existence of a Nash equilibrium in general games (see Fudenberg and Tirole, 1991, p. 34). But this proof is of no avail here since we have seen that, at least, for CSF homogeneous of degree zero, payoffs are not continuous at $\mathbf{g} = \mathbf{0}$.

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In this section, we present two approaches to the existence of a symmetric equilibrium that will cover many of the cases seen before. The strategy of the proof in both cases is identical to the one used in Proposition 2.2. First, we look for a candidate equilibrium found by solving the first-order conditions, and later we show that this candidate is indeed an equilibrium. Throughout this section, we will assume that the CSF is symmetric, twice differentiable whenever there is, at least, a positive effort and $\partial p_i(\mathbf{g})/\partial g_i > 0$. We will refer to these assumptions as $\boldsymbol{D}$.

The first approach is taken from Malueg and Yates (2006), and it works for contestants with identical valuations and CSF homogeneous of degree zero. We have seen that there are plenty of CSFs satisfying this assumption.

Again, let us start by looking at the first-order conditions of payoff maximization. Since the CSF is homogeneous of degree zero, $\partial p_i(\mathbf{g}^*)/\partial g_i$ is homogeneous of degree -1. (See Sydsæter and Hammond, 2012, chapter 12, p. 432.) This implies that we can write

$$\frac{\partial p_i(\mathbf{g})}{\partial g_i} = \frac{1}{g_i} \frac{\partial p_i\left(\frac{g_1}{g_i}, \frac{g_2}{g_i}, \ldots, \frac{g_n}{g_i}\right)}{\partial g_i}.$$

Since we are looking for a symmetric equilibrium, all ratios g_j/g_i are 1. The first-order condition is written as

$$\frac{\partial p_i(1, 1, \ldots, 1)}{\partial g_i} V = g^*. \tag{2.13}$$

Now g^*, which is strictly positive because $\partial p_i(\mathbf{g})/\partial g_i > 0$, is our candidate equilibrium. Exercise 2.5 asks the reader to prove that $(0, 0, \ldots, 0)$ cannot be an equilibrium; therefore our candidate is the only candidate. Thus, if a symmetric equilibrium exists, it is given by (2.13). Malueg and Yates gave sufficient conditions that guarantee the existence of such an equilibrium, mainly:

(1) There exist $\bar{g}$ such that $\partial^2 p_i(g, 1, \ldots, 1)/\partial g_i^2 \begin{cases} < 0 & \forall g > \bar{g} \\ > 0 & \forall g < \bar{g} \end{cases}$.
   Condition (1) allows for nonconcave CSF as long as at some point concavity is recovered.
(2) $1/n - \partial p_i(1, \ldots, 1)/\partial g_i > p_i(0, 1, \ldots, 1)$.
   Condition (2) guarantees that choosing zero effort is not a profitable deviation.

PROPOSITION 2.3 *Under conditions (1)–(2) above, there is a unique symmetric equilibrium for any contest with identical valuations and a CSF homogeneous of degree zero satisfying* $\boldsymbol{D}$.

Proof The interested reader can see the complete proof in Malueg and Yates (2006); we just give here a guide to the proof's steps.

From the analysis above, it is only left to prove that when all contestants but i choose g^*, g^* is the best reply of i to $\mathbf{g}^*_{-i}$.

(1) When all other contestants are choosing g^*, payoff functions are continuous.
   Since $g_i \in [0, V]$, a payoff-maximizing effort, call it $\tilde{g}$, exists. This maximum

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can be located only in three places: at the first-order condition if the maximum is interior or at the bounds, namely, 0 or V, the latter clearly not an optimal choice.

(2) Condition (2) implies that 0 cannot be a maximum because $\pi_i(\mathbf{g}^*) = V(1/n - \partial p_i(1, 1, \dots, 1)/\partial g_i) > p_i(0, 1, \dots, 1)V = \pi_i(0, \mathbf{g}^*_{-i})$.

(3) Condition (1), implies that there are at most two strategies that satisfy the first-order condition; we already know that g^* is one of them; let $\hat{g}$ be the other possible one.

(4) Finally, the proof ends by showing that if $\hat{g}$ exist, $\pi_i(\hat{g}, \mathbf{g}^*_{-i}) < \pi_i(g^*, \mathbf{g}^*_{-i})$.

□

As an example, let $n = 2$ and consider the generalized Ratio Form CSF

$$p_i(g_i, g_j) = \frac{g_i^\epsilon}{g_i^\epsilon + g_j^\epsilon};\ p_j(g_j, g_i) = 1 - p_i(g_i, g_j)$$

then our candidate is

$$g_1^* = g_2^* = \frac{\epsilon}{4}V.$$

For this particular case, condition (1) holds whenever $\epsilon \le 2$ and condition (2) also holds (see Exercise 2.6).

What about CSFs that are not homogeneous of degree zero? Let us follow the footsteps of the previous proof. First, let us find a symmetric candidate. Let $\alpha(y)$ be the elasticity of p_i when $g_i = y > 0$ for all $i \in \{1, \dots, n\}$.

$$\alpha(y) = \frac{\partial p_i(y, y, \dots, y, \dots, y)}{\partial g_i} ny.$$

When all contestants choose the same effort y, the first-order condition can be written as

$$\frac{\alpha(y)}{n} V = y. \tag{2.14}$$

From now on, and without loss of generality, we will take $y \in [0, V]$ because, as we said earlier, no rational contestant will spend more in the contest than the value of the prize. Now we assume the following:

(I) α is continuous for all $y > 0$.
(II) $\lim_{y \to 0} \alpha(y)$ is well defined. Call it $\alpha(0)$.
(III) $p_i(0, \boldsymbol{g}_{-i}) = \beta$ a constant for $\boldsymbol{g}_{-i} = (y, y, \dots, y)$ with $\beta < 1/n$.
(IV) $\alpha(0) > 0$ and $\alpha(y) < 1 - n\beta$ for all $y \in [0, V]$.

How does this assumption look like when CSF is Logit? Defining $\epsilon(g_i)$ as the elasticity of ϕ, i.e.,

$$\epsilon(g_i) = \frac{g_i \phi'(g_i)}{\phi(g_i)},$$

we see that $\alpha(y) = \epsilon(y)(n-1)/n$ and, given that $\beta = 0$, part (IV) of the previous assumption says that $\epsilon(0) > 0$ and $\epsilon(g_i) \le n/(n-1)$. This is the assumption used by

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Pérez-Castrillo and Verdier (1992). A constant elasticity $\phi(g_i)$ with $\epsilon \leq 1$ (e.g., the Ratio CSF) fulfills this assumption. When $\beta \neq 0$ like in the CSF proposed by Nitzan, we have that if a is the constant multiplying $1/n$ in the CSF, $\beta = a/n$ and the second part of (IV) above says that $\alpha(y) < 1 - n\beta = 1 - a$. And similarly with the CSF of Beviá and Corchón.

With these assumptions in hand, we focus on (2.14). We see that for $y = 0$, $\alpha(0)V/n > 0$ and for $y = V$, $\alpha(V)V < nV$ since $\alpha(y) < 1 - n\beta$. Thus, the intermediate value theorem applied to (2.14) says that there is a positive number $y^* \in (0, V)$ such that

$$\frac{\alpha(y^*)}{n} V = y^*.$$

The last assumption imposed is:

(V) $\partial \pi_i(\varepsilon, \mathbf{y}_{-i})/\partial g_i > 0$ for ε sufficiently close to zero, $\partial \pi_i(\varepsilon, \mathbf{y}_{-i})/\partial g_i < 0$ when ε is large enough. And there is at most a unique g_i such that $\partial^2 \pi_i(g_i, \mathbf{y}_{-i})/\partial g_i^2 = 0$.

This assumption is similar to condition (2) above of Malueg and Yates. The Ratio Form CSF indeed fulfills these five properties (see Exercise 2.8).

PROPOSITION 2.4 *There is a symmetric equilibrium for any contest with identical valuations and a CSF satisfying (I)–(V) above and* ***D***.

Proof It is only left to prove that when all contestants but i choose y^*, y^* is the best reply of i. The key fact is that when all other contestants are choosing y^*, payoff functions are continuous. Since $g_i \in [0, V]$, a payoff-maximizing effort, call it $\bar{g}$, exists. This maximum can be located only in three places: at the first-order condition if the maximum is interior or at the bounds, namely, 0 or V, the latter clearly not an optimal choice. Setting $\bar{g} = y^*$, the payoffs when contestant i plays y are

$$\frac{V}{n} - \frac{\alpha(y^*)}{n} V > \beta V \text{ if and only if } 1 - n\beta > \alpha(y^*).$$

It is only left to prove that there is only one strategy that satisfies the first-order condition. For this, it is enough to invoke condition (V) above. Thus, when all contestants but i choose y^*, y^* is the unique best reply for i and we have found a Nash equilibrium. □

In some cases, we can prove that the equilibrium is unique. For instance, when the CSF is Logit with concave and strictly increasing ϕ, the first-order conditions are

$$V \frac{\phi'(y)(n-1)}{\phi(y)n^2} = 1, \; i \in \{1, 2, \ldots, n\}. \tag{2.15}$$

Since the left-hand side of (2.15) is strictly decreasing in y, the symmetric equilibrium is unique. More generally, if the function α is decreasing, the solution to (2.14) is unique so the symmetric equilibrium is unique too.

The map of the conditions under which an equilibrium exists is still incomplete, even if we assume the generalized Ratio Form CSF. It seems intuitively obvious that when the elasticity of efforts, ϵ, is very large, this CSF tends to the All Pay Auction, so only equilibria in mixed strategies exist. When $n = 2$, Baye, Kovenock, and de

Vries (1994) showed that the symmetric two-player contest with $\epsilon \in (2, \infty)$ allows a mixed-strategy Nash equilibrium. Further characterization of these mixed equilibria is given by Alcalde and Dahm (2010) and Ewerhart (2015, 2017). See Exercise 2.12 for the case of an additively separable utility function.

2.8 Exercises

2.1 Compute the Nash equilibrium of a contest with n contestants, linear cost, $V_i = V$ for all i, and the Logit CSF when $\phi(g_i) = g_i^{\epsilon}$, $\epsilon \in (0, 1]$.

2.2 Compute the Nash equilibrium of a contest with n contestants, linear cost, $V_i = V$ for all i, and the Logit CSF when $\phi(g_i) = g_i + k$.

2.3 Compute the Nash equilibrium of a contest with n contestants, linear cost, $V_i = V$ for all i, and the CSF proposed by Nitzan (1991a).

2.4 Compute the Nash equilibrium of a contest with n contestants, linear cost, $V_i = V$ for all i, and the CSF proposed by Beviá and Corchón (2015).

2.5 Show that in a symmetric contest with a CSF homogeneous of degree zero and strictly increasing in g_i, $(0,,,0)$ cannot be an equilibrium.

2.6 Show that if $p_i(\mathbf{g}) = g_i^{\epsilon} / \sum_{j=1}^{n} g_j^{\epsilon}$, the sufficient conditions of Malueg and Yates (2006) hold if $\epsilon \leq n/(n-1)$.

2.7 Suppose that the best reply of a symmetric game, denoted by B, is continuous in $\mathbb{R}_+$. Let $R(y) = B(y, \ldots, y, \ldots, y)$ be the best response when all contestants except i choose y. Suppose that R is such that

(a) $\exists \underline{x}$ such that $\forall y \in (0, \underline{x})$, $R(y)(n-1) > y$

(b) $\exists \bar{x}$ such that $\forall y \in (\bar{x}, \infty)$, $R(y) = 0$.

(1) Show that under (a) and (b) above, a symmetric Nash equilibrium exists (equilibrium is not necessarily unique). (Hint: Use the intermediate value theorem.)

(2) Give a micro foundation to (a) and (b) (hint for a) find the slope of B differentiating the first-order condition).

2.8 Show that the Ratio Form CSF fulfills assumptions (I)–(V) in the main text.

2.9 Compute the symmetric Nash equilibrium of a Possibly Indecisive Contest (a contest where the probability that no one receives the prize is not zero) in which cost is linear and the CSF is $p_i(\mathbf{g}) = g_i / (\sum_{j=1}^{n} g_j + a)$, $a > 0$.

2.10 There are n identical firms that can acquire each other prior to a contest. Let $\pi(m)$ be payoffs from the contests when there are m firms. Acquisition price is $P = a\pi(n) + (1-a)\pi(n-1)$ with $a \in [0, 1]$, i.e., the acquisition price is a linear convex combination of payoffs with n firms (status quo) and $n-1$ firms (when the acquisition takes place). If acquisition takes place, total payoffs for a firm are the payoffs from the

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contests with $n-1$ firms minus the price paid by the acquisition. Study conditions on π and a under which acquisition does not take place.

2.11 Prove that, with the Logit CSF with a differentiable, strictly increasing and concave ϕ, an increase of the prize increases efforts but an increase in the number of contestants decreases efforts (see Nti, 1997 and Corchón, 2007).

2.12 Show the existence of equilibrium when contestants have an additively separable utility function $u(p_i V) - c(g_i)$, where p_i is the Ratio CSF, u is concave, and c is convex. See Dickson, MacKenzie, and Sekeris (2022).

2.13 Consider a two-person contest with payoff functions $\pi_1(\mathbf{g}) = p(g_1, g_2)V - g_1$, and $\pi_2(\mathbf{g}) = (1 - p(g_1, g_2))V - g_2$. Assume differentiability as much as you need. From the first-order condition, compute the infinitesimal effect of a change in V on total effort and find conditions under which this effect is positive.

2.14 A contest with a variable prize: Consider a contest with n individuals and a variable prize, $V = \sum_{i=1}^{n} g_i^{\gamma}$, with $\gamma \in (0,1)$. Assuming a Ratio Form CSF and a linear cost of effort, find the Nash equilibrium. Give an interpretation of this game.

2.15 Consider n experts, with $n \geq 2$, making predictions about the outcome of a random variable that can take on m different values with probabilities $p_1, p_2, \ldots, p_m$ (common knowledge among experts). Assume that $p_1 \geq p_2 \geq \cdots \geq p_m$. Experts get a reward from the public (prestige, money, etc.). In particular, assume that if n_I experts announce I and I occurs, the reward for each of these experts is V/n_I. The rest of the experts get zero. An equilibrium in the prediction market is a list of natural numbers $(n_1^*, n_2^*, \ldots, n_m^*)$ such that $\sum_{I=1}^{m} n_I^* = n$ and for all $I \in \{1, 2, \ldots, m\}$, if $n_I^* > 0$,

$$\frac{V}{n_I^*} p_I \geq \frac{V}{n_J^* + 1} p_J, \forall J \in \{1, 2, \ldots, m\}, J \neq I$$

so no expert announcing I has an incentive to switch and announce any other prediction. Show that there is an equilibrium in which experts announce different predictions.

2.16 Consider a piece of land with a value of V that may be invaded by a large number of identical potential invaders. The number of actual invaders is denoted as n. The probability of a successful invasion is given by

$$\frac{g(n)}{g(n) + kg(n)},$$

where $g(n)$ measures the strength of the invaders, and $kg(n)$ represents the strength of those opposing the invasion (such as police, armed forces, or current owners) which is assumed to be proportional to the strength of the invaders. If the land is successfully conquered, it will be divided equally among the invaders.

(1) Determine the payoffs for a potential invader in this scenario.

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(2) Let R be the opportunity cost of an invader, which is assumed to be independent of the number of invaders. Determine the equilibrium number of invaders by taking into account the opportunity cost R.
(3) Give historical examples of land invasions, see Falcone and Rosenberg (2023).

2.17 Suppose a contest where the objective function of each contestant is to maximize the difference in payoffs (where payoffs are defined as has been assumed throughout this book). Assume two contestants and a Logit CSF with $\phi(g_i) = g_i^\epsilon$ with $\epsilon \in (0, 1]$.

(1) Find the Nash equilibrium of such a game.
(2) Suppose now that the contestant's objective is to maximize the ratio of the payoffs. Find the Nash equilibrium of such a game.

For a motivation of the assumption on the objective function of contestants, see the natural selection model of Schaffer (1989).

2.18 Suppose rent-seeking activities can be taxed with a constant unit tax rate of t on expenses. The CSF is a generalized Ratio Form. All contestants value the prize at V.

(1) Write the payoff of a typical contestant.
(2) Use the first-order condition and symmetry to find the unique symmetric equilibrium.
(3) Find the equilibrium payoffs. Comment on how payoffs depend on t. Interpret this relationship.
(4) Calculate the total revenue raised by taxes. Determine the value of t that maximizes total revenue. Compute the sum of payoffs and total revenue.

2.19 Suppose a Logit CSF in which $\phi(g_i) = \max(ag_i - \frac{b}{2}g_i^2, 0)$. This function is increasing if and only if $g_i \leq a/b$.

(1) Find reasons why an increase in expenses might decrease the probability of winning the prize.
(2) Find the unique symmetric equilibrium. (Hint: Disregard the nonnegativity constraint in $\phi(g_i)$ and show that a symmetric solution to the first-order condition satisfies this constraint.)

2.20 An organization with n identical individuals is going to be "purged." In particular, $k \leq n - 2$ individuals are selected at random and expelled from the organization with probability $1 - q$. They enjoy zero utility. The remaining individuals enter into a contest with a prize V and a Ratio Form CSF.

(1) Assuming that q is given, find the k for which all individuals support the purge. Comment on the solution
(2) Now assume that $q = (n - k)/n$ (favorable cases divided by total cases). Find the k such that all individuals support the purge.

2.21 Suppose that the CSF is Ratio Form, agents are identical and have constant absolute risk aversion. Write the payoff functions and the first-order condition of

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payoff maximization of a typical contestant. Give an example in which a symmetric division of the prize will not increase the aggregate investment when the players are risk neutral but it is possible for a symmetric division to increase the aggregate investment when the players are risk averse. (See example 4 in Brookins and Jindapon, 2022.)

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