

ON AFFINE TRANSFORMATIONS OF A RIEMANNIAN MANIFOLD*

JUN-ICHI HANO

In this paper we establish some theorems about the group of affine transformations on a Riemannian manifold. First we prove a decomposition theorem (Theorem 1) of the largest connected group of affine transformations on a simply connected complete Riemannian manifold, which corresponds to the decomposition theorem of de Rham [4]¹⁾ for the manifold. In the case of the largest group of isometries, a theorem of the same type is found in de Rham's paper [4] in a weaker form. Using Theorem 1 we obtain a sufficient condition for an infinitesimal affine transformation to be a Killing vector field (Theorem 2). This result includes K. Yano's theorem [13] which states that on a compact Riemannian manifold an infinitesimal affine transformation is always a Killing vector field. His proof of the theorem depends on an integral formula which is valid only for a compact manifold. Our method is quite different and is based on a result [11] of K. Nomizu.

The author expresses his deep thanks to Dr. Nomizu who suggested these problems to him.

I. Preliminaries

1. Let M be a differentiable manifold of class C^∞ .²⁾ The set \mathfrak{X} of all tangent vector fields defined on M is a module over the ring \mathfrak{F} of all differentiable functions on M .

An affine connection is defined by a homomorphism over $\mathfrak{F} : X \rightarrow \mathcal{V}_X$ from \mathfrak{X} into the module of linear mappings (over the field of all real numbers) of T , which satisfies the following condition

Received March 10, 1955.

* The subject of this paper was prepared while the author was a Yukawa Fellow at Osaka University.

¹⁾ Numbers in brackets refer to Bibliography at the end of this paper.

²⁾ As we only consider manifolds, tangent vector fields, tensor fields and mappings which are "differentiable of class C^∞ ," we always omit this adjective. We deal only with connected manifolds. For the terminology concerning manifolds, we follow C. Chevalley [3].

$$\nabla_X(f \cdot Y) = f \cdot \nabla_X \cdot Y + (X \cdot f) \cdot Y,$$

where X and Y are in T and f in F .³⁾ ∇_X is the so-called covariant differentiation along X .

The torsion tensor field T , of type $(1, 2)$, and the curvature tensor field R , of type $(1, 3)$, are expressed as follows:

$$\begin{aligned} T(X, Y) &= \nabla_X \cdot Y - \nabla_Y \cdot X - [X, Y] \\ R(X, Y) \cdot Z &= \nabla_X \cdot \nabla_Y \cdot Z - \nabla_Y \cdot \nabla_X \cdot Z - \nabla_{[X, Y]} Z, \end{aligned}$$

for any X, Y and Z in T [10].

When M has a Riemannian metric defined by a positive definite symmetric quadratic tensor field G , there is one and only one affine connection such that its torsion tensor field T is zero and the covariant differentiation of the fundamental tensor field G along any X in \mathfrak{X} is zero. This connection is called the Riemannian connection associated to a Riemannian metric G .

Let M_i ($i = 0, 1, \dots, r$) be a manifold with an affine connection and let M be the direct product of M_i ($i = 0, 1, \dots, r$). For any point $p = (p_0, p_1, \dots, p_r)$, where $p_i \in M_i$, the mapping $\iota_i(p) : p'_i \rightarrow (p_0, \dots, p_{i-1}, p'_i, p_{i+1}, \dots, p_r)$ from M_i into M defines the submanifold $M_i(p)$.⁴⁾ The tangent space $T(p)$ at p is the direct sum of $T_i(p)$ ($i = 0, 1, \dots, r$), where $T_i(p)$ is the tangent space at p of the submanifold $M_i(p)$. Any X in \mathfrak{X} can be written uniquely as a sum of X_i , where $X_i(p) \in T_i(p)$. We call X_i the i -component of X . Moreover from a tangent vector field X_i^* on M_i we can define a tangent vector field $X_i : p \rightarrow d\iota_i(p) \cdot X_i^*$ on M . Let \mathfrak{X}_i be the set of all extended tangent vector fields which are so obtained from vector fields on M_i . In the submanifold $M_i(p)$ through p , we can define a connection which is isomorphic with the given connection of M_i by the mapping $\iota_i(p)$.

We can now define the product connection on M as follows. For any X and Y in \mathfrak{X}

$$(1) \quad \nabla_X \cdot Y = \sum_{i=0}^r (\nabla_{iX_i} \cdot Y_i + \sum_{i \neq j} [X_i, Y_j]_j),$$

where the lower index i (resp. j) of a vector field means the i (resp. j)-component and $(\nabla_{iX_i} \cdot Y_i)(p)$ is calculated on the submanifold $M_i(p)$ for each i .

³⁾ This definition of an affine connection is due to Koszul [10].

⁴⁾ Clearly two mappings $\iota_i(p)$ and $\iota_i(p')$ coincide if $M_i(p) = M_i(p')$.

When each connection on M_i is a Riemannian connection associated to a fundamental tensor field G_i , the product connection on M is also a Riemannian connection which is associated to the product Riemannian metric G .

For any $p \in M$, $M_i(p)$ ($i = 0, 1, \dots, r$) is a totally geodesic submanifold, that is, any geodesic curve tangent to $M_i(p)$ at a point is contained in $M_i(p)$.

Let T_i and T be the torsion tensor fields of M_i and M respectively, and R_i and R the curvature tensor fields of M_i and M respectively. Then it is easily seen that

$$(2) \quad T(X, Y) = \sum_{i=0}^r T_i(X_i, Y_i)$$

$$(3) \quad R(X, Y)Z = \sum_{i=0}^r R_i(X_i, Y_i)Z_i,$$

where $(T_i(X_i, Y_i))(p)$ and $(R_i(X_i, Y_i)Z_i)(p)$ are calculated in $M_i(p)$. Indeed, T and R being tensors, the above equalities have only to be verified for $X_i \in \mathfrak{X}_i$, $Y_j \in \mathfrak{X}_j$ and $Z_k \in \mathfrak{X}_k$.

2. Any tangent vector field V on a manifold M generates a local one-parameter group of local transformations φ_t ($-\varepsilon < t < \varepsilon$)⁵⁾ such that

$$\lim_{t \rightarrow 0} \frac{1}{t} (\varphi_t \circ f - f) = V \cdot f,$$

for any f in \mathfrak{F} . A vector field V on M with an affine connection is called an infinitesimal affine transformation if it generates a local one-parameter group of local affine transformations, that is,

$$(4) \quad d\varphi_t(\nabla_X \cdot Y) = \Delta_{d\varphi_t(X)} \cdot (d\varphi_t(Y))$$

for any X and Y in \mathfrak{X} , and for any t ($-\varepsilon < t < \varepsilon$). When M has a Riemannian metric, we say that V is a Killing vector field if it generates a local one-parameter group of local isometries, that is,

$$(5) \quad G(d\varphi_t(X), d\varphi_t(Y)) = G(X, Y)$$

for any X and Y in \mathfrak{X} , and for any t ($-\varepsilon < t < \varepsilon$).

A necessary and sufficient condition for a vector field V to be an infinitesimal affine transformation is that we have

⁵⁾ More precisely, for each point p , there exist a neighbourhood $U(p)$ and a positive number ε such that φ_t ($-\varepsilon < t < \varepsilon$) defined on $U(p)$.

$$(6) \quad \mathcal{F}_X \cdot \mathcal{F}_Y \cdot (V) - \mathcal{F}_{\nabla_X Y} \cdot (V) + \mathcal{F}_X \cdot (T(V, Y)) - T(V, \mathcal{F}_X \cdot Y) + R(V, X)Y = 0$$

for any X and Y in \mathfrak{X} . Using the covariant differentiation and the contraction, we can write the above equality as follows:

$$\langle \mathcal{F} \mathcal{F} V, Y \otimes X \rangle + (\mathcal{F}_X \cdot T)(V, Y) + T(\mathcal{F}_X \cdot V, Y) + R(V, X)Y = 0.$$

A vector field V is a Killing vector field if and only if it satisfies

$$(7) \quad G(\mathcal{F}_X \cdot V, Y) + G(X, \mathcal{F}_Y \cdot V) = 0$$

for any X and Y in \mathfrak{X} . The equalities (6) and (7) can be obtained by computing the Lie derivatives along V of the both sides of (4) and (5) respectively.

LEMMA 1. *Let $M = M_0 \times \dots \times M_r$ be a product manifold with a product affine connection, and let V be an infinitesimal affine transformation on M with the i -component V_i . Then V_i is an infinitesimal affine transformation on M and the restriction of V_i on $M_i(p)$ is an infinitesimal affine transformation on $M_i(p)$ for each i .*

Proof. From (1), for a vector Z such that $Z(p) \in T_i(p)$ at each point, we have

$$(\mathcal{F}_X \cdot Z_i)(p) \in T_i(p),$$

for any $X \in \mathfrak{X}$. From (2) and (3) we have

$$(T(V, X))_i = T(V_i, X) \quad \text{and} \quad (R(V, X)Y)_i = R(V_i, X)Y,$$

for any X and Y in \mathfrak{X} . The i -component of the left hand side of (6) is equal to the left hand side of (6) in which V is replaced by V_i . This shows that V_i is an infinitesimal affine transformation on M .

Next, from (1) we can see easily

$$\mathcal{F}_{X_i} \cdot Y_i = \mathcal{F}_{iX_i} \cdot Y$$

and this vector field is contained in \mathfrak{X}_i for X_i and Y_i in \mathfrak{X}_i . Moreover for a vector field Z such that $Z(p) \in T_i(p)$ at each point p , we have

$$\mathcal{F}_{X_i} \cdot Z = \mathcal{F}_{iX_i} \cdot Z$$

for $X_i \in \mathfrak{X}_i$. From (2) and (3) it follows that

$$\begin{aligned} (T(X, Y))_i &= T_i(X_i, Y_i) \\ (R(X, Y) \cdot Z)_i &= R_i(X_i, Y_i) \cdot Z_i, \end{aligned}$$

for any X and Y in \mathfrak{X} . Therefore, the i -component of the left hand side of (6) for $X = X_i$ in \mathfrak{X}_i and $Y = Y_i$ in \mathfrak{X}_i is equal to

$$\begin{aligned} & \nabla_{iX_i} \cdot \nabla_{iY_i} \cdot (V_i) + \nabla_{i\nabla_{iX_i} \cdot Y_i} \cdot (V_i) + \nabla_{iX_i} \cdot (T_i(V_i, Y_i)) \\ & - T_i(V_i, \nabla_{iX_i} \cdot (Y_i)) + R_i(V_i, X_i) \cdot Y_i. \end{aligned}$$

Since each term is calculated on $M_i(p)$, this shows that the restriction of V on $M_i(p)$ is an infinitesimal affine transformation on $M_i(p)$.

When V is a Killing vector field the situation is just the same as above.

Remark. It is almost evident that the vector field V_i in \mathfrak{X}_i which is the extended vector field of an infinitesimal affine transformation (resp. a Killing vector field) of M_i is also an infinitesimal affine transformation (resp. a Killing vector field) of M .

We say that a manifold with an affine connection (or a Riemannian manifold) M is complete if every geodesic curve can be extended for any large value of the canonical parameter. When the completeness is satisfied on M , any infinitesimal affine transformation (or a Killing vector field) generates a one-parameter group of affine transformations (isometries) from M onto itself [7].

3. Let M be a manifold with an affine connection. The group $A(M)$ of all affine transformation of M onto itself is a Lie group with respect to the compact-open topology [5], [6], [9]. When M has a Riemannian metric, the group $I(M)$ of all isometries of M onto itself is a closed subgroup of $A(M)$. $I(M)$ is also a Lie group [8].

The mapping from $A(M) \times M$ onto M , which gives the transformation law, is differentiable as is known from a theorem of S. Bochner and D. Montgomery [1]. Any one-parameter subgroup in $A(M)$ (resp. $I(M)$) induces an infinitesimal affine transformation (resp. a Killing vector field) on M .

II. Decomposition Theorem of $A_0(M)$ and $I_0(M)$

4. In this section we always assume that M is a simply connected complete Riemannian manifold.

The homogeneous holonomy group $\mathcal{H}(p)$ at p operates on the tangent space $T(p)$ and is completely reducible. If $T(p)$ is irreducible, we call M irreducible.

The following theorem is due to de Rham [4]:

1) A simply connected complete Riemannian manifold M is isometric to the direct product of a Euclidean space M_0 and simply connected complete irreducible Riemannian manifolds M_i with $\dim M_i \geq 2$ ($i=1, \dots, r$), and this decomposition is unique up to the order and isometries.

2) The tangent space $T(p)$ at each point p is the direct sum of the $\mathcal{P}(p)$ -invariant subspace $T_0(p), T_1(p), \dots, T_r(p)$, where $T_0(p)$ is the subspace of all $\mathcal{P}(p)$ -invariant vectors, $T_i(p)$ ($i=1, \dots, r$) are irreducible and any two subspaces $T_i(p)$ and $T_j(p)$ ($i \neq j, i, j=0, 1, \dots, r$) are mutually orthogonal. This decomposition is unique.

If we identify M with the direct product $M_0 \times M_1 \times \dots \times M_r$, the subspace $T_i(p)$ is the tangent space at p of the submanifold $M_i(p)$ which corresponds to M_i in the manner mentioned in 1.

We call these decompositions the de Rham decomposition of the manifold, or of the tangent space $T(p)$.

5. The aim of this section is to prove the following

THEOREM 1. *Let M be a simply connected complete Riemannian manifold, and $M = M_0 \times M_1 \times \dots \times M_r$ be the de Rham decomposition of M . Then the group $A_0(M)$ is isomorphic to the direct product $A_0(M_0) \times A_0(M_1) \times \dots \times A_0(M_r)$, and the group $I_0(M)$ is isomorphic to the direct product $I_0(M_0) \times I_0(M_1) \times \dots \times I_0(M_r)$, where $A_0(M)$ and $A_0(M_i)$ (resp. $I_0(M)$ and $I_0(M_i)$) are the connected components of the identity in $A(M)$ and $A(M_i)$ (resp. $I(M)$ and $I(M_i)$) respectively.*

To prove this theorem the following lemma, which is given by K. Nomizu [11], [12] is useful.

LEMMA 2. *Let $T(p) = T_0(p) + T_1(p) + \dots + T_r(p)$ be the de Rham decomposition of the tangent space. Then*

$$d\varphi(T_i(p)) = T_i(\varphi(p)),$$

for any $\varphi \in A_0(p)$ and for any $p \in M$.

Proof of Theorem 1. Let φ_i be an element in $A_0(M_i)$ ($i=0, 1, \dots, r$), a transformation $\varphi(p) = (\varphi_0(p_0), \dots, \varphi_i(p_i), \dots, \varphi_r(p_r))$, where $p = (p_0, \dots, p_i, \dots, p_r)$ is an element in $A_0(M)$. The mapping from $A_0(M_0) \times A_0(M_1) \times \dots \times A_0(M_r)$ into $A_0(M)$ which maps $(\varphi_0, \varphi_1, \dots, \varphi_r)$ to φ is an isomorphism. Then to prove the theorem, we have only to show that the induced

isomorphism from the direct sum of the Lie algebras associated to $A_0(M_i)$ ($i=0, 1, \dots, r$) into the Lie algebras \mathfrak{a} associated to $A_0(M)$ is onto. The Lie algebra \mathfrak{a} is the Lie algebra of all infinitesimal affine transformations on M . The set \mathfrak{a}_i of all infinitesimal affine transformations contained in \mathfrak{A}_i , namely the image of the Lie algebra associated to $A_0(M_i)$ by the induced isomorphism, is an ideal for each i .

Let V be an infinitesimal affine transformation in \mathfrak{a} and V_i the i -component of V . We now fix an arbitrary point $p^* = (p_0^*, \dots, p_i^*, \dots, p_r^*)$ and consider the mapping $\iota_i(p^*)$ from M_i onto $M_i(p^*)$ as stated in 2. From Lemma 1 V_i and the restriction of V_i on $M_i(p^*)$ are infinitesimal affine transformations on M and $M_i(p^*)$ respectively, and there is an infinitesimal affine transformation V'_i on M_i such that $d\iota_i(p^*)(V'_i)$ is equal to the restriction of V_i on $M_i(p^*)$. Let V_i^* be the infinitesimal affine transformation on M which is the extension of V'_i . Clearly V_i^* is contained in \mathfrak{A}_i . We shall prove that $V_i^* = V_i$.

We now consider the difference $V_i - V_i^*$. Clearly it is an infinitesimal affine transformation on M , and $(V_i - V_i^*)(p)$ is zero when p is on $M_i(p^*)$. From our assumption that M is complete, $V_i - V_i^*$ generates a one-parameter subgroup φ_t in $A_0(M)$. First, as $(V_i - V_i^*)(p) = 0$ on $M_i(p^*)$, each transformation φ_t leaves every point on $M_i(p^*)$ fixed. And as $(V_i - V_i^*)(p^*) \in T_i(p)$ at any point on M , we have $\varphi_t(p) \in M_i(p)$, namely, each φ_t maps $M_i(p)$ onto itself. Next we consider the submanifold $M_{\hat{i}}(p) = \{p' ; p' \in M, p' = (p'_0, \dots, p'_{i-1}, p_i, p'_{i+1}, \dots, p'_r)$, where $p'_j \in M_j (j \neq i)\}$ through $p = (p_0^*, \dots, p_{i-1}^*, p_i, p_{i+1}^*, \dots, p_r^*)$ which is on $M_i(p^*)$. The tangent space of $M_{\hat{i}}(p)$ at p' is $T_{\hat{i}}(p') = T_0(p') + \dots + T_{i-1}(p') + T_{i+1}(p') + \dots + T_r(p')$. Let q be an arbitrary point on $M_{\hat{i}}(p)$. As $M_{\hat{i}}(p)$ is a totally geodesic submanifold and is complete, p and q can be joined by a geodesic curve $\sigma(s)$ on $M_{\hat{i}}(p)$, and the tangent vector $X(p)$ of $\sigma(s)$ at p is contained in $T_{\hat{i}}(p)$. According to Lemma 2, we have $d\varphi_t \cdot X(p) \in T_{\hat{i}}(p)$ for any real number t , and $d\varphi_t \cdot X(p)$ is the tangent vector of a geodesic curve $\varphi_t \cdot \sigma(s)$ at p . Therefore the geodesic curve $\varphi_t \cdot \sigma(s)$ lies on $M_{\hat{i}}(p)$, and $\varphi_t(q)$ is in $M_{\hat{i}}(p)$. This shows $\varphi_t(M_{\hat{i}}(p)) \subset M_{\hat{i}}(p)$ for any p on $M_i(p^*)$. Any point q on M is contained in one and only one $M_{\hat{i}}(p)$ through some p on $M_i(p^*)$ and, on the other hand, in $M_i(q)$. Hence $\varphi_t(q) \in M_{\hat{i}}(p) \cap M_i(q)$. But $M_i(q)$ and $M_{\hat{i}}(p)$ have one and only one common point q , and hence $\varphi_t(q) = q$. Thus we conclude that φ_t is the identity transfor-

mation for any t and $V_i = V_i^*$ on M . This shows that any element V in \mathfrak{a} can be written as a sum of elements in \mathfrak{a}_i ($i = 0, 1, \dots, r$), and our isomorphism is onto.

III. An Application of the Decomposition Theorem

6. Let M be a Riemannian manifold with the fundamental tensor field G . For any φ in $A(M)$, $d\varphi \cdot G$ is also a positive definite symmetric tensor field and as $d\varphi$ is commutative with the covariant differentiation, $\nabla(d\varphi \cdot G) = 0$. Then $(d\varphi \cdot G)(p)$ is invariant by the operations of the homogeneous holonomy group $\mathcal{H}(p)$. If M is irreducible, there is a positive constant $c(\varphi)$ such that $d\varphi \cdot G = c^2(\varphi) \cdot G$, and $\varphi \rightarrow c(\varphi)$ is a continuous representation of $A(M)$ into the multiplicative group of all positive real numbers [11], [12]. This follows from the fact that, the invariant positive definite bilinear form on $T(p)$ is uniquely determined up to a positive constant, as the homogeneous holonomy group is irreducible. From this fact the following lemma is easily obtained.

LEMMA 3. *When a Riemannian manifold M is irreducible and complete, an infinitesimal affine transformation V is a Killing vector field if and only if there is a non-trivial orbit on which the length of V is bounded.*

Proof. Let φ_t be the one-parameter group generated by V . Then $V(\varphi_t(p)) = d\varphi_t \cdot V(p)$ and $|V(\varphi_t(p))| = c(\varphi_t)|V(p)|$, where $|X(p)|$ denotes the length of a vector $X(p)$. If V is a Killing vector field, $c(\varphi_t) = 1$ for every t and surely the length of V is bounded on any orbit $\{\varphi_t(p)\}$. Conversely, let $\{\varphi_t(p)\}$ be a non-trivial orbit on which the length of V is bounded. Then $V(p)$ is not zero and the function $c(\varphi_t)$ of t is bounded. As already mentioned, $c(\varphi_t)$ is a continuous representation of the additive group of all real numbers. Therefore $c(\varphi_t)$ must be equal to 1 for every t , and V is a Killing vector field.

7. A Riemannian manifold is called locally flat when the homogeneous holonomy group is discrete.

If there is an absolutely parallel vector field⁶⁾ on a complete Riemannian manifold, it is a Killing vector field. We define a *translation* as an isometry which lies on the one-parameter group of isometries generated by such a Killing

⁶⁾ A vector field X is an absolutely parallel vector field when for any two points p and q , $X(p)$ and $X(q)$ are parallel along any piece-wise differentiable curve which joins p and q . This condition is equivalent to the condition $\nabla X = 0$.

vector field.

For the necessity in the next section we prepare

LEMMA 4. *If the length of an infinitesimal affine transformation V on a complete locally flat Riemannian manifold is bounded, then V is a Killing vector field and generates a one-parameter group of translations.*

Proof. The universal covering manifold \tilde{M} of M has a natural Riemannian metric induced from that of M , with respect to which \tilde{M} is also complete and locally flat. As is well known,⁷⁾ a complete locally flat Riemannian manifold M is isometric to a Euclidean space with the usual metric when it is simply connected, and the group of all affine transformations $A(\tilde{M})$ is the usual affine group. If we take a system of cartesian coordinates x^1, x^2, \dots, x^n on \tilde{M} , then an infinitesimal affine transformation \tilde{V} is given by

$$\sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} \cdot x^j + b_i \right) \frac{\partial}{\partial x^i}$$

where a_{ij} and b_i are arbitrary real numbers. It is easily seen that the length $\sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} \cdot x^j + b_i \right)^2}$ of V is bounded on M if and only if each a_{ij} ($i, j = 1, 2, \dots, n$) is zero.

For a Killing vector field V on M whose length is bounded, there is a Killing vector field \tilde{V} on \tilde{M} such that the projection from \tilde{M} to M maps \tilde{V} on V , and its length is bounded on M . Then \tilde{V} and accordingly V are absolutely parallel vector fields on \tilde{M} and M respectively.

COROLLARY. *If M is a compact locally flat Riemannian manifold, $A_0(M)$ coincides with $I_0(M)$ and is abelian.*

8. THEOREM 2. *Let M be a complete Riemannian manifold. If the length of an infinitesimal affine transformation V is bounded on M , then V is a Killing vector field.*

Proof. Let \tilde{M} be the universal covering manifold of M and let $\tilde{M} = \tilde{M}_0 \times \tilde{M}_1 \times \dots \times \tilde{M}_r$ be the de Rham decomposition of \tilde{M} . It has a Riemannian connection which is naturally induced from that of M , and is complete with respect to it. If \tilde{V} is the infinitesimal affine transformation

⁷⁾ For example, cf. [2] Chapitre III, II.

which is mapped on V by the projection from \tilde{M} onto M , then the length of \tilde{V} is also bounded on \tilde{M} . As \tilde{M} is simply connected and complete, we can apply the arguments in II.

Since the length of \tilde{V} is bounded on \tilde{M} , the length of the i -component \tilde{V}_i of \tilde{V} is also bounded on \tilde{M} . As we have seen in the proof of Theorem 1, \tilde{V}_i is the extension of a certain infinitesimal affine transformation \tilde{V}'_i on \tilde{M}_i , and the length of \tilde{V}'_i is clearly bounded. We see that \tilde{V}'_i is a Killing vector field from Lemma 3 when $1 \leq i \leq r$, and from Lemma 4 when $i=0$. As we already remarked, the extension \tilde{V}_i of a Killing vector field \tilde{V}'_i on \tilde{M}_i is also a Killing vector field on \tilde{M} for each i . Hence \tilde{V} and accordingly V are Killing vector fields on \tilde{M} and M respectively.

On a compact Riemannian manifold, the length of any vector field is of course bounded. Hence we have

THEOREM 3. (K. Yano) *On a compact Riemannian manifold M , every infinitesimal affine transformation is a Killing vector field. Therefore $A_0(M)$ coincides with $I_c(M)$.*

BIBLIOGRAPHY

- [1] S. Bochner and D. Montgomery: Groups of differentiable and real or complex analytic transformations, *Ann. of Math.*, vol. **46** (1945), pp. 685-694.
- [2] E. Cartan: *Leçons sur la géométrie des espaces de Riemann*, Paris, (1946).
- [3] C. Chevalley: *Theory of Lie groups I*, Princeton University Press, (1946).
- [4] G. de Rham: Sur la réductibilité d'un espace de Riemann, *Comm. Math. Helv.*, vol. **26** (1952), pp. 328-344.
- [5] J. Hano and A. Morimoto: Note on the group of affine transformations of an affinely connected manifold, *Nagoya Math. Jour.*, vol. **8** (1955), pp. 85-95.
- [6] S. Kobayashi: Groupe de transformations qui laissent invariante une connexion infinitésimale, *Comptes rendus*, **238** (1954), pp. 644-645.
- [7] S. Kobayashi: Espaces à connexion de Cartan complets, *Proc. of the Jap. Acad.*, vol. **30** (1953), pp. 709-710.
- [8] S. Myers and N. Steenrod: The group of isometries of a Riemannian manifold, *Ann. of Math.*, vol. **40** (1939), pp. 400-416.
- [9] K. Nomizu: On the group of affine transformations of an affinely connected manifold, *Proc. of the Amer. Math. Soc.*, vol. **4** (1953), pp. 816-823.
- [10] K. Nomizu: Invariant affine connections on homogeneous spaces, *Amer. Jour. of Math.*, vol. **76** (1954), pp. 33-65.

- [11] K. Nomizu : Sur les transformations affines d'une variété riemannienne, *Comptes rendus*, **237** (1953), pp. 1308-1310.
- [12] K. Nomizu : Studies on Riemannian homogeneous spaces, In this Journal.
- [13] K. Yano : On harmonic and Killing vector fields, *Ann. of Math.*, vol. **55** (1952), pp. 38-45.

*Mathematical Institute,
Nagoya University*