Canad. Math. Bull. Vol. **17** (4), 1974

ON 2-SUMMING OPERATORS

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In this note all Banach space are assumed to be real and separable and their norms will be denoted by $\| \ \|$. The canonical bilinear form between a Banach space B and its topological dual B' will be denoted by $\langle x, y\rangle$, $x \in B$, $y \in B'$. Recall that if B_1 and B_2 are Banach spaces a linear operator $A:B_1 \to B_2$ is said to be p-summing $(0<p<\infty)$ if there is a constant $\rho>0$ such that for any finite family $x_1, x_2, \ldots, x_n$ in B_1 we have

$$\left(\sum_i \|Ax_i\|^p\right)^{1/p} \leq \rho \sup_{y \in U_1} \left(\sum_i |\langle x_i, y\rangle|^p\right)^{1/p}$$

where U_1 is the unit ball in B_1'. The fundamental result of Pietsch [4] states that A is p-summing if and only if there is a finite Borel measure ν on U_1 (here U_1 has the weak* topology and is therefore compact) such that

$$\|Ax\|^p \leq \int_{U_1} |\langle x, y\rangle|^p \, d\nu(y)$$

for all $x \in B_1$. If A is p-summing and $0<p\leq q<\infty$ then A is q-summing [2]. In the case $A:H_1 \to H_2$ where H_1 and H_2 are Hilbert spaces then A is 2-summing if and only if A is Hilbert-Schmidt and in this case A is p-summing for all $0<p<\infty$ [3]. The operator A can then be written in the form

$$Ax = \sum_1^\infty \lambda_n (x, x_n) y_n$$

where $\{x_n\}$ and $\{y_n\}$ are orthonormal bases in H_1 and H_2 respectively, and $\{\lambda_n\}$ is a sequence of positive numbers such that $\sum_n \lambda_n^2 < \infty$ [1, Chap. 1].

Setting $z_n = \lambda_n x_n$ we see that $Ax = \sum_n (x, z_n) y_n$ where

$$\sup_{y \in U_2} \sum (y_n, y)^2 = \sup_{y \in U_2} \|y\|^2 = 1 < \infty$$

and

$$\sum_n (x, z_n)^2 = \sum_n \lambda_n^2 (x, x_n)^2 = \int_{U_1} (x, y)^2 \, d\nu(y)$$

where $\nu \geq 0$ is the finite Borel measure on U_1 given by $\sum_n \lambda_n^2 \delta_{x_n}$, δ_{x_n} = unit mass at x_n. We give now an analogous representation for 2-summing operators.

THEOREM 1. *A linear operator $A:B_1 \to B_2$ is 2-summing if and only if A can be written in the form $Ax = \sum_n \langle x, y_n\rangle z_n$, the convergence being in B_2. The sequence*

Received by the editors September 22, 1972 and, in revised form, February 21, 1973.
[1] Research supported by NRC Grant No. A-7840.

$\{z_n\}\subset B_2$ satisfies $\sup_{y\in U_2}\sum_n \langle z_n, y\rangle^2<\infty$ where U_2 is the unit ball in B_2' and the sequence $\{y_n\}\subset B_1'$ has the property that

$$\sum_n \langle x, y_n\rangle^2 = \int_{U_1} \langle x, y\rangle^2\, d\nu(y)$$

for all $x\in B_1$. Here $\nu\geq 0$ is a finite Borel measure on U_1.

Proof. If A is 2-summing, then

$$\|Ax\|^2 \leq \int_{U_1} \langle x, y\rangle^2\, d\nu(y)$$

for some finite Radon measure $\nu\geq 0$ on U_1.

For each $x\in B_1$, we let f_x be the equivalence class in $L^2(\nu)$ of the function $y\to \langle x, y\rangle$ defined on U_1. Denote by B_1^* the linear subspace of $L^2(\nu)$ defined by $B_1^*=\{f_x : x\in B_1\}$. If H_ν is the closure of B_1 in $L^2(\nu)$, then H_ν is a Hilbert space with inner product

$$(f_1, f_2) = \int_{U_1} f_1 f_2\, d\nu$$

and norm $|f|_1=(f,f)^{1/2}$. Consider now the map A^* defined on B_1^* with values in B_2 by $A^*f_x=Ax$. Then A^* is a linear and well-defined: if $f_{x_1}=f_{x_2}$ in $L^2(\nu)$, i.e.,

$$|f_{x_1}-f_{x_2}|_1 = |f_{x_1-x_2}|_1 = 0,$$

then

$$\|A^*f_{x_1}-A^*f_{x_2}\|^2 = \|Ax_1-Ax_2\|^2 = \|A(x_1-x_2)\|^2 \leq \int_{U_1}\langle x_1-x_2, y\rangle^2\, d\nu(y) = |f_{x_1-x_2}|_1^2 = 0$$

so that $A^*f_{x_1}=A^*f_{x_2}$. Moreover,

$$\|A^*f_x\|^2 = \|Ax\|^2 \leq \int \langle x, y\rangle^2\, d\nu(y) = |f_x|_1^2$$

so that A^* is continuous from B_1^* into B_2. Since B_1^* is dense in H_ν the map A^* extends to a continuous linear map $A^*: H_\nu\to B_2$.

Note also that H_ν is a separable Hilbert space since B_1 is separable and is a dense subset of H_ν. Therefore every dense linear manifold (in particular B_1^*) contains a countable orthonormal basis by the Gramm-Schmidt process. Let $\{f_{x_n}\}_n$ be such a basis. Then every $f\in H_\nu$ can be represented uniquely in the form

$$f = \sum_n (f, f_{x_n}) f_{x_n},$$

and since A^* is continuous,

$$A^*f = \sum_n (f, f_{x_n}) A^* f_{x_n},$$

https://doi.org/10.4153/CMB-1974-087-5 Published online by Cambridge University Press

the convergence being in B_2. Set $z_n = A^* f_{x_n} \in B_2$. If $A^{*\prime}: B_2' \to H_\nu$ is the adjoint map of A^* and $y \in B_2'$, then

$$\sum_n \langle z_n, y\rangle^2 = \sum_n \langle A^* f_{xn}, y\rangle^2 = \sum_n (f_{xn}, A^{*\prime} y)^2 = |A^{*\prime} y|_1^2$$

so that

$$\sup_{y\in U_2} \sum_n \langle z_n, y\rangle^2 = \sup_{y \in U_2} |A^{*\prime}|_1^2 = \|A^{*\prime}\|^2 < \infty$$

since A^*, hence $A^{*\prime}$ is continuous. Note also that the map

$$x \to (f_x, f_{xn}) = \int_{U_1} \langle x, y\rangle \langle x_n, y\rangle \, d\nu(y)$$

is a continuous linear form on B_1 so there exists $y_n \in B_2'$ such that $(f_x, f_{x_n}) = \langle x, y_n\rangle$ for all x in B_1, $n=0, 1, 2, \ldots$. Finally, if $x \in B_1$ then

$$\sum_n \langle x, y_n\rangle^2 = \sum_n (f_x, f_{xn})^2 = |f_x|_1^2 = \int_{U_1} \langle x, y\rangle^2 \, d\nu(y) < \infty.$$

Thus if $x \in B_1$, then $Ax = A^* f_x = \sum \langle x, y_n\rangle z_n$ where the sequences $\{y_n\}$ and $\{z_n\}$ have the properties announced in the theorem. We turn now to the sufficiency. Suppose $A: B_1 \to B_2$ has the form

$$Ax = \sum_n \langle x, y_n\rangle z_n$$

where the sequences $\{y_n\}$ and $\{z_n\}$ and the measure $\nu \geq 0$ satisfy the conditions stated in the theorem. Let

$$\rho = \sup_{y\in U_2} \sum \langle z_n, y\rangle^2$$

which is finite by assumption. Let $x \in B_1$. Then

$$\|Ax\|^2 = \sup_{y\in U_2} |\langle Ax, y\rangle|^2 = \sup_{y \in U_2} \left| \sum_n \langle x, y_n\rangle \langle z_n, y\rangle \right|^2$$

$$\leq \sup_{y\in U_2} \left[\left(\sum_n \langle x, y_n\rangle^2 \right) \left(\sum_n \langle z_n, y\rangle^2 \right) \right] \leq \rho \sum_n \langle x, y_n\rangle^2 = \rho \int_{U_1} \langle x, y\rangle^2 \, d\nu(y).$$

Thus A is 2-summing and the proof of the theorem is complete.

We say that a sequence $\{y_n\} \subset B'$ is strongly 2-summable if

$$\lim_{m\to\infty} \sup_{\|x\|\leq 1} \sum_{n=m}^{\infty} \langle x, y_n\rangle^2 = 0.$$

COROLLARY. *Let B_1 be a separable Banach space such that the unit ball in B_1'' is metrizable in the weak* topology (e.g. if B_1' is separable or B_1 reflexive). Let $A: B_1 \to B_2$ be p-summing* ($0 < p \leq 2$). *Then A can be written in the form*

$$Ax = \sum_n \langle x, y_n\rangle x_n$$

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where $\{x_n\}\subset B_2$ *is* 2-*summable and* $\{y_n\}\subset B_1'$ *is strongly* 2-*summable. Moreover,*

$$A_m x = \sum_{n=1}^{m} \langle x, y_n\rangle x_n \xrightarrow{m} Ax$$

uniformly in $\|x\|\leq 1$.

Proof. Let K be the unit ball in B_1''. If A is p-summing ($0<p\leq 2$) then A is 2-summing, and using the notation in the proof of Theorem 1 we have that for each $x\in B_1''$ the function $y\to\langle x, y\rangle$ defined on U_1 is in H_ν. Indeed, since the unit ball in B_1 is weak* dense in K [7, p. 114] and K is assumed to be metrizable, for each $x\in B_1''$ we can find a bounded sequence $\{x_n\}\subset B_1$ such that $\langle x_n, y\rangle\to\langle x, y\rangle$ for all $y\in U_1$ and therefore $x_n\to x$ in H_ν. It follows as in the proof of Theorem 1 that for each $x\in B_1$,

$$Ax = \sum_n \langle x, y_n\rangle x_n$$

where the sequence $\{x_n\}\subset B_2$ is 2-summable and the sequence $\{y_n\}\subset B_1'$ satisfies

$$\sum_n \langle x, y_n\rangle^2 = \int_{U_1} \langle x, y\rangle^2\, d\nu(y)$$

for each $x\in B_1''$. Now K is compact in the weak* topology and for each n the function $x\to\langle x, y_n\rangle$ is a continuous function on K which we denote by $f_n(x)$. Similarly, the function

$$x\to\int_{U_1} \langle x, y\rangle^2\, d\nu(y) \equiv g(x)$$

is continuous on K as K is metrizable. Since

$$\sum_{k=1}^{n} \langle x, y_k\rangle^2 = \sum_{K=1}^{n} f_k^2(x) \uparrow g(x)$$

it follows from Dini's theorem that the convergence is uniform. Hence

$$\lim_{m\to\infty} \sup_{\|x\|\leq 1} \sum_{n=m}^{\infty} \langle x, y_n\rangle^2 \leq \lim_{m\to\infty} \sup_{x\in K} \sum_{n=m}^{\infty} f_n^2(x) = 0$$

and the sequence $\{y_n\}$ is strongly 2-summable. Finally, note that

$$\|Ax-A_m x\|^2 = \sup_{\|y\|\leq 1} |\langle Ax, y\rangle-\langle A_m x, y\rangle|^2 = \sup_{\|y\|\leq 1} \left|\sum_{n=m}^{\infty} \langle x, y_n\rangle\langle x_n, y\rangle\right|^2$$

$$\leq \sup_{\|y\|\leq 1} \left(\sum_{n=m}^{\infty} \langle x, y_n\rangle^2\right)\left(\sum_{n=m}^{\infty} \langle x_n, y\rangle^2\right) \leq M\left(\sum_{n=m}^{\infty} \langle x, y_n\rangle^2\right)$$

where

$$M = \sup_{\|y\|\leq 1} \sum_{n=1}^{\infty} \langle x_n, y\rangle^2 < \infty.$$

Therefore

$$\|A-A_m\|^2 = \sup_{\|x\|\leq 1} \|Ax-A_m x\|^2 \leq M \sup_{\|x\|\leq 1} \left(\sum_{n=m}^{\infty} \langle x, y_n\rangle^2\right) \to 0$$

as $m\to\infty$ since $\{y_n\}$ is strongly 2-summable, and the proof is complete.

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Denote by U_1' the locally compact Hausdorff space $U_1\backslash\{0\}$. Let $\mu\geq 0$ be a Radon measure on U_1' (i.e., a Borel measure μ on U_1' such that $\mu(K)<\infty$ for all compact $K\subset U_1'$), having the property that

$$\int_{U_1'}\langle x, y\rangle^2\, d\mu(y) < \infty$$

for each $x\in B_1$. If $A:B_1\to B_2$ is a linear operator, we say that A is 2-integral bounded if there is a Radon measure μ as above such that

$$\|Ax\|^2 \leq \int_{U_1'}\langle x, y\rangle^2\, d\mu(y)$$

for all $x\in B_1$. This is a natural generalization of a 2-summing operator. The following theorem characterizes 2-integral bounded operators.

THEOREM 2. *An operator $A:B_1\to B_2$ is 2-integral bounded if and only if there is a sequence $\{y_n\}\subset B_1'$ such that*

$$\|Ax\|^2 \leq \sum \langle x, y_n\rangle^2 < \infty$$

for all $x\in B_1$. Moreover, every 2-integral bounded operator $A:B_1\to B_2$ can be written in the form

$$Ax = \sum_{n=1}^{\infty}\langle x, y_n\rangle x_n$$

where $\{x_n\}\subset B_2$ is 2-summable and $\{y_n\}\subset B_1'$ satisfies

$$\sum_n \langle x, y_n\rangle^2 \leq C\,\|x\|^2$$

for all $x\in B_1$. Here $C\geq 0$ is a finite constant.

Proof. For $x\in B_1$ denote by $f_x(y)$ the function $y\to\langle x, y\rangle$ on U_1'. We show first that if μ is a Radon measure on U_1' satisfying

$$\int_{U_1'}\langle x, y\rangle^2\, d\mu(y) = \mu(f_x^2) < \infty$$

for all $x\in B_1$, then there exists a finite constant $C\geq 0$ such that $\mu(f_x^2)\leq C\,\|x\|^2$ for all $x\in B_1$. Let H_μ be the closure of the linear subspace $\{f_x{:}x\in B_1\}$ in $L^2(U_1', \mu)$. Then H_μ is a separable Hilbert space and the map $x\to f_x$ is linear and everywhere defined from B_1 into H_μ. If $x_n\to x$ in B_1 and $f_{x_n}\to f$ in H_μ, then there is a subsequence $\{n'\}\subset\{n\}$ such that $f_{x_{n'}}\to f$ a.e. μ. But $f_{x_{n'}}(y)=\langle x_{n'}, y\rangle$ converges to $\langle x, y\rangle=f_x(y)$ everywhere. Hence $f_x(y)=f(y)$ a.e. μ, and therefore $f=f_x$ in H_μ. This shows that the map $x\to f_x$ has a closed graph and is therefore continuous. Hence there is a finite $C\geq 0$ such that $\mu(f_x^2)\leq C\,\|x\|^2$ for all $x\in B_1$. Suppose now $\|Ax\|^2\leq\mu(f_x^2)$ for all $x\in B_1$. Let $\{e_n\}_n$ be an orthonormal basis for H_μ so that

$$\int\langle x, y\rangle^2\, d\mu(y) = \mu(f_x^2) = \sum_n (f_x, e_n)^2$$

https://doi.org/10.4153/CMB-1974-087-5 Published online by Cambridge University Press

from Parseval's equality where (,) is the inner product in H_μ. But for each n, $|(f_x, e_n)|^2 \le (f_x, f_x)(e_n, e_n) = \mu(f_x^2) \le C\|x\|^2$ so that the map $x \to (f_x, e_n) = \langle x, y_n\rangle$ for some $y_n \in B_1'$. Thus

$$\|Ax\|^2 \le \mu(f_x^2) = \sum_n \langle x, y_n\rangle^2 \le C\|x\|^2$$

for all $x \in B_1$. Conversely, suppose

$$\|Ax\|^2 \le \sum_n \langle x, y_n\rangle^2 < \infty$$

for all $x \in B_1'$ where $\{y_n\} \in B_1'$. We may choose positive constants $\lambda_n > 0$ with $\lambda_n \downarrow 0$ such that $y_n' = \lambda_n y_n$ satisfies $y_n' \in U_1'$ and $\|y_n'\| \to 0$ as $n \to \infty$. Denoting by μ the Radon measure

$$\sum_{n=1}^{\infty} \lambda_n^{-2} \delta_{y_n'}$$

on U_1' ($\delta_{y_n'}$=unit mass at $\{y_n'\}$) we have

$$\|Ax\|^2 \le \sum_n \langle x, y_n\rangle^2 = \sum \lambda_n^{-2}\langle x, y_n'\rangle^2 = \mu(f_x^2)$$

and A is 2-integral bounded. This proves the first sentence in Theorem 2. Repeating the same proof as in Theorem 1 yields a representation of A in the form

$$Ax = \sum \langle x, y_n\rangle x_n$$

where $\{x_n\} \subset B_2$ is 2-summable and

$$\sum \langle x, y_n\rangle^2 = \mu(f_x^2) \le C\|x\|^2.$$

This completes the proof of Theorem 2.

REMARK. It is clear that the results obtained in this paper hold equally well for complex Banach spaces: one has only to replace expressions of the form $(\ )^2$ by $|\ |^2$ in the theorems and proofs.

The author would like to thank the referee for many helpful suggestions in clarifying the presentation of the paper.

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