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THE HAUSDORFF MOMENT PROBLEM

BY
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1. **Introduction.** Suppose throughout that

$$0 \le \lambda_0 < \cdots < \lambda_n, \qquad \lambda_n \to \infty, \qquad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty,$$

and that $\{\mu_n\}$ $(n \ge 0)$ is a sequence of real numbers. The (generalized) Hausdorff moment problem is to determine necessary and sufficient conditions for there to be a function x in some specified class satisfying

$$\mu_n = \int_0^1 t^{\lambda_n}\, dx(t) \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

Let

$$D_0 = d_0 = 1, \qquad D_n = d_0 + d_1 + \cdots + d_n = \left(1 + \frac{1}{\lambda_1}\right) \cdots \left(1 + \frac{1}{\lambda_n}\right).$$

Define the divided difference $[\mu_k, \ldots, \mu_n]$ inductively by $[\mu_k] = \mu_k$,

$$[\mu_k, \ldots, \mu_n] = \frac{[\mu_k, \ldots, \mu_{n-1}] - [\mu_{k+1}, \ldots, \mu_n]}{\lambda_n - \lambda_k} \quad \text{for} \quad 0 \le k < n.$$

For $0 \le k \le n$, $0 \le t \le 1$, let

$$\lambda_{nk} = \lambda_{k+1} \cdots \lambda_n [\mu_k, \ldots, \mu_n],$$
$$\lambda_{nk}(t) = \lambda_{k+1} \cdots \lambda_n [t^{\lambda_k}, \ldots, t^{\lambda_n}]$$

with the convention that products such as $\lambda_{k+1} \cdots \lambda_n = 1$ when $k = n$. Let

$$M_{pn} = \begin{cases} \left(\sum_{k=0}^{n} |\lambda_{nk}|^p \left(\frac{D_n}{d_k}\right)^{p-1} \right)^{1/p} & \text{if} \quad 1 \le p < \infty, \\ \max_{0 \le k \le n} |\lambda_{nk}| \dfrac{D_n}{d_k} & \text{if} \quad p = \infty; \end{cases}$$

$$M_p = \sup_{n \ge 0} M_{pn}.$$

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Let C be the normed linear space of functons x continuous on $[0, 1]$ with norm $\|x\|_C = \sup_{0\le t\le 1} |x(t)|$. Let BV be the space of functions of bounded variation on $[0, 1]$. A function $x \in BV$ is said to be normalized if $x(0)=0$ and $2x(t)=x(t+)+x(t-)$ for $0<t<1$. For $p\ge 1$, let L_p be the normed linear space of measurable functions x on $(0, 1)$ with finite norm $\|x\|_p$ where

$$\|x\|_p = \begin{cases} \left(\int_0^1 |x(t)|^p\,dt\right)^{1/p} & \text{when} \quad 1\le p<\infty, \\ \operatorname*{ess.\,sup}_{0<t<1} & \text{when} \quad p=\infty. \end{cases}$$

It is known that $M_1<\infty$ if and only if there is a functon $\alpha \in BV$ satisfying

$$\text{(1)} \qquad \mu_n = \int_0^1 t^{\lambda_n}\,d\alpha(t) \quad \text{for} \quad n=0, 1, 2, \ldots.$$

The case $\lambda_0=0$ of this result was established by Hausdorff [5], [6] and Schoenberg [12] subsequently gave a different proof. The case $\lambda_0>0$ was proved by Leviatan [9] (see also Endl [4]).

It can be deduced from theorems of Leviatan [9, Theorem 2.3; 10, Theorem 1 and Theorem 2] (see also Berman [3]) and identity (5) (below) that, for $1<p\le\infty$, $M_p<\infty$ if and only if there is a function $\beta \in L_p$ satisfying

$$\text{(2)} \qquad \mu_n = \int_0^1 t^{\lambda_n}\beta(t)\,dt \quad \text{for} \quad n=0, 1, 2, \ldots.$$

The case $\lambda_n=n$ for $n=0, 1, 2, \ldots$ of this result is due to Hausdorff [7]. In this case we have that for $0\le k\le n$, $0\le t\le 1$,

$$\lambda_{nk}(t) = \binom{n}{k} t^k(1-t)^{n-k}, \qquad \lambda_{nk} = \binom{n}{k}\Delta^{n-k}\mu_k$$

where $\Delta^0\mu_k=\mu_k$, $\Delta^n\mu_k=\Delta^{n-1}\mu_k-\Delta^{n-1}\mu_{k+1}$.

In this paper we give new and reasonably self-contained proofs of the above results. Our proofs involve functional analysis and differ radically from those of the above-mentioned authors. Unlike previous proofs, ours do not treat the cases $\lambda_0=0$ and $\lambda_0>0$ separately.

In addition, we show that if (1) holds with α normalized, then $M_1=\int_0^1 |d\alpha(t)|$ when $\lambda_0=0$, and $M_1=\int_0^1 |d\alpha(t)|-|\alpha(0+)|$ when $\lambda_0>0$. We also show that if (2) holds for $1<p\le\infty$, then $M_p=\|\beta\|_p$. Finally, we show that M_{pn} increases with n and hence that $M_p=\lim_{n\to\infty} M_{pn}$ for $1\le p\le\infty$. The cases $\lambda_n=n$ for $n=0, 1, 2, \ldots$ of these results are derived in a book by Shohat and Tamarkin [13, pp. 97–101]. This book, incidentally, gives an excellent and extensive review of the classical moment problem. Another good reference book on the subject is one by Akhiezer [1].

2. **Preliminary results.** The following simple identities and inequalities are known:

(3) $$\mu_s=\sum_{k=0}^{n}\lambda_{nk}\left(1-\frac{\lambda_s}{\lambda_{k+1}}\right)\cdots\left(1-\frac{\lambda_s}{\lambda_n}\right)\quad\text{for}\quad 0\le s\le n.$$ [6, (5)]

(4) $$0\le\lambda_{ns}(t)\le\sum_{k=0}^{n}\lambda_{nk}(t)\le 1\quad\text{for}\quad 0\le t\le 1,\qquad 0\le s\le n.$$ [10, Lemma 1]

(5) $$\int_0^1\lambda_{nk}(t)\,dt=\frac{d_k}{D_n}\quad\text{for}\quad 0\le k\le n.$$ [6, p. 294]

We require some lemmas.

Lemma 1. *If $M_1<\infty$, then*

$$\mu_s=\lim_{n\to\infty}\sum_{k=0}^{n}\lambda_{nk}\left(\frac{D_k}{D_n}\right)^{\lambda_s}\quad\textit{for}\quad s=0,1,2,\ldots.$$

Proof. Let $\lambda>0$, $u_n=e^{-\lambda/\lambda_n}$,

$$\phi_n(\lambda)=\sum_{k=0}^{n}\lambda_{nk}u_{k+1}\cdots u_n,$$

and let

$$\psi_n=\sum_{k=0}^{n}\lambda_{nk}v_{k+1}\cdots v_n$$

where $v_n=e^{-\gamma_n/\lambda_n}$ for sufficiently large n and $\gamma_n\to\lambda$ as $n\to\infty$.

Let $0<\varepsilon<\lambda$. Then, for $\delta>0$, $|\gamma-\lambda|<\varepsilon$, we have that

$$|e^{-\delta\lambda}-e^{-\delta\gamma}|\le\delta\,|\gamma-\lambda|\,e^{-\delta(\lambda-\varepsilon)}\le\frac{\varepsilon}{\lambda-\varepsilon}.$$

Choose a positive integer N so large that $|\gamma_n-\lambda|<\varepsilon$ for $n>N$. Then, for $n>N$, we have that

$$|\psi_n-\phi_n(\lambda)|\le M_1\sum_{k=0}^{N-1}|v_{k+1}\cdots v_n-u_{k+1}\cdots u_n|+\frac{\varepsilon}{\lambda-\varepsilon}\sum_{k=N}^{n}|\lambda_{nk}|.$$

Since $u_n\to 0$ and $v_n\to 0$ as $n\to\infty$, it follows that

$$\limsup_{n\to\infty}|\psi_n-\phi_n(\lambda)|\le\frac{M_1\varepsilon}{\lambda-\varepsilon},$$

and hence that

$$\lim_{n\to\infty}(\psi_n-\phi_n(\lambda))=0.$$

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Note that when $v_n = 1 - \lambda_s/\lambda_n$, then, by (3), the corresponding $\psi_n = \mu_s$ for $n \geq s$. Thus

$$\lim_{n\to\infty} \phi_n(\lambda_s) = \mu_s.$$

The desired conclusion is now obtained by considering the ψ_n corresponding to

$$v_n = \left(1+\frac{1}{\lambda_n}\right)^{-\lambda_s}.$$

LEMMA 2.

(i) *If* (1) *is satisfied by a function* $\alpha \in BV$, *then* $M_1 \leq \int_0^1 |d\alpha(t)|$.

(ii) *If* $1 < p \leq \infty$ *and* (2) *is satisfied by a function* $\beta \in L_p$, *then* $M_p \leq \|\beta\|_p$.

Proof. Part (i). We have that

$$\lambda_{nk} = \int_0^1 \lambda_{nk}(t)\, d\alpha(t) \quad \text{for} \quad 0 \leq k \leq n,$$

and thus, by (4), that

$$\sum_{k=0}^{n} |\lambda_{nk}| \leq \int_0^1 |d\alpha(t)| \sum_{k=0}^{n} \lambda_{nk}(t) \leq \int_0^1 |d\alpha(t)|.$$

Hence

$$M_1 \leq \int_0^1 |d\alpha(t)|.$$

Part (ii). We now have that

$$\lambda_{nk} = \int_0^1 \lambda_{nk}(t)\beta(t)\, dt \quad \text{for} \quad 0 \leq k \leq n.$$

Hence, by (5),

$$|\lambda_{nk}| \leq \int_0^1 \lambda_{nk}(t)\, |\beta(t)|\, dt \leq \frac{d_k}{D_n} \operatorname*{ess.\,sup}_{0<t<1} |\beta(t)|.$$

Next, if $1 < p < \infty$, then, by Hölder's inequality and (5),

$$\begin{aligned}|\lambda_{nk}|^p &\leq \int_0^1 \lambda_{nk}(t)\, |\beta(t)|^p\, dt \left(\int_0^1 \lambda_{nk}(t)\, dt\right)^{p-1} \\ &= \left(\frac{d_k}{D_n}\right)^{p-1} \int_0^1 \lambda_{nk}(t)\, |\beta(t)|^p\, dt;\end{aligned}$$

and so, by (4),

$$\sum_{k=0}^{n} |\lambda_{nk}|^p \left(\frac{D_n}{d_k}\right)^{p-1} \leq \int_0^1 |\beta(t)|^p\, dt \sum_{k=0}^{n} \lambda_{nk}(t) \leq \int_0^1 |\beta(t)|^p\, dt.$$

Consequently, if $1 < p \leq \infty$, then $M_p \leq \|\beta\|_p$.

LEMMA 3. *If a normalized function $x \in BV$ is such that*

$$\int_0^1 t^{\lambda_n}\, dx(t) = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

then $x(t) = x(0+)$ for $0 < t \le 1$. If, in addition, $\lambda_0 = 0$, then $x(0+) = 0$.

Proof. Suppose first that $\lambda_0 = 0$. A known consequence of the hypothesis [11, p. 337] is that

$$\int_0^1 t^n\, dx(t) = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

Hence, by a standard result [14, Theorem 6.1], $x(t) = 0$ for $0 \le t \le 1$.

Suppose next that $\lambda_0 > 0$. Then, by hypothesis,

$$\int_0^1 t^{\lambda_n - \lambda_0}\, dy(t) = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

where $y(t) = \int_0^t u^{\lambda_0}\, dx(u)$. Since y is normalized [14, Theorem 8b], we have, by the part already proved, that $y(t) = 0$ for $0 \le t \le 1$. Let $0 < \varepsilon \le t \le 1$. Then

$$0 = \int_\varepsilon^t u^{\lambda_0}\, dx(u) = t^{\lambda_0} x(t) - \varepsilon^{\lambda_0} x(\varepsilon) = \int_\varepsilon^t u^{\lambda_0 - 1} x(u)\, du$$

and so x is absolutely continuous in $[\varepsilon, 1]$. Therefore $0 = \int_\varepsilon^t u^{\lambda_0} x'(u)\, du$ and consequently $x'(u) = 0$ a.e. in $(\varepsilon, 1)$. It follows that $x(t) = x(\varepsilon)$ for $0 < \varepsilon \le t \le 1$, and hence that $x(t) = x(0+)$ for $0 < t \le 1$.

This completes the proof of Lemma 3.

3. **The main results.** The proofs of both parts of the following theorem are based on proofs in Shohat and Tamarkin's book [13, pp. 99–101] of the case $\lambda_n = n$ for $n = 0, 1, 2, \ldots$. Hildebrandt [8] originally proved this case of part (i) by a similar method.

THEOREM 1.
(i) *If $M_1 < \infty$, then there is a normalized function $\alpha \in BV$ such that* (1) *is satisfied and $\int_0^1 |d\alpha(t)| \le M_1$.*

(ii) *If $1 < p \le \infty$ and $M_p < \infty$, then there is a function $\beta \in L_p$ such that* (2) *is satisfied and $\|\beta\|_p \le M_p$.*

Proof. Define Λ to be the linear space of functions P such that

$$P(t) = \sum_{k=0}^{m} a_k t^{\lambda_k} \quad \text{for} \quad 0 \le t \le 1, \tag{6}$$

where m is an arbitrary non-negative integer and $a_0, a_1, \ldots, a_m$ are real constants. Define the *moment operator* μ on Λ by setting

$$\mu(P)=\sum_{k=0}^{m} a_k\mu_k$$

when P is given by (6).

Suppose that $M_p<\infty$ where $1\le p\le\infty$. Let $P\in\Lambda$ and let $B_n\in\Lambda$ be given by

$$B_n(t)=\sum_{k=0}^{n} \lambda_{nk}(t)P\left(\frac{D_k}{D_n}\right) \quad \text{for} \quad o\le t\le 1.$$

Then

$$\mu(B_n)=\sum_{k=0}^{n} \lambda_{nk}P\left(\frac{D_k}{D_n}\right), \tag{7}$$

and hence, by Lemma 1,

$$\lim_{n\to\infty} \mu(B_n)=\mu(P), \tag{8}$$

since, by Hölder's inequality, $M_1\le M_p$.

Part (i). It follows from (7) that

$$|\mu(B_n)|\le M_1\|P\|_C$$

and hence, by (8), that

$$|\mu(P)|\le M_1\|P\|_C.$$

Thus μ is a bounded linear functional on a linear subspace of C. Hence, by the Hahn–Banach theorem [11, Theorem 5.16] and the Riesz representation theorem for bounded linear functionals on C [2, p. 61], there is a normalized function $\alpha\in BV$ such that, for every $P\in\Lambda$,

$$\mu(P)=\int_0^1 P(t)\,d\alpha(t) \quad \text{and} \quad \int_0^1 |d\alpha(t)|\le M_1.$$

In particular, taking $P(t)=t^{\lambda_n}$, we get that

$$\mu_n=\int_0^1 t^{\lambda_n}\,d\alpha(t) \quad \text{for} \quad n=0,1,2,\ldots.$$

Part (ii). Let $(1/p)+(1/q)=1$ where $1<p\le\infty$. Applying Hölder's inequality to (7) we get that

$$|\mu(B_n)|\le M_p\left(\sum_{k=0}^{n}\frac{d_k}{D_n}\left|P\left(\frac{D_k}{D_n}\right)\right|^q\right)^{1/q}.$$

Since

$$\max_{0\le k\le n}\frac{d_k}{D_n}=\max_{0\le k\le n}\frac{D_k}{D_n}\frac{1}{1+\lambda_k}\to 0 \quad \text{as} \quad n\to\infty,$$

the term multiplying M_p in the inequality tends to $\int_0^1(|P(t)|^q\,dt)^{1/q}$. In view of (8), it follows that

$$|\mu(P)| \le M_p\,\|P\|_q.$$

Thus μ is a bounded linear functional on a linear subspace of L_q. Hence, by the Hahn–Banach theorem and the Riesz representation theorem for bounded linear functionals on L_q [2, pp. 64, 65], there is a function $\beta \in L_p$ such that, for every $P \in \Lambda$,

$$\mu(P) = \int_0^1 P(t)\beta(t)\,dt \quad \text{and} \quad \|\beta\|_p \le M_p.$$

In particular, taking $P(t) = t^{\lambda_n}$, we get that

$$\mu_n = \int_0^1 t^{\lambda_n}\beta(t)\,dt \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

This completes the proof of Theorem 1.

Combining Lemma 2 and Theorem 1 we obtain:

THEOREM 2.

(i) $M_1 < \infty$ *if and only if* (1) *is satisfied by a function* $\alpha \in BV$.

(ii) *For* $1 < p \le \infty$, $M_p < \infty$ *if and only if* (2) *is satisfied by a function* $\beta \in L_p$.

The next two theorems give more precise information about M_p.

THEOREM 3.

(i) *If* (1) *is satisfied by a normalized function* $\alpha \in BV$, *then*

$$\text{(a)} \quad M_1 = \int_0^1 |d\alpha(t)| \quad \textit{when} \quad \lambda_0 = 0,$$

$$\text{(b)} \quad M_1 = \int_0^1 |d\alpha(t)| - |\alpha(0+)| \quad \textit{when} \quad \lambda_0 > 0.$$

(ii) *If* $1 < p \le \infty$ *and* (2) *is satisfied by a function* $\beta \in L_p$, *then* $M_p = \|\beta\|_p$.

Proof. Part (i). By Lemma 2(i), we have that $M_1 \le \int_0^1 |d\alpha(t)| < \infty$. Hence by Theorem 1(i), there is a normalized function $\tilde{\alpha} \in BV$ such that $\mu_n = \int_0^1 t^{\lambda_n}\,d\tilde{\alpha}(t)$ for $n = 0, 1, 2, \ldots$ and $\int_0^1 |d\tilde{\alpha}(t)| \le M_1$.

If $\lambda_0 = 0$, then, by Lemma 3, $\tilde{\alpha}(t) = \alpha(t)$ for $0 \le t \le 1$, and hence $M_1 = \int_0^1 |d\alpha(t)|$.

Suppose that $\lambda_0 > 0$, and let $\gamma(0) = 0$, $\gamma(t) = \alpha(t) - \alpha(0+)$ for $0 < t \le 1$. Then $\mu_n = \int_0^1 t^{\lambda_n}\,d\gamma(t)$ for $n = 0, 1, 2, \ldots$ and hence, by Lemma 2(i), $M_1 \le \int_0^1 |d\gamma(t)|$. Further, by Lemma 3, $\gamma(t) = \tilde{\alpha}(t) - \tilde{\alpha}(0+)$ for $0 < t \le 1$, and so, since $\gamma(0+) = \gamma(0) = 0$, we have that $M_1 \le \int_0^1 |d\gamma(t)| \le \int_0^1 |d\tilde{\alpha}(t)| \le M_1$. Hence $M_1 = \int_0^1 |d\gamma(t)| = \int_0^1 |d\alpha(t)| - |\alpha(0+)|$.

Part (ii). By Lemma 2(ii), we have that $M_p \leq \|\beta\|_p < \infty$. Hence, by Theorem 1(ii), there is a function $\tilde{\beta} \in L_p$ such that $\mu_n = \int_0^1 t^{\lambda_n}\tilde{\beta}(t)\,dt$ for $n = 0, 1, 2, \ldots$ and $\|\tilde{\beta}\|_p \leq M_p$. By Lemma 3, $\int_0^t \beta(u)\,du = \int_0^t \tilde{\beta}(u)\,du$ for $0 \leq t \leq 1$, and hence $\beta(t) = \tilde{\beta}(t)$ a.e. in (0, 1). It follows that $M_p \leq \|\beta\|_p = \|\tilde{\beta}\|_p \leq M_p$, so that $M_p = \|\beta\|_p$. This completes the proof of Theorem 3.

THEOREM 4. *If* $1 \leq p \leq \infty$, *then* $M_{pn} \leq M_{p,n+1}$ *for* $n \geq 0$ *and* $\lim_{n\to\infty} M_{pn} = M_p$.

Proof. Let $0 \leq k \leq n$. Then

$$\lambda_{n+1,k} = \lambda_{k+1} \cdots \lambda_{n+1} \frac{[\mu_k, \ldots, \mu_n] - [\mu_{k+1}, \ldots, \mu_{n+1}]}{\lambda_{n+1} - \lambda_k}$$

$$= \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_k}\lambda_{nk} - \frac{\lambda_{k+1}}{\lambda_{n+1} - \lambda_k}\lambda_{n+1,k+1}$$

and hence

$$\lambda_{nk} = \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)\lambda_{n+1,k} + \frac{\lambda_{k+1}}{\lambda_{n+1}}\lambda_{n+1,k+1}. \tag{8}$$

It follows that

$$\frac{\lambda_{nk}}{d_k} = \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)\frac{\lambda_{n+1,k}}{d_k} + \left(\frac{1}{\lambda_{n+1}} + \frac{\lambda_k}{\lambda_{n+1}}\right)\frac{\lambda_{n+1,k+1}}{d_{k+1}}$$

and hence that

$$M_{\infty,n} \leq M_{\infty,n+1}\left(1 + \frac{1}{\lambda_{n+1}}\right)\frac{D_n}{D_{n+1}} = M_{\infty,n+1}.$$

Finally, for $1 \leq p < \infty$, application of Hölder's inequality to (8) yields that

$$D_n^{p-1}\,|\lambda_{nk}|^p\, d_k^{1-p} \leq \left\{\left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)|\lambda_{n+1,k}|^p\, d_k^{1-p} + \frac{\lambda_{k+1}}{\lambda_{n+1}}|\lambda_{n+1,k+1}|^p\, d_{k+1}^{1-p}\right\} D_{n+1}^{p-1},$$

since

$$1 - \frac{\lambda_k}{\lambda_{n+1}} + \frac{\lambda_{k+1}}{\lambda_{n+1}}\frac{d_{k+1}}{d_k} = 1 + \frac{1}{\lambda_{n+1}} = \frac{D_{n+1}}{D_n}.$$

Summing the above inequality for $k = 0, 1, \ldots, n$, we get that

$$M_{pn}^p \leq M_{p,n+1}^p - \frac{\lambda_0}{\lambda_{n+1}}|\lambda_{n+1,0}|^p\, d_0^{1-p} D_{n+1}^{p-1} \leq M_{p,n+1}^p.$$

This completes the proof of Theorem 4.

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