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An existence theorem for the generalized complementarity problem

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Given a closed, convex cone S, in R^n, its polar S^* and a mapping g from R^n into itself, the generalized nonlinear complementarity problem is to find a $z \in R^n$ such that

$$g(z) \in S^* , \quad z \in S ,$$
$$\langle g(z), z \rangle = 0 .$$

Many existence theorems for the problem have been established under varying conditions on g. We introduce new mappings, denoted by $J(S)$-functions, each of which is used to guarantee the existence of a solution to the generalized problem under certain coercivity conditions on itself. A mapping $g : S \to R^n$ is a $J(S)$-function if

$$g(z) - g(0) \in S^* , \quad z \in S ,$$
$$\langle g(z)-g(0), z \rangle = 0 ,$$

imply that $z = 0$. It is observed that the new class of functions is a broader class than the previously studied ones.

1. Introduction

The generalized nonlinear complementarity problem is to find a $z \in R^n$ satisfying

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$$(1.1)\qquad \begin{aligned} & g(z) \in S^* , \quad z \in S , \\ & \langle g(z), z\rangle = 0 , \end{aligned}$$

where S is a closed, convex cone in R^n, S^* the polar cone, and g is a mapping from R^n into itself. Problem (1.1) has been studied by Habetler and Price [2], Karamardian [3, 5], and Moré [6], who have given many theorems for the existence and uniqueness of a solution to the problem.

In this paper we introduce a nonlinear generalization of $J(S)$-matrices (which are known as J-matrices in [7]), denoted by $J(S)$-functions, and study problem (1.1) defined by members of this class of functions. We show that the classes of P- and regular functions become proper subclasses of this class when S is taken as R^n_+. We also show that a function which is strongly S-copositive on S is a $J(S)$-function.

We establish the following: let S be pointed, and let $g : S \to R^n$ be a $J(S)$-function. Then there is a solution to (1.1) if

(i) the map $G(z) = g(z) - g(0)$ is continuous and positively homogeneous of some degree on S, and

(ii) the system

$$0 \neq z \in S , \quad G(z)+p \in S^* , \quad \langle G(z)+p, z\rangle = 0$$

is inconsistent for some $p \in \operatorname{int} S^*$.

2. Notations and definitions

For brevity, we shall use much of the notation of [7].

A nonempty subset S of R^n is a closed, convex cone if S is closed, and $\alpha x + \beta y$ belongs to S for all $\alpha, \beta \geq 0$, and $x, y \in S$. The polar cone of S is the cone S^* defined by

$$S^* = \{x \in R^n : \langle x, y\rangle \geq 0 \text{ for all } y \in S\} .$$

A cone is said to be pointed if whenever $x \neq 0$ is in the cone, $-x$ is not in the cone. For a closed, convex cone S, the interior of S^*, denoted by $\operatorname{int} S^*$, is nonempty if, and only if, S is pointed. The trivial cone $S = \{0\}$ is excluded from the discussion.

A map $g : R_+^n \to R^n$ is a uniform P-function if there exists a scalar $c > 0$ such that, for any $x \neq y$ in R_+^n, there is an index $k = k(x, y)$ with

$$\left(x_k - y_k\right)\left(g_k(x) - g_k(y)\right) \geq c\|x-y\|^2 .$$

A map $G : R_+^n \to R^n$ with $G(0) = 0$ is a regular function if the system

$$G_i(x) + t = 0 \quad \text{for} \quad i \in I_+(x) ,$$

$$G_i(x) + t \geq 0 \quad \text{for} \quad i \in I_0(x) ,$$

$$0 \neq x \geq 0 , \quad t \geq 0 ,$$

is inconsistent. Here $I_+(x)$ and $I_0(x)$ denote the set of indices corresponding to the positive and zero components of x, respectively.

A map $g : S \to R^n$ is strongly S-copositive on S if there exists a scalar $c > 0$ such that, for all $x \in S$,

$$\langle g(x) - g(0), x \rangle \geq c\|x\|^2 .$$

A map $g : S \to R^n$ is a $J(S)$-function if

$$g(x) - g(0) \in S^* , \quad x \in S ,$$

$$\langle g(x) - g(0), x \rangle = 0 ,$$

imply that $x = 0$.

A map $G : S \to R^n$ is positively homogeneous of degree d over S if, for every $x \in S$,

$$G(\lambda x) = \lambda^d G(x) \quad \text{for all} \quad \lambda \geq 0 .$$

A square matrix A is a $J(S)$-matrix (termed as J-matrix in [7]) if

$$Ax \in S^* , \quad x^T Ax = 0 , \quad x \in S ,$$

imply that $x = 0$.

3. Main results

LEMMA 3.1. *Let S be a closed, convex cone in R^n, and let $g : S \to R^n$.*

(a) If g is strongly S-copositive on S, then g is a $J(S)$ function.

(b) g is a $J(R^n_+)$-function whenever either

(i) g is a uniform P-function on R^n_+, or

(ii) the map $G(x) = g(x) - g(0)$ is a regular function.

Proof. *(a)* Let g be strongly S-copositive on S. Then $x \in S$, $g(x) - g(0) \in S^*$, $\langle g(x)-g(0), x\rangle = 0$ imply that

$$0 = \langle g(x)-g(0), x\rangle \geq c\|x\|^2 ,$$

and consequently, $x = 0$.

(b) If g is a uniform P-function on R^n_+, then, for every $0 \neq x \geq 0$, we have an index k (depending upon x) with

$$x_k\left(g_k(x)-g_k(0)\right) \geq c\|x\|^2 > 0 .$$

The conclusion *(b)* for uniform P-function is then obvious.

To prove the second part of *(b)* we proceed as follows: let $G(x) = g(x) - g(0)$. The consistency of the system

$$x \geq 0 , \quad g(x) - g(0) \geq 0 , \quad \langle g(x)-g(0), x\rangle = 0 ,$$

implies that $x_i\left(g_i(x)-g_i(0)\right) = 0$ for all $1 \leq i \leq n$. If $x \neq 0$, we will have $G_i(x) = 0$ for $i \in I_+(x)$ and $G_i(x) \geq 0$ for $i \in I_0(x)$. Now taking $t = 0$ in the definition of regular function, we get a contradiction to the regularity of $G(x)$.

REMARK 3.2. It is interesting to examine the following two examples in order to see that the classes of uniform P- and regular functions are proper subclasses of the class of $J(S)$-functions when $S = R^n_+$. The mapping $g(x) = \left[x_1+x_2^2, x_2\right]^T$ defined over R^2_+ is a $J\left(R^2_+\right)$-function, but

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it is not a uniform P-function, since if $x^k = [1-2/k, k+1/k^2]^T$ and $y^k = [1, k]^T$, $k = 2, 3, \ldots$, then

$$\left(x_1^k-y_1^k\right)\left(g_1\left(x^k\right)-g_1\left(y^k\right)\right) = -2/k^5 < 0 ,$$

$$\left(x_2^k-y_2^k\right)\left(g_2\left(x^k\right)-g_2\left(y^k\right)\right) = 1/\left(4k^2+1\right) . \|x^k-y^k\|^2 .$$

The other example is the mapping $g(x) = \left[x_2^2-x_1, x_2\right]^T$ which is a $J\left(R_+^2\right)$-function, but not regular.

We shall need the following results.

LEMMA 3.3 [2, Lemma 5.1]. *Let* S *be a pointed, closed, convex cone in* R^n, *and let* $p \in \text{int } S^*$. *Then the set*

$$V = \{x : x \in S, \langle p, x\rangle = 1\}$$

is compact.

THEOREM 3.4 [7, Theorem 3.6]. *If* $F : R^n \to R^n$ *is a continuous mapping on the nonempty, compact, convex set* C *in* R^n, *then there is an* x^0 *such that*

$$\langle F\left(x^0\right), x-x^0\rangle \geq 0 \quad \textit{for all} \quad x \in C .$$

Now we give the following existence theorem.

THEOREM 3.5. *Let* S *be a pointed, closed, convex cone in* R^n, *and let* $g : S \to R^n$ *be a* $J(S)$*-function. Then there is a solution to* (1.1) *if*

(i) *the map* $G(z) = g(z) - g(0)$ *is continuous and positively homogeneous of degree* $d > 0$ *on* S, *and*

(ii) *there exists a vector* $p \in \text{int } S^*$ *such that the system*

(A) $0 \neq z \in S$, $G(z) + p \in S^*$, $\langle G(z)+p, z\rangle = 0$ *is inconsistent.*

Proof. It is easy to show that the product set $K = S \times R_+$ is a pointed, closed, convex cone in R^{n+1}, and its polar is $K^* = S^* \times R_+$. It is also obvious that $(p, 1) \in \text{int } K^*$. Let

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$$C = \{(z, t) : z \in S, t \geq 0, \langle p, z\rangle + t = 1\} .$$

The set C is a nonempty, convex subset of K, and by Lemma 3.3, it is compact. Define the map $F : K \to R^{n+1}$ by

$$F(z, t) = \begin{bmatrix} G(z)+t\big(p+g(0)\big) \\ t \end{bmatrix} .$$

It follows from Theorem 3.4 that there exists $(\bar{z}, \bar{t})$ in C such that, for all $(z, t) \in C$,

$$\langle G(\bar{z})+\bar{t}\big(p+g(0)\big), z-\bar{z}\rangle + \bar{t}(t-\bar{t}) \geq 0 .$$

But this means that

$$\langle G(\bar{z})+\bar{t}\big(p+g(0)\big), \bar{z}\rangle + \bar{t}^2 = \min_{(z,t)\in C} \langle G(\bar{z})+\bar{t}\big(p+g(0)\big), z\rangle + \bar{t}.t .$$

Now using the Kuhn-Tucker necessary conditions of optimality [1] for cone domains, we have a ζ_0 in R such that

$$G(\bar{z}) + \bar{t}\big(p+g(0)\big) + \zeta_0 p \in S^* , \quad \bar{t} + \zeta_0 \geq 0 ,$$

(1.2) $$\langle G(\bar{z})+\bar{t}\big(p+g(0)\big)+\zeta_0 p, \bar{z}\rangle = 0 , \quad \bar{t}\big(\bar{t}+\zeta_0\big) = 0 ,$$

$$\bar{z} \in S , \quad \bar{t} \geq 0 , \quad \langle p, \bar{z}\rangle + \bar{t} = 1 .$$

We claim that $\bar{t} > 0$. If not so, then suppose that $\bar{t} = 0$. This with (1.2) implies that $\bar{z} \neq 0$, and consequently, we have the system

$$G(\bar{z}) + \zeta_0 p \in S^* , \quad \zeta_0 \geq 0 ,$$

$$\langle G(\bar{z})+\zeta_0 p, \bar{z}\rangle = 0 ,$$

consistent for $0 \neq \bar{z} \in S$. Since g is a $J(S)$-function, ζ_0 can not be equal to zero. Also, $\zeta_0 \nmid 0$ since, in that case, taking the positively homogeneous property of G into consideration we can have a vector $0 \neq \bar{y} = \bar{z}/\big(\zeta_0\big)^{1/d} \in S$ satisfying the system *(A)*. Hence, $\bar{t} > 0$, and consequently, we have $\bar{t} + \zeta_0 = 0$. Now substituting $\zeta_0 = -\bar{t}$ in (1.2), it can be easily shown that $z^0 = \bar{z}/(\bar{t})^{1/d}$ is the desired solution.

REMARK 3.6. If we take $S = R^n_+$ and $p = e\alpha$, $\alpha > 0$, with

$e^T = (1, 1, \ldots, 1)$, and assume that $G(z) = g(z) - g(0)$ is regular, then Theorem 3.5 reduces to Theorem 3.1 in [4]. If we take $g(z) = M(z) + q$ where $M : S \to R^n$ is a nonlinear map with $M(0) = 0$, and q is a vector in R^n , then Theorem 3.5 yields Theorem 3.1 in [5]. Theorem 4.2 of Parida and Sahoo [7] follows as a special case of Theorem 3.5.

REMARK 3.7. We consider it interesting to provide the following example. Let $g : R^3 \to R^3$ be an affine map defined by $g(x) = [x_1 - x_2, x_1 + x_2, x_3 - 1]^T$. The problem is to find a solution to the system

$$y = g(x) \ , \quad x_1^2 + x_2^2 - d^2 x_3^2 \le 0 \ , \quad x_3 \ge 0 \ ,$$

$$y_1^2 + y_2^2 - y_3^2/d^2 \le 0 \ , \quad y_3 \ge 0 \ , \quad d \ne 0 \ ,$$

$$x^T y = 0 \ .$$

It can be cast in the form (1.1) as follows: find $x \in R^n$ such that

$$x \in S \ , \quad g(x) \in S^* \ , \quad \langle g(x), x \rangle = 0 \ ,$$

where

$$S = \left\{ x \in R^3 : x^T B x \le 0, \ x_3 \ge 0 \right\} ,$$

$$S^* = \left\{ y \in R^3 : y^T B^{-1} y \le 0, \ y_3 \ge 0 \right\} ,$$

and

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -d^2 \end{bmatrix} , \quad d \ne 0 \ .$$

It can be shown that g , as given above, with $p = [0, 0, 1]^T$ satisfies the conditions of Theorem 3.5. So there exists a solution to the problem. Indeed, we find that $x_1 = 0$, $x_2 = 0$, $x_3 = 1$ is a solution.

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