

STABILITY OF CONTACT METRIC MANIFOLDS AND
UNIT VECTOR FIELDS OF MINIMUM ENERGY

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In this paper we obtain criteria of stability for η -Einstein K -contact manifolds, for Sasakian manifolds of constant φ -sectional curvature and for 3-dimensional Sasakian manifolds. Moreover, we show that a stable compact Einstein contact metric manifold M is Sasakian if and only if the Reeb vector field ξ minimises the energy functional. In particular, the Reeb vector field of a Sasakian manifold M of constant φ -holomorphic sectional curvature $+1$ minimises the energy functional if and only if M is not simply connected.

1. INTRODUCTION

Let (M, g) be a compact Riemannian manifold. The identity map $id_M : (M, g) \rightarrow (M, g)$ is a harmonic map; that is, a critical point of the energy functional. Following Nagano [15], the Riemannian manifold M is said to be *stable* if the identity map is stable, that is, the second variation of the energy functional at id_M is non-negative, otherwise M is said to be *unstable*. Smith [20] proved that: *if (M, g) is a compact Einstein manifold of dimension n , then M is stable if and only if $\lambda_1 \geq (2r/n)$, where λ_1 is the first eigenvalue of the Laplace-Beltrami operator acting on functions and r is the scalar curvature. Others results on the stability of M in terms of λ_1 can be found, for example, in [10, 21, 23, 24]. Moreover, in [20], it was proved that every holomorphic map between Kaehler manifolds is stable, in particular a compact Kaehler manifold is stable (see also [15]). Even if many concepts of Kaehlerian geometry have their correspondences in Sasakian geometry, in the theory of the harmonic maps in Sasakian geometry, and more in general in contact metric geometry, we still have few results ([4, 10, 14]). From the instability theorem of Xin (see, for example, [25, p.162]) follows that the unit sphere S^n , $n = 2m + 1 \geq 3$, which is a classical example of Sasakian manifold, is unstable. Concerning the study of the stability of a contact metric manifold, we know only the following result ([10, Theorem 3.1]): *if M is a compact Sasakian $(2m + 1)$ -manifold of constant φ -sectional curvature $c \leq 1$ and λ_1 satisfies $\lambda_1 < c(m + 1) + 3m - 1$, then M is unstable. In [4] the authors found that:**

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a compact cosymplectic manifold (which is a particular almost contact metric manifold) is stable.

On the other hand, in the last ten years, the theory of the harmonic unit vector fields, namely the unit vector fields $U : (M, g) \rightarrow (T^1M, g_s)$, g_s being the Sasaki metric, which are critical points of the energy functional E defined on the set $\mathfrak{X}^1(M)$ of all unit vector fields, has been largely developed by many authors (see, for example, [26, 29, 11, 16, 17] and the references in [12]). Brito [8] proved that the Hopf vector fields on the unit sphere S^3 are the unique absolute minimisers of the energy functional $E : \mathfrak{X}^1(S^3) \rightarrow \mathbb{R}$. Wood [29] (see also [11]) proved that, for the unit sphere S^{2m+1} with $m > 1$, the Hopf vector fields are unstable. We note that a Hopf vector field on S^{2m+1} is precisely the Reeb vector field ξ of a natural Sasakian structure on S^{2m+1} . In general, in dimension greater than three, to our knowledge, there are no examples of unit vector fields which realise an absolute minimum for the energy and neither criteria of existence of minima for the energy (see [12, 18]).

In this paper we investigate the stability of a compact contact metric manifold (M, g, η, ξ) in terms of the first eigenvalue μ_1 of the Laplacian acting on 1-forms and in terms of stability of the Reeb vector field ξ . As a consequence of more general results, we obtain criteria of stability for η -Einstein K -contact manifolds (Theorem 3.1), for Sasakian manifolds of constant φ -sectional curvature c (Theorem 3.2) and for 3-dimensional Sasakian manifolds (Theorem 3.3). Finally, in Section 4 we show that a stable compact Einstein contact metric manifold M is a Sasakian manifold if and only if the Reeb vector field ξ minimises the energy functional $E : \mathfrak{X}^1(M) \rightarrow \mathbb{R}$. In particular, we get that the Reeb vector field of a compact Sasakian manifold M of constant φ -holomorphic sectional curvature $+1$ minimises the energy functional if and only if M is not simply connected (Theorem 4.2).

2. THE SECOND VARIATION FORMULA

Let (M, g) and (N, h) be two Riemannian manifolds with M compact and n -dimensional. A smooth map $f : (M, g) \rightarrow (N, h)$ is said to be harmonic if it is a critical point of the energy functional

$$E : C^\infty(M, N) \rightarrow \mathbb{R}, f \mapsto E(f) = \int_M e(f) \, dv,$$

where $e(f) = \|df\|^2/2$ is the energy density and $\|df\|$ is the norm of the differential of f with respect to the metrics g and h . More precisely, f is a harmonic map if $(dE(f_t)/dt)|_{t=0} = 0$ for every smooth variation f_t of f . If f is a harmonic map, the Hessian form of the energy E at f is defined by the second variation formula [20]:

$$(2.1) \quad (\text{Hess}E)_f(V, V) = \frac{d^2}{dt^2} E(f_t)|_{t=0} = \int_M h(V, J_f V) v_g,$$

where V is a vector field along f . The operator J_f , called the *Jacobi operator* of f , is a second order selfadjoint elliptic differential operator acting on the space $\Gamma(f^{-1}TN)$ of the vector fields along f , and is defined by

$$(2.2) \quad J_f V = \bar{\Delta}_f V - \text{Ric}_f V.$$

The operator $\bar{\Delta}_f$, called the *rough Laplacian* along f , is defined by

$$\bar{\Delta}_f V = - \sum_{i=1}^n (\bar{\nabla}_{e_i} \bar{\nabla}_{e_i} V - \bar{\nabla}_{\nabla_{e_i} e_i} V), \quad V \in \Gamma(f^{-1}TN),$$

where $\bar{\nabla}$ is the connection (on the vector bundle $f^{-1}TN$) induced by the Levi-Civita connection of (N, h) , and $\{e_i\}_{i=1, \dots, n}$ is a local orthonormal frame on M . Moreover, denoting by R_h the curvature tensor of (N, h) ,

$$(2.3) \quad \text{Ric}_f V = \sum_{i=1}^n R_h(f_* e_i, V) f_* e_i.$$

A harmonic map f is said to be *stable* if $(\text{Hess}E)_f$ is positive semi-definite, and *unstable* if there exists a vector field V along f such that $(\text{Hess}E)_f(V, V) < 0$. The identity map $id_M : (M, g) \rightarrow (M, g)$ is a trivial example of a harmonic map, but the theory of the second variation is much more complicated. From (2.1)–(2.3) we readily deduce that the second variation formula of the energy for id_M is given by

$$(2.4) \quad (\text{Hess}E)_{id}(X, X) = \int_M g(J_{id} X, X) v_g = \int_M g(\bar{\Delta} X - QX, X) v_g \quad \forall X \in \mathfrak{X}(M),$$

where

$$\bar{\Delta} X = \bar{\Delta}_{id} X = - \sum_{i=1}^n (\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X),$$

Q is the Ricci operator and $J_{id} = \bar{\Delta} - Q$ is the Jacobi operator of the identity map. The graph of the identity map is the diagonal of $M \times M$, which is a minimal submanifold. Then, Nagano [15] proved that M is stable, (that is, the identity map id_M is stable), if and only if the diagonal in $M \times M$ is stable as minimal submanifold. In particular a compact Kaehler manifold is stable (see also [20]).

3. STABILITY OF η -EINSTEIN CONTACT MANIFOLDS

A $(2m + 1)$ -dimensional manifold M , $m \geq 1$, is said to be a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^m \neq 0$. Given η , there exists a unique vector field ξ , called the *Reeb vector field* (or the *characteristic vector field*) such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. Furthermore, a Riemannian metric g is said to be an *associated metric* if there exists a tensor φ of type $(1,1)$ such that

$$(3.1) \quad \eta = g(\xi, \cdot), \quad d\eta = g(\cdot, \varphi \cdot), \quad \varphi^2 = -I + \eta \otimes \xi.$$

(η, g, ξ, φ) , or (η, g) , is called a *contact metric structure*, or *contact Riemannian structure* and (M, η, g) a *contact metric (or Riemannian) manifold*. Let (M, η, g) be a contact metric manifold of dimension $2m+1$, $m \geq 1$. We denote by ∇ the Levi-Civita connection, by ρ the Ricci tensor, by Q the corresponding Ricci operator and by r the scalar curvature. The tensor $h = \mathcal{L}_\xi \varphi / 2$, where \mathcal{L} denotes the Lie derivative, is symmetric and plays a fundamental role in contact metric geometry [2]. It satisfies the equation ([2, p.67])

$$(3.2) \quad \nabla \xi = -\varphi - \varphi h.$$

The scalar torsion $\|\tau\|$, $\tau = \mathcal{L}_\xi g$, introduced in [9], and the tensor h are related by $\tau = 2g(h\varphi, \cdot)$. M is said to be *K-contact* if ξ is a Killing vector field, that is, $\tau = 0$, or equivalently $h = 0$. Moreover M is said to be a *Sasakian manifold* if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \quad \forall X, Y \in \mathfrak{X}(M).$$

Any Sasakian manifold is K-contact and the converse also holds when the manifold is of dimension three. For further information about contact metric geometry we refer to [2].

From now on we denote by μ_1 the first eigenvalue of the Laplacian Δ_1 acting on 1-forms. We recall that Δ_1 also acts on vector fields via duality and it is related to the rough Laplacian $\bar{\Delta}$ and the Ricci operator Q by the well-known Wietzenböck's formula (see, for example, [1, p. 57])

$$(3.3) \quad \Delta_1 = \bar{\Delta} + Q.$$

The 1-form η is an eigenform of the Laplacian Δ_1 corresponding to the eigenvalue $4m$. In fact, in [17] the first author proved that

$$(3.4) \quad \bar{\Delta} \xi = 4m\xi - Q\xi,$$

hence, by (3.3),

$$(3.5) \quad \Delta_1 \xi = 4m\xi$$

and so

$$\Delta_1 \eta = \Delta_1 g(\xi, \cdot) = g(\Delta_1 \xi, \cdot) = 4mg(\xi, \cdot) = 4m\eta,$$

by which we deduce that

$$(3.6) \quad \mu_1 \leq 4m.$$

Now, we suppose that M is a η -Einstein manifold, that is, the Ricci tensor is of the form

$$(3.7) \quad \rho = ag + b\eta \otimes \eta \quad (\text{or equivalently } Q = aI + b\eta \otimes \xi)$$

where $a, b \in C^\infty(M, \mathbb{R})$. It is known that for $m \geq 2$, on any η -Einstein K-contact manifold the functions a and b are constant ([2, p. 105]). (κ, μ) -spaces of dimension $2m + 1$, with

$\mu = 2(1 - m)$, are examples of non Sasakian η -Einstein manifolds ([3, p.206]). On a general contact metric manifold the Ricci curvature in the direction of ξ is given by ([2, p. 92])

$$(3.8) \quad \rho(\xi, \xi) = 2m - \text{tr}h^2, \quad \text{tr}h^2 = \frac{\|\tau\|^2}{4}.$$

From (3.7) we have $\rho(\xi, \xi) = a + b$ and hence

$$(3.9) \quad a + b = 2m - \frac{\|\tau\|^2}{4}.$$

Moreover, from (3.7), we have

$$(3.10) \quad r = \text{tr}\rho = (2m + 1)a + b.$$

Consequently, we get

$$(3.11) \quad a = \frac{r}{2m} - 1 + \frac{\|\tau\|^2}{8m} \quad \text{and} \quad b = -\frac{r}{2m} + (2m + 1)\left(1 - \frac{\|\tau\|^2}{8m}\right).$$

Assuming M compact, by (2.4) and (3.3), we get

$$(3.12) \quad \begin{aligned} (\text{Hess}E)_{id}(X, X) &= \int_M g(\Delta_1 X, X)v_g - 2 \int_M \rho(X, X)v_g \\ &= \int_M g(\Delta_1 X, X)v_g - 2 \int_M \left(\frac{r}{2m} - 1 + \frac{\|\tau\|^2}{8m}\right) \|X\|^2 v_g \\ &\quad + 2 \int_M \left[\frac{r}{2m} - (2m + 1)\left(1 - \frac{\|\tau\|^2}{8m}\right)\right] [\eta(X)]^2 v_g. \end{aligned}$$

Now, let ω be an eigenform of Δ_1 corresponding to the eigenvalue μ_1 : $\Delta_1 \omega = \mu_1 \omega$. Denote by X_0 the vector field dual of ω . If the scalar curvature r satisfies the inequality

$$r \leq 2m(2m + 1)\left(1 - \frac{\|\tau\|^2}{8m}\right), \quad \text{that is} \quad b \geq 0,$$

from (3.12) we have

$$(3.13) \quad (\text{Hess}E)_{id}(X_0, X_0) \leq \int_M \left[\mu_1 - \left(\frac{r}{m} - 2 + \frac{\|\tau\|^2}{4m}\right)\right] \|X_0\|^2 v_g.$$

Further, using (3.1), (3.7) and (3.11), we can write

$$(3.14) \quad QX_0 = \left(2m - \frac{\|\tau\|^2}{4}\right)X_0 + \left[-\frac{r}{2m} + (2m + 1)\left(1 - \frac{\|\tau\|^2}{8m}\right)\right]\varphi^2 X_0.$$

Since

$$g(\varphi^2 X_0, X_0) = -g(\varphi^2 X_0, \varphi^2 X_0) + g(\varphi^2 X_0, \eta(X_0)\xi) = -\|\varphi X_0\|^2,$$

if the scalar curvature satisfies the inequality

$$r \geq 2m(2m + 1)\left(1 - \frac{\|\tau\|^2}{8m}\right), \quad \text{that is } b \leq 0,$$

by (3.14), (3.12) becomes

$$\begin{aligned} (3.15) \quad (\text{Hess}E)_{id}(X_0, X_0) &= \int_M \left[\mu_1 - \left(4m - \frac{\|\tau\|^2}{2}\right) \right] \|X_0\|^2 v_g \\ &\quad + \int_M \left[-\frac{r}{m} + 2(2m + 1)\left(1 - \frac{\|\tau\|^2}{8m}\right) \right] \|\varphi X_0\|^2 v_g \\ &\leq \int_M \left[\mu_1 - \left(4m - \frac{\|\tau\|^2}{2}\right) \right] \|X_0\|^2 v_g. \end{aligned}$$

Therefore, by (3.6), (3.13) and (3.15), we obtain at once the following

PROPOSITION 3.1. *Let (M, η, g) be a compact contact metric manifold of dimension $2m + 1$. Then, the first eigenvalue μ_1 of the Laplacian acting on 1-forms satisfies the inequality $\mu_1 \leq 4m$. If, in addition, M is η -Einstein and μ_1 satisfies one of the following conditions:*

- (i) $\mu_1 < (r/m) - 2 + (\|\tau\|^2/4m) \leq 4m - (\|\tau\|^2/2),$
- (ii) $\mu_1 < 4m - (\|\tau\|^2/2) \leq (r/m) - 2 + (\|\tau\|^2/4m),$

then M is unstable.

In [20] Smith proved that if (M, g) is a compact Einstein manifold of dimension n , $\rho = (r/n)g$, then M is stable if and only if $\lambda_1 \geq (2r/n)$, where λ_1 is the first eigenvalue of the Laplace-Beltrami operator Δ acting on functions. The following Proposition gives a version of Smith's result in contact metric geometry.

PROPOSITION 3.2. *Let (M, g) be a compact Einstein manifold of dimension $(2m + 1)$, homothetic to a contact metric manifold $(M, \tilde{\eta}, \tilde{g})$, that is, $\tilde{g} = \alpha g$, $\alpha = \text{const} > 0$. Then, the first eigenvalue μ_1 of the Laplacian acting on 1-forms satisfies:*

- (i) $\mu_1 \leq 4m\alpha;$
- (ii) M is stable $\Leftrightarrow \mu_1 \in \left[4m\alpha\left(1 - (\|\tau\|^2/8m)\right), 4m\alpha \right]$, where $\|\tau\| = \|\mathcal{L}_\xi g\|$ is constant, $\xi = \alpha\tilde{\xi};$
- (iii) if ξ is Killing, M is stable $\Leftrightarrow \mu_1 = 4m\alpha;$
- (iv) if $r > 0$, $\mu_1 < 4m\alpha\left(1 - (\|\tau\|^2/8m)\right) \Leftrightarrow \lambda_1 \in \left[(2m + 1)\alpha\left(1 - (\|\tau\|^2/8m)\right), 4m\alpha\left(1 - (\|\tau\|^2/8m)\right) \right].$

PROOF: (i) We have already observed in (3.5) that the Reeb vector field $\tilde{\xi}$ is an eigenvector of the Laplacian $\tilde{\Delta}_1$ corresponding to the eigenvalue $4m$: $\tilde{\Delta}_1\tilde{\xi} = 4m\tilde{\xi}$. Setting $\xi = \alpha\tilde{\xi}$ we get

$$\Delta_1\xi = \alpha\tilde{\Delta}_1\xi = 4m\alpha^2\tilde{\xi} = 4m\alpha\xi,$$

and hence $\mu_1 \leq 4m\alpha$.

(ii) We first suppose that M is stable. Then, by (2.4) and (3.3) we have

$$(3.16) \quad 0 \leq (\text{Hess}E)_{id_M}(X, X) = \int_M g(\Delta_1 X - 2QX, X)v_g$$

for any $X \in \mathfrak{X}(M)$. Since g and \tilde{g} are homothetic, we have:

$$\tilde{\rho} = \rho(X, Y) = \frac{r}{2m+1}g = \frac{r}{(2m+1)\alpha}\tilde{g},$$

that is, $(M, \tilde{\eta}, \tilde{g})$ is an Einstein contact metric manifold. Thus, by (3.11), (M, \tilde{g}) has constant scalar curvature

$$\tilde{r} = 2m(2m+1)\left(1 - (\|\tau\|^2/8m)\right),$$

where $\tau = \mathcal{L}_\xi g = \mathcal{L}_{\tilde{\xi}}\tilde{g}$, $\xi = \alpha\tilde{\xi}$. Consequently

$$\|\tau\| \text{ is constant, } r = 2m(2m+1)\alpha\left(1 - \frac{\|\tau\|^2}{8m}\right) \text{ and } \rho = 2m\alpha\left(1 - \frac{\|\tau\|^2}{8m}\right)g.$$

Let X_0 be a vector field eigenvector of the Laplacian Δ_1 corresponding to the eigenvalue μ_1 : $\Delta_1 X_0 = \mu_1 X_0$. From (3.16), we obtain

$$\left[\mu_1 - 4m\alpha\left(1 - \frac{\|\tau\|^2}{8m}\right)\right] \int_M \|X_0\|^2 v_g \geq 0, \text{ that is, } \mu_1 \geq 4m\alpha\left(1 - \frac{\|\tau\|^2}{8m}\right)$$

and so, by (i),

$$\mu_1 \in \left[4m\alpha\left(1 - \frac{\|\tau\|^2}{8m}\right), 4m\alpha\right].$$

Vice versa, we assume

$$\mu_1 \in \left[4m\alpha\left(1 - \frac{\|\tau\|^2}{8m}\right), 4m\alpha\right].$$

Let $f \in C^\infty(M, \mathbb{R})$, $df \neq 0$, be an eigenfunction of the Laplace-Beltrami operator Δ corresponding to the eigenvalue λ_1 : $\Delta f = \lambda_1 f$. Then

$$\Delta_1(df) = d\Delta f = \lambda_1 df$$

and thus

$$\lambda_1 \geq \mu_1 \geq 4m\alpha\left(1 - \frac{\|\tau\|^2}{8m}\right)$$

where $2m\alpha\left(1 - (\|\tau\|^2/8m)\right)$ is the Einstein constant of (M, g) and so, by the above mentioned Smith's result, M is stable.

(iii) If ξ is a Killing vector field, then $\tau = 0$ and so, by means of (ii), we get that M is stable if and only if $\mu_1 = 4m\alpha$.

(iv) If (M, g) has positive scalar curvature, since (M, g) is Einstein with $\rho = 2m\alpha(1 - (\|\tau\|^2/8m))g$, by Lichnerowicz–Obata Theorem ([25, p. 182]) λ_1 satisfies the inequality

$$(3.17) \quad \lambda_1 \geq (2m + 1)\alpha\left(1 - \frac{\|\tau\|^2}{8m}\right).$$

If $\mu_1 < 4m\alpha(1 - (\|\tau\|^2/8m))$, by the result of (ii), we get that M is unstable and so (3.17) and Smith’s result gives

$$\lambda_1 \in \left[(2m + 1)\alpha\left(1 - \frac{\|\tau\|^2}{8m}\right), 4m\alpha\left(1 - \frac{\|\tau\|^2}{8m}\right) \right].$$

Of course, $\lambda_1 < 4m\alpha(1 - (\|\tau\|^2/8m))$ implies $\mu_1 < 4m\alpha(1 - (\|\tau\|^2/8m))$. □

Next, we consider a compact n -dimensional Riemannian manifold (M, g) and its unit tangent sphere bundle (T^1M, g_s) equipped with the Sasaki metric g_s . Every unit vector field $U \in \mathfrak{X}^1(M)$ defines a map $U : (M, g) \rightarrow (T^1M, g_s)$ and the energy of U is the energy of the corresponding map ([26, 29]):

$$(3.18) \quad E(U) = \frac{1}{2} \int_M \|\mathrm{d}U\|^2 v_g = \frac{n}{2} \mathrm{vol}(M, g) + \frac{1}{2} \int_M \|\nabla U\|^2 v_g.$$

U is called a *harmonic vector field* if it is critical for the energy functional $E : \mathfrak{X}^1(M) \rightarrow \mathbb{R}$. The corresponding critical point condition has been determined in [26, 29] and is given by:

$$\overline{\Delta}U \text{ is collinear to } U.$$

When U is a harmonic vector field, the Hessian form at U (of the second variation formula) takes the form [26, 11]

$$(\mathrm{Hess}E)_U(W, W) = \int_M (\|\nabla W\|^2 - \|W\|^2 \|\nabla U\|^2) v_g$$

where $W \in U^\perp$. U is called *stable* if $(\mathrm{Hess}E)_U(W, W) \geq 0$ for any $W \in U^\perp$, otherwise it is said to be *unstable*.

If (M, η, g) is a contact metric manifold, in [17] was proved that ξ is a harmonic vector field if and only if it is an eigenvector of the Ricci operator. In such case M is called an *H-contact manifold*. We now suppose that (M, η, g) is a compact η -Einstein contact manifold. From (3.7) it follows that M is *H-contact*. If ξ is unstable, then there exists a vector field $W \in \mathrm{Ker}\eta = \xi^\perp$ such that

$$(3.19) \quad (\mathrm{Hess}E)_\xi(W, W) = \int_M (\|\nabla W\|^2 - \|W\|^2 \|\nabla \xi\|^2) v_g < 0.$$

Since $\|\xi\| = 1$, we obtain $g(\overline{\Delta}\xi, \xi) = \|\nabla \xi\|^2$. Therefore, by (3.4) and (3.8), we have

$$\|\nabla \xi\|^2 = 4m - \rho(\xi, \xi) = 2m + \frac{\|\tau\|^2}{4}$$

and (3.19) gives

$$(3.20) \quad \int_M \left[\|\nabla W\|^2 - \left(2m + \frac{\|\tau\|^2}{4} \right) \|W\|^2 \right] v_g < 0.$$

Then, by (2.4) and (3.20), we obtain

$$(3.21) \quad \begin{aligned} (\text{Hess}E)_{id}(W, W) &= \int_M \left[\|\nabla W\|^2 - \rho(W, W) \right] v_g \\ &< \int_M \left[\left(2m + \frac{\|\tau\|^2}{4} \right) \|W\|^2 - \rho(W, W) \right] v_g. \end{aligned}$$

Using (3.7), since $W \in \text{Ker } \eta$, we have $\rho(W, W) = a\|W\|^2$. Consequently, (3.21) becomes

$$(3.22) \quad (\text{Hess}E)_{id}(W, W) < \int_M \left[2m + 1 - \frac{r}{2m} + (2m - 1) \frac{\|\tau\|^2}{8m} \right] \|W\|^2 v_g.$$

From (3.22), we get the following

PROPOSITION 3.3. *Let M be a compact η -Einstein contact manifold of dimension $(2m + 1)$, with scalar curvature*

$$r \geq 2m(2m + 1) + (2m - 1) \frac{\|\tau\|^2}{4}.$$

If ξ is unstable, then M is unstable.

Boyer, Galicki and Matzeu [6] study η -Einstein geometry as a class of distinguished Riemannian metrics on contact metric manifolds. In particular they show that there are many compact η -Einstein Sasakian manifolds. We recall that a compact η -Einstein K-contact manifold with scalar curvature $r > -2m$ is Sasakian (see Boyer and Galicki [5], or [2, p. 106]). Moreover, on any η -Einstein K-contact manifold the functions a and b and the scalar curvature r are constant. Then, Propositions 3.1, 3.2 and 3.3 imply the following

THEOREM 3.1. *Let (M, η, g) be a compact η -Einstein K-contact manifold of dimension $2m + 1$, $m > 1$.*

- (1) *If $\mu_1 < \min\{(r/m) - 2, 4m\}$, then M is an unstable η -Einstein Sasakian manifold.*
- (2) *If $r \geq 2m(2m + 1)$ and ξ is unstable, then M is an unstable η -Einstein Sasakian manifold.*
- (3) *If $r = 2m(2m + 1)$, then M is Sasaki-Einstein and*
 - (i) $\mu_1 = 4m \Leftrightarrow M$ is stable,
 - (ii) $\mu_1 < 4m \Leftrightarrow (2m + 1) \leq \lambda_1 < 4m$.

The unit sphere S^{2m+1} is an example of Sasakian space of constant sectional curvature $+1$. We denote by (η_0, g_0) the standard Sasakian structure on S^{2m+1} . Let (M, g) be an odd dimensional compact Riemannian manifold of constant sectional curvature $+1$. Then, (M, g) is a spherical space form $(S^{2m+1}/\Gamma, g)$, where Γ is a finite subgroup of $O(2m + 2)$ in which only the identity element has $+1$ as an eigenvalue, and g is the Riemannian metric on the quotient space S^{2m+1}/Γ induced by g_0 . The unitary group $U(m + 1)$ can be considered as the subgroup of all elements of $O(2m + 2)$ which preserve η_0 , and Γ is conjugate in $O(2m + 2)$ to a subgroup of $U(m + 1)$, then M inherits a contact 1-form η from η_0 (Wolf [28]). Since the metric g is induced from g_0 , then (η, g) is a Sasakian structure on the spherical space form M (Tanno [22, p. 505]). So, we have that: *every odd-dimensional compact Riemannian manifold of constant sectional curvature $+1$ is a Sasakian manifold*. This fact corrects a remark of Gherghe, Ianus and Pastore ([10, p. 51]). Now, we consider a Sasakian space form $M^{2m+1}(c)$, that is, a $(2m + 1)$ -dimensional Sasakian manifold of constant φ -sectional curvature c . Such spaces have constant scalar curvature $r = (m/2)[(2m + 1)(c + 3) + c - 1]$, and are η -Einstein ([2, p. 113]) with

$$a = \frac{(m + 1)c + 3m - 1}{2} \quad \text{and} \quad b = \frac{(m + 1)(1 - c)}{2}.$$

Moreover, $c = +1$ if and only if M^{2m+1} has constant sectional curvature $+1$. On the other hand, Urakawa ([24, p. 572]) proved that the first eigenvalue λ_1 of the Laplace-Beltrami operator on $(S^{2m+1}/\Gamma, g)$, $\Gamma \neq \{id_M\}$, is bigger than or equal to $4m$, that is, $(S^{2m+1}/\Gamma, g)$, $\Gamma \neq \{id_M\}$, is stable. Therefore, these results and Theorem 3.1 yield

THEOREM 3.2. *Let (M, η, g) be a compact Sasakian space form of constant φ -sectional curvature c , $\dim M = 2m + 1 \geq 3$.*

- (i) *If $\mu_1 < \min\{(m + 1)c + 3m - 1, 4m\}$, then M is unstable.*
- (ii) *If $c \geq 1$ and ξ is unstable, then M is unstable.*
- (iii) *If $c = 1$, then: $\mu_1 = 4m \Leftrightarrow M$ is stable \Leftrightarrow the fundamental group $\pi_1(M) \neq \{0\}$.*

In Theorem 3.2 we can also consider the case $m = 1$ because in such case M is η -Einstein and the functions $a = (c + 1)$ and $b = (1 - c)$ are constant.

REMARK 3.1.

- (a) In [10, Theorem 3.1], the authors considered a Sasakian space form $M^{2m+1}(c)$ with $c \leq 1$ and proved that $\lambda_1 < (m + 1)c + 3m - 1$ implies that M is unstable. Of course, our Theorem 3.2(i) extends and improves such result.
- (b) We recall that $\mu_1(S^{2m+1}) = 2m + 1$ and $\mu_2(S^{2m+1}) = 4m$. Then, if M is a compact Sasakian manifold of constant curvature $+1$, with $\pi_1(M) \neq \{0\}$, from Theorem 3.2 we deduce that $\mu_1(M) = \mu_2(S^{2m+1})$.

- (c) By (3.21) we easily deduce that, if (M, η, g) is a compact K-contact $(2m + 1)$ -manifold with Ricci tensor $\rho \geq 2mg$, then the instability of the Reeb vector field implies the instability of M .

The results given in Theorems 3.1 and 3.2 can be sharpened in dimension three. In fact, any K-contact three-manifold is Sasakian and η -Einstein. More precisely the Ricci tensor is given by

$$\rho = \left(\frac{r}{2} - 1\right)g + \left(-\frac{r}{2} + 3\right)\eta \otimes \eta.$$

Then, by the same arguments used in the proofs of Theorems 3.1 and 3.2, we deduce the following

THEOREM 3.3. *Let (M, η, g) be a compact Sasakian manifold of dimension three. If holds one of the following conditions*

- (i) $\mu_1 < r - 2 \leq 4$,
- (ii) $\mu_1 < 4 \leq r - 2$,
- (iii) $r \geq 6$ and ξ is unstable,

then M is a unstable. Moreover, if $r = 6$, then: $\mu_1 = 4 \Leftrightarrow M$ is stable $\Leftrightarrow \pi_1(M) \neq \{0\}$.

4. EXISTENCE OF MINIMA FOR THE ENERGY $E|_{\mathfrak{X}^1(M)}$

Brito [8] proved that the Hopf vector fields on the unit sphere S^3 , that is, the unit vector fields tangent to the fiber of any Hopf fibration, are the unique minimisers of the energy functional $E : \mathfrak{X}^1(S^3) \rightarrow \mathbb{R}$. In particular they are stable. On the other hand, S^3 is unstable, so the converse of Theorem 3.3 (iii) is not true. However, the Hopf vector fields on the unit sphere S^{2m+1} , $m > 1$, are unstable ([29, 11]). Besides, to our knowledge, in dimension > 3 , there are no examples of unit vector fields which realise an absolute minimum for the energy and neither criteria of existence of minima for the energy (see [12, 18]). Theorems of Section 3 give results for which the instability of the Reeb vector field implies the instability of the contact metric manifold, or, equivalently, the stability of the contact metric manifold gives the stability of the Reeb vector field. So, it is natural to study the existence of minima for the energy $E|_{\mathfrak{X}^1(M)}$ when M is a stable compact Riemannian manifold. Urakawa [23] classified all the compact simply connected irreducible Riemannian symmetric spaces which are stable (such spaces are in particular Einstein).

In this section we discuss the existence of minima for the energy $E : \mathfrak{X}^1(M) \rightarrow \mathbb{R}$, where (M, g) is supposed to be a stable compact Einstein manifold of dimension n , $\rho = \kappa g$. Since M is stable, by (2.4), we have

$$\begin{aligned} (4.1) \quad (\text{Hess}E)_{id}(X, X) &= \int_M g(\bar{\Delta}X - QX, X)v_g \\ &= \int_M (\|\nabla X\|^2 - \rho(X, X))v_g \geq 0 \quad \forall X \in \mathfrak{X}(M). \end{aligned}$$

Then the energy $E : \mathfrak{X}^1(M) \rightarrow \mathbb{R}$, given by (3.18), satisfies

$$(4.2) \quad E(U) = \frac{1}{2} \int_M \|\nabla U\|^2 v_g + \frac{n}{2} \text{vol}(M, g) \geq \frac{\kappa + n}{2} \text{vol}(M, g) \quad \forall U \in \mathfrak{X}^1(M).$$

Now, we recall that a vector field X is called to be a *Jacobi vector field* if it is a solution of the equation $J_{id}X = 0$. Jacobi vector fields are studied in [30] and there labelled “geodesic vector fields”. Then, if U_0 is a unit Jacobi vector field, we have

$$\int_M \|\nabla U_0\|^2 v_g = \int_M g(\bar{\Delta}U_0, U_0)v_g = \int_M \rho(U_0, U_0)v_g = \kappa \text{vol}(M, g).$$

Hence,

$$E(U_0) = \frac{\kappa + n}{2} \text{vol}(M, g) \leq E(U) \quad \forall U \in \mathfrak{X}^1(M),$$

that is, U_0 minimises the energy. Vice versa, let U be a unit vector field such that

$$E(U) = \frac{\kappa + n}{2} \text{vol}(M, g).$$

Then, by (4.2), $\int_M \|\nabla U\|^2 v_g = \kappa \text{vol}(M, g)$, and thus, by (4.1), we get

$$(\text{Hess}E)_{id}(U, U) = 0.$$

Since M is stable, we can expand U into the infinite sum:

$$U = \sum_{i=1}^{\infty} E_i, \quad \text{with } J_{id}E_i = a_i E_i \quad \text{and} \quad \int_M g(E_i, E_j)v_g = 0 \quad \forall i \neq j, \text{ where } a_i \geq 0.$$

Then, we have

$$J_{id}U = \sum_{i=p+1}^{\infty} a_i E_i, \quad \text{where } p = \dim \text{Ker } J_{id} \quad \text{and } a_i > 0 \quad \forall i \geq p + 1,$$

and thus

$$0 = (\text{Hess}E)_{id}(U, U) = \int_M g(J_{id}U, U)v_g = \sum_{i=p+1}^{\infty} a_i \int_M g(E_i, E_i)v_g$$

implies that $E_i = 0$ for any $i \geq p + 1$. So, $J_{id}U = 0$, that is, U is a Jacobi vector field. If $U \in \mathfrak{X}^1(M)$ is a unit vector field whose flow is volume-preserving, that is, $\text{div}U = 0$, then it is easy to get that U is a Killing vector field if and only if U is a Jacobi vector field (see, for example, [19, p. 171]). Hence we have

PROPOSITION 4.1. *Let (M, g) be a stable compact Einstein manifold. Then, the unit Jacobi vector fields are the unique unit vector fields which minimise the energy. In particular, if $U \in \mathfrak{X}^1(M)$ is volume-preserving, then U minimises the energy if and only if U is a Killing vector field.*

Moreover, we obtain

THEOREM 4.1. *Let (M, η, g) be a stable compact Einstein contact metric manifold. Then, the Reeb vector field ξ minimises the energy if and only if M is a Sasakian manifold.*

PROOF: First, we note that the flow of the Reeb vector field ξ of a contact metric manifold is volume-preserving. In fact, by (3.2), choosing a (local) orthonormal basis E_i of eigenvectors of the tensor h , we have

$$\operatorname{div} \xi = \sum_{i=1}^{2m} g(\nabla_{E_i} \xi, E_i) = \sum_{i=1}^{2m} g(-\varphi E_i - \varphi h E_i, E_i) = 0.$$

Besides, by a result of Boyer–Galicki [5], a compact Einstein K -contact manifold is Sasaki-Einstein. Then, the result follows from Proposition 4.1. \square

Finally, we obtain the following result.

THEOREM 4.2. *Let (M, η, g) be a compact Sasakian manifold of constant sectional curvature $+1$, $\dim M = 2m + 1$, $m > 1$. Then, the Reeb vector field ξ minimises the energy if and only if M is not simply connected.*

PROOF: We first show that if ξ minimises the energy, then necessarily M is not simply connected. In fact, if M were simply connected, then M would be the unit sphere S^{2m+1} . On the other hand, on the sphere S^{2m+1} the Hopf vector fields are exactly the unit Killing vector fields [27]. So, ξ would be a Hopf vector field and we know that Hopf vector fields on S^{2m+1} , $m > 1$, are unstable. This would imply that ξ does not minimise the energy, that is, a contradiction. The converse follows from Theorems 3.2 and 4.1. \square

REMARK 4.1. (*On the volume functional*) Let (M, g) be a compact Riemannian manifold and $U \in \mathfrak{X}^1(M)$. Then, U determines a map between (M, g) and (T^1M, g_s) and $\operatorname{vol}(U)$ is the volume of the corresponding submanifold $U(M)$ of (T^1M, g_s) . U is said to be *minimal* if it is a critical point of the functional $\operatorname{vol}: \mathfrak{X}^1(M) \rightarrow \mathbb{R}$. The study of the volume of unit vector fields begins with the pioneering work of Gluck–Ziller [13], where the motivation was to find the optimal unit vector fields on the unit sphere S^3 . They proved that unit vector fields of minimum volume on S^3 are precisely the Hopf vector fields, equivalently the unit Killing vector fields, and no others. For more details and the state of the art for the minimality of unit vector fields, we can refer to the survey [12]. The Reeb vector field ξ of a compact Sasakian manifold is minimal, concerning its stability Borrelli [7] proved that if ξ is E-stable, that is, stable with respect to the energy, then there exists a number $k_s \in]0, \infty]$, called *stability number*, such that the vector field $\tilde{\xi} = \sqrt{k} \xi \in \mathfrak{X}^1(M, \tilde{g} = (1/k)g)$ is stable with respect to the volume if and only if $0 < k \leq k_s$. However, his paper does not contain examples of Sasakian manifolds with ξ E-stable. Applying Theorem 4.2 and the result of Borrelli, we obtain that: *if (M, η, g) is a compact Sasakian manifold of constant sectional curvature $+1$,*

$\dim M = 2m + 1$, $m > 1$ and $\pi_1(M) \neq \{0\}$, then there exists $k_s \in]0, \infty]$, such that the vector field $\tilde{\xi} = \sqrt{k}\xi \in \mathfrak{X}^1(M, \bar{g} = (1/k)g)$ is stable with respect to the volume if and only if $0 < k \leq k_s$.

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