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SOME SCHWARZ TYPE INEQUALITIES FOR SEQUENCES OF OPERATORS IN HILBERT SPACES

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We give some inequalities of Cauchy–Bunyakovsky–Schwarz type for sequences of bounded linear operators in Hilbert spaces with applications.

1. INTRODUCTION

Let $(H;(\cdot,\cdot))$ be a real or complex Hilbert space and $B(H)$ the Banach algebra of all bounded linear operators that map H into H.

We recall that a self-adjoint operator $A \in B(H)$ is positive in $B(H)$ if and only if $(Ax,x) \geqslant 0$ for any $x \in H$. The binary relation $A \geqslant B$ if and only if $A - B$ is a positive self-adjoint operator, is an *order relation* on $B(H)$. We remark that for any $A \in B(H)$ the operators $U := AA^*$ and $V := A^*A$ are positive self adjoint operators on H and $\|U\| = \|V\| = \|A\|^2$.

In [1], the author has proved the following inequality of Cauchy–Bunyakovsky–Schwarz type in the order of $B(H)$.

THEOREM 1. *Let $A_1, \dots, A_n \in B(H)$ and $z_1, \dots, z_n \in \mathbb{K}$ $(\mathbb{R}, \mathbb{C})$. Then the following inequality holds:*

$$\sum_{i=1}^{n} |z_i|^2 \sum_{i=1}^{n} A_i A_i^* \geqslant \left(\sum_{i=1}^{n} z_i A_i\right)\left(\sum_{i=1}^{n} \overline{z_i} A_i^*\right) \geqslant 0. \tag{1.1}$$

PROOF: For the sake of completeness, we give here a simple proof of this inequality. For any $i, j \in \{1, \dots, n\}$ one has in the order of $B(H)$:

$$(\overline{z_i} A_j - \overline{z_j} A_i)(\overline{z_i} A_j - \overline{z_j} A_i)^* \geqslant 0,$$

that is,

$$(\overline{z_i} A_j - \overline{z_j} A_i)(z_i A_j^* - z_j A_i^*) \geqslant 0,$$

where

$$|z_i|^2 A_j A_j^* + |z_j|^2 A_i A_i^* \geqslant \overline{z_i} z_j A_j A_i^* + \overline{z_j} z_i A_i A_j^* \tag{1.2}$$

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for any $i, j \in \{1, \dots, n\}$.

If we sum (1.2) over i from 1 to n we deduce

$$\left(\sum_{i=1}^{n} |z_i|^2\right) A_j A_j^* + |z_j|^2 \left(\sum_{i=1}^{n} A_i A_i^*\right) \geqslant z_j A_j \left(\sum_{i=1}^{n} \overline{z_i} A_i^*\right) + \left(\sum_{i=1}^{n} z_i A_i\right) \overline{z_j} A_j^*, \tag{1.3}$$

for any $j \in \{1, \dots, n\}$.

If we sum (1.3) over j from 1 to n, we deduce

$$\sum_{i=1}^{n} |z_i|^2 \sum_{j=1}^{n} A_j A_j^* + \sum_{j=1}^{n} |z_j|^2 \left(\sum_{i=1}^{n} A_i A_i^*\right) \geqslant \sum_{j=1}^{n} z_j A_j \sum_{i=1}^{n} \overline{z_i} A_i^* + \left(\sum_{i=1}^{n} z_i A_i\right) \left(\sum_{j=1}^{n} \overline{z_j} A_j^*\right), \tag{1.4}$$

that is,

$$\sum_{k=1}^{n} |z_k|^2 \sum_{k=1}^{n} A_k A_k^* \geqslant \sum_{k=1}^{n} z_k A_k \sum_{k=1}^{n} \overline{z_k} A_k^* = \left(\sum_{k=1}^{n} z_k A_k\right) \left(\sum_{k=1}^{n} z_k A_k\right)^* \geqslant 0, \tag{1.5}$$

and the theorem is proved. ∎

The following version of the Cauchy–Bunyakovsky–Schwarz inequality for norms also holds [**1**].

Corollary 1. *With the assumptions in Theorem 1, one has*

$$\sum_{k=1}^{n} |z_k|^2 \left\| \sum_{k=1}^{n} A_k A_k^* \right\| \geqslant \left\| \sum_{k=1}^{n} z_k A_k \right\|^2. \tag{1.6}$$

Proof: The operators:

$$A := \sum_{k=1}^{n} |z_k|^2 \sum_{k=1}^{n} A_k A_k^*, \quad B := \left(\sum_{k=1}^{n} z_k A_k\right) \left(\sum_{k=1}^{n} \overline{z_k} A_k^*\right)$$

are obviously self-adjoint, positive and by (1.1), $A \geqslant B \geqslant 0$. Thus $\|A\| \geqslant \|B\|$ and since,

$$\|A\| = \sum_{k=1}^{n} |z_k|^2 \left\| \sum_{k=1}^{n} A_k A_k^* \right\|$$

and

$$\|B\| = \left\| \sum_{k=1}^{n} z_k A_k \right\|^2$$

the corollary is proved. ∎

For other related results, see [**2**].

The main aim of this paper is to point out other inequalities similar to (1.6).

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2. Norm Inequalities

The following result holds.

Theorem 2. *Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $A_1, \dots, A_n \in B(H)$. Then one has the inequalities:*

$$\left\|\sum_{i=1}^{n} \alpha_i A_i\right\|^2 \leqslant \begin{cases} \max\limits_{i=\overline{1,n}} |\alpha_i|^2 \sum\limits_{i=1}^{n} \|A_i\|^2 \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2p}\right)^{1/p} \left(\sum\limits_{i=1}^{n} \|A_i\|^{2q}\right)^{1/q} & \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \sum\limits_{i=1}^{n} |\alpha_i|^2 \max\limits_{i=\overline{1,n}} \|A_i\|^2 \end{cases}$$
$$+ \begin{cases} \max\limits_{1\leqslant i\neq j\leqslant n} \{|\alpha_i|\,|\alpha_j|\} \sum\limits_{1\leqslant i\neq j\leqslant n} \|A_i A_j^*\| \\ \left[\left(\sum\limits_{i=1}^{n} |\alpha_i|^r\right)^2 - \sum\limits_{i=1}^{n} |\alpha_i|^{2r}\right]^{1/r} \left(\sum\limits_{1\leqslant i\neq j\leqslant n} \|A_i A_j^*\|^s\right)^{1/s} \\ \text{if } r > 1,\ \frac{1}{r} + \frac{1}{s} = 1; \\ \left[\left(\sum\limits_{i=1}^{n} |\alpha_i|\right)^2 - \sum\limits_{i=1}^{n} |\alpha_i|^2\right] \max\limits_{1\leqslant i\neq j\leqslant n} \|A_i A_j^*\|, \end{cases} \tag{2.1}$$

where (2.1) *should be seen as all the 9 possible configurations.*

Proof: We have

$$\begin{aligned} 0 &\leqslant \left(\sum_{i=1}^{n} \alpha_i A_i\right)\left(\sum_{i=1}^{n} \alpha_i A_i\right)^* = \left(\sum_{i=1}^{n} \alpha_i A_i\right)\left(\sum_{j=1}^{n} \overline{\alpha_j} A_j^*\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha_j} A_i A_j^* = \sum_{i=1}^{n} |\alpha_i|^2 A_i A_i^* + \sum_{1\leqslant i\neq j\leqslant n} \alpha_i \overline{\alpha_j} A_i A_j^*. \end{aligned} \tag{2.2}$$

Taking the norm in (2.2) and observing that $\|UU^*\| = \|U\|^2$ for any $U \in B(H)$, one has the inequality

$$\begin{aligned} \left\|\sum_{i=1}^{n} \alpha_i A_i\right\|^2 &= \left\|\sum_{i=1}^{n} |\alpha_i|^2 A_i A_i^* + \sum_{1\leqslant i\neq j\leqslant n} \alpha_i \overline{\alpha_j} A_i A_j^*\right\| \\ &\leqslant \sum_{i=1}^{n} |\alpha_i|^2 \|A_i A_i^*\| + \sum_{1\leqslant i\neq j\leqslant n} |\alpha_i|\,|\alpha_j|\,\|A_i A_j^*\| \\ &= \sum_{i=1}^{n} |\alpha_i|^2 \|A_i\|^2 + \sum_{1\leqslant i\neq j\leqslant n} |\alpha_i|\,|\alpha_j|\,\|A_i A_j^*\|. \end{aligned} \tag{2.3}$$

Using Hölder's inequality, we may write that:

$$(2.4)\qquad \sum_{i=1}^{n}|\alpha_i|^2\|A_i\|^2 \leqslant \begin{cases} \max\limits_{i=\overline{1,n}}|\alpha_i|^2\sum\limits_{i=1}^{n}\|A_i\|^2 \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{2p}\right)^{1/p}\left(\sum\limits_{i=1}^{n}\|A_i\|^{2q}\right)^{1/q} & \text{if } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \sum\limits_{i=1}^{n}|\alpha_i|^2\max\limits_{i=\overline{1,n}}\|A_i\|^2. \end{cases}$$

Also, Hölder's inequality for double sums produces

$$(2.5)\qquad \sum_{1\leqslant i\neq j\leqslant n}|\alpha_i||\alpha_j|\left\|A_iA_j^*\right\| \leqslant \begin{cases} \max\limits_{1\leqslant i\neq j\leqslant n}\{|\alpha_i||\alpha_j|\}\sum\limits_{1\leqslant i\neq j\leqslant n}\left\|A_iA_j^*\right\| \\ \left(\sum\limits_{1\leqslant i\neq j\leqslant n}|\alpha_i|^r|\alpha_j|^r\right)^{1/r}\left(\sum\limits_{1\leqslant i\neq j\leqslant n}\left\|A_iA_j^*\right\|^s\right)^{1/s} \\ \text{if } r>1,\ \dfrac{1}{r}+\dfrac{1}{s}=1; \\ \sum\limits_{1\leqslant i\neq j\leqslant n}|\alpha_i||\alpha_j|\max\limits_{1\leqslant i\neq j\leqslant n}\left\|A_iA_j^*\right\|, \end{cases}$$

$$= \begin{cases} \max\limits_{1\leqslant i\neq j\leqslant n}\{|\alpha_i||\alpha_j|\}\sum\limits_{1\leqslant i\neq j\leqslant n}\left\|A_iA_j^*\right\| \\ \left[\left(\sum\limits_{i=1}^{n}|\alpha_i|^r\right)^2-\sum\limits_{i=1}^{n}|\alpha_i|^{2r}\right]^{1/r}\left(\sum\limits_{1\leqslant i\neq j\leqslant n}\left\|A_iA_j^*\right\|^s\right)^{1/s} \\ \text{if } r>1,\ \dfrac{1}{r}+\dfrac{1}{s}=1; \\ \left[\left(\sum\limits_{i=1}^{n}|\alpha_i|\right)^2-\sum\limits_{i=1}^{n}|\alpha_i|^2\right]\max\limits_{1\leqslant i\neq j\leqslant n}\left\|A_iA_j^*\right\|, \end{cases}$$

Using (2.3) and (2.4), (2.5) one deduces the desired inequality (2.1). ∎

The following corollaries are natural consequences.

Corollary 2. *With the assumptions of Theorem 2, one has the inequality*

$$(2.6)\qquad \left\|\sum_{i=1}^{n}\alpha_iA_i\right\| \leqslant \max_{i=\overline{1,n}}|\alpha_i|\left(\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\|\right)^{1/2}.$$

Proof: Follows by the first line in (2.1) on taking into account that

$$\max_{1\leqslant i\neq j\leqslant n}\{|\alpha_i||\alpha_j|\} \leqslant \max_{i=\overline{1,n}}|\alpha_i|^2,$$

and

$$\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\| = \sum_{i=1}^{n}\|A_i\|^2 + \sum_{1\leqslant i\neq j\leqslant n}\left\|A_iA_j^*\right\|.$$ ∎

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Corollary 3. *With the assumptions in Theorem 2, one has the inequality:*

$$\left\|\sum_{i=1}^{n}\alpha_i A_i\right\|^2 \leqslant \left(\sum_{i=1}^{n}|\alpha_i|^{2p}\right)^{1/p}\left[\left(\sum_{i=1}^{n}\|A_i\|^{2q}\right)^{1/q} + (n-1)^{1/p}\left(\sum_{1\leqslant i\neq j\leqslant n}\|A_iA_j^*\|^q\right)^{1/q}\right], \tag{2.7}$$

where $p > 1$, $1/p + 1/q = 1$.

Proof: Using the Cauchy–Bunyakovsky–Schwarz inequality for positive numbers

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leqslant n\sum_{i=1}^{n} a_i^2$$

we may write that

$$\left(\sum_{i=1}^{n}|\alpha_i|^p\right)^2 - \sum_{i=1}^{n}|\alpha_i|^{2p} \leqslant n\sum_{i=1}^{n}|\alpha_i|^{2p} - \sum_{i=1}^{n}|\alpha_i|^{2p} = (n-1)\sum_{i=1}^{n}|\alpha_i|^{2p}.$$

Now, using the second line in (2.1) for $r = p$, $s = q$, we deduce the desired result (2.7). ∎

Corollary 4. *With the assumptions in Theorem 2, one has the inequality*

$$\left\|\sum_{i=1}^{n}\alpha_i A_i\right\|^2 \leqslant \sum_{i=1}^{n}|\alpha_i|^2\left[\max_{i=\overline{1,n}}\|A_i\|^2 + (n-1)\max_{1\leqslant i\neq j\leqslant n}\|A_iA_j^*\|\right]. \tag{2.8}$$

Proof: Follows by the third line of (2.1) on taking into account that

$$\left(\sum_{i=1}^{n}|\alpha_i|\right)^2 - \sum_{i=1}^{n}|\alpha_i|^2 \leqslant (n-1)\sum_{i=1}^{n}|\alpha_i|^2.$$

∎

Another interesting particular case is embodied in the following corollary as well.

Corollary 5. *With the assumptions in Theorem 2, one has the inequality*

$$\left\|\sum_{i=1}^{n}\alpha_i A_i\right\|^2 \leqslant \sum_{i=1}^{n}|\alpha_i|^2\left[\max_{i=\overline{1,n}}\|A_i\|^2 + \left(\sum_{1\leqslant i\neq j\leqslant n}\|A_iA_j^*\|^2\right)^{1/2}\right]. \tag{2.9}$$

Proof: It is obvious that

$$\left[\left(\sum_{i=1}^{n}|\alpha_i|^2\right)^2 - \sum_{i=1}^{n}|\alpha_i|^4\right]^{1/2} \leqslant \sum_{i=1}^{n}|\alpha_i|^2.$$

Thus, combining the third line in the first bracket in (2.1) with the second line for $r = s = 2$ in the second bracket, the inequality (2.9) is obtained. ∎

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REMARK 1. If one is interested in obtaining bounds in terms of $\sum_{i=1}^{n}|\alpha_i|^2$, there are other possibilities as shown below. Obviously, since

$$\max_{1\leqslant i\neq j\leqslant n}\{|\alpha_i|\,|\alpha_j|\}\leqslant \max_{i=\overline{1,n}}|\alpha_i|^2\leqslant \sum_{i=1}^{n}|\alpha_i|^2.$$

then, by (2.1), in choosing the third line in the first bracket with the first line in the second bracket, one would obtain

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|^2\leqslant \sum_{i=1}^{n}|\alpha_i|^2\left[\max_{i=\overline{1,n}}\|A_i\|^2+\sum_{1\leqslant i\neq j\leqslant n}\|A_iA_j^*\|\right]. \tag{2.10}$$

Also, it is evident that

$$\left[\left(\sum_{i=1}^{n}|\alpha_i|^r\right)^2-\sum_{i=1}^{n}|\alpha_i|^{2r}\right]^{1/r}\leqslant\left(\sum_{i=1}^{n}|\alpha_i|^r\right)^{2/r}$$

By the monotonicity of the power mean $\left(\left(\sum_{i=1}^{n}\Big/n\right)a_i^m\right)^{1/m}$ as a function of $m\in\mathbb{R}$, we have

$$\left(\frac{\sum_{i=1}^{n}|\alpha_i|^r}{n}\right)^{1/r}\leqslant\left(\frac{\sum_{i=1}^{n}|\alpha_i|^2}{n}\right)^{1/2},\quad 1<r\leqslant 2,$$

giving

$$\left(\sum_{i=1}^{n}|\alpha_i|^r\right)^{2/r}\leqslant n^{2/r-1}\sum_{i=1}^{n}|\alpha_i|^2.$$

Thus, using the third line in the first bracket of (2.1) combined with the second line in the second bracket for $1<r\leqslant 2$, $1/s+1/r=1$, we deduce

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|^2\leqslant \sum_{i=1}^{n}|\alpha_i|^2\left[\max_{i=\overline{1,n}}\|A_i\|^2+n^{(2/r)-1}\left(\sum_{1\leqslant i\neq j\leqslant n}\|A_iA_j^*\|^s\right)^{1/s}\right]. \tag{2.11}$$

Note that for $r=s=2$, we recapture (2.9).

The following particular result also holds.

PROPOSITION 1. *Let $\alpha_1,\dots,\alpha_n\in\mathbb{K}$ and $A_1,\dots,A_n\in B(H)$ with the property that $A_iA_j^*=0$ for any $i\neq j$, $i,j\in\{1,\dots,n\}$. Then one has the inequality;*

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\|\leqslant\begin{cases}\max_{i=\overline{1,n}}|\alpha_i|\left(\sum_{i=1}^{n}\|A_i\|^2\right)^{1/2},\\ \left(\sum_{i=1}^{n}|\alpha_i|^{2p}\right)^{1/(2p)}\left(\sum_{i=1}^{n}\|A_i\|^{2q}\right)^{1/(2q)} & \text{if } p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1;\\ \left(\sum_{i=1}^{n}|\alpha_i|^2\right)^{1/2}\max_{i=\overline{1,n}}\|A_i\|.\end{cases} \tag{2.12}$$

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If by $M(\alpha, \mathbf{A})$ we denote any of the bounds provided by (2.1), (2.6), (2.7), (2.8), (2.9), (2.10) or (2.11), then we may state the following proposition as well.

Proposition 2. *Under the assumptions of Theorem 2, we have:*

(i) *For any $x \in H$*

$$\left\| \sum_{i=1}^{n} \alpha_i A_i x \right\|^2 \leqslant \|x\|^2 M(\alpha, \mathbf{A}). \tag{2.13}$$

(ii) *For any $x, y \in H$,*

$$\left| \sum_{i=1}^{n} \alpha_i \langle A_i x, y \rangle \right|^2 \leqslant \|x\|^2 \|y\|^2 M(\alpha, \mathbf{A}). \tag{2.14}$$

Proof:

(i) Obviously,

$$\begin{aligned} \left\| \sum_{i=1}^{n} \alpha_i A_i x \right\|^2 = \left\| \left(\sum_{i=1}^{n} \alpha_i A_i \right)(x) \right\|^2 \leqslant \left\| \sum_{i=1}^{n} \alpha_i A_i \right\|^2 \|x\|^2 \\ \leqslant M(\alpha, \mathbf{A}) \|x\|^2. \end{aligned}$$

(ii) We have

$$\left| \sum_{i=1}^{n} \alpha_i \langle A_i x, y \rangle \right|^2 = \left| \left\langle \sum_{i=1}^{n} \alpha_i A_i x, y \right\rangle \right|^2 \leqslant \left\| \sum_{i=1}^{n} \alpha_i A_i x \right\|^2 \|y\|^2,$$

which, by (i), gives the desired result (2.14). ∎

3. Inequalities for Vectors in Hilbert Spaces

We consider the non zero vectors $y_1, \dots, y_n \in H$. Define the operators

$$A_i : H \to H, \quad A_i x = \frac{(x, y_i)}{\|y_i\|} \cdot y_i, \quad i \in \{1, \dots, n\}.$$

Since

$$\|A_i\| = \sup_{\|x\|=1} \|A_i x\| = \sup_{\|x\|=1} \left|(x, y_i)\right| = \|y_i\|, \quad i \in \{1, \dots, n\} \tag{3.1}$$

then A_i are bounded linear operators in H. Also, since

$$(A_i x, x) = \left(\frac{(x, y_i) y_i}{\|y_i\|}, x \right) = \frac{|(x, y_i)|^2}{\|y_i\|} \geqslant 0, \quad x \in H, \quad i \in \{1, \dots, n\} \tag{3.2}$$

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and

$$(A_i x, z) = \left(\frac{(x, y_i) y_i}{\|y_i\|}, z\right) = \frac{(x, y_i)(y_i, z)}{\|y_i\|},$$
$$(x, A_i z) = \left(x, \frac{(z, y_i) y_i}{\|y_i\|}\right) = \frac{(x, y_i)\overline{(z, y_i)}}{\|y_i\|} = \frac{(x, y_i)(y_i, z)}{\|y_i},$$

giving

$$(3.3) \qquad (A_i x, z) = (x, A_i z), \quad x, z \in H, \quad i \in \{1, \ldots, n\},$$

we may conclude that A_i $(i = 1, \ldots, n)$ are positive self-adjoint operators on H.

Since, for any $x \in H$, one has

$$\begin{aligned} \|(A_i A_j)(x)\| &= \|(A_i)(A_j x)\| = \left\|A_i\left(\frac{(x, y_j) y_j}{\|y_j\|}\right)\right\| \\ &= \frac{|(x, y_j)|}{\|y_j\|}\|A_i y_j\| = \frac{|(x, y_j)|}{\|y_j\|} \cdot \frac{|(y_j, y_i)|\|y_j\|}{\|y_j\|} \\ &= \frac{|(x, y_j)|\,|(y_j, y_i)|}{\|y_j\|}, \quad i, j \in \{1, \ldots, n\}, \end{aligned}$$

we deduce that

$$(3.4) \qquad \|A_i A_j\| = \sup_{\|x\|=1} \frac{|(x, y_j)||(y_j, y_i)|}{\|y_j\|} = |(y_i, y_j)|; \quad i, j \in \{1, \ldots, n\}.$$

If $(y_i)_{i=\overline{1,n}}$ is an orthogonal family on H, then $\|A_i\| = 1$ and $A_i A_j = 0$ for $i, j \in \{1, \ldots, n\}$, $i \neq j$.

The following inequality for vectors holds.

THEOREM 3. *Let $x, y_1, \ldots, y_n \in H$ and $\alpha_n, \ldots, \alpha_n \in \mathbb{K}$. Then one has the inequalities:*

$$(3.5) \quad \left\|\sum_{i=1}^{n} \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i\right\|^2 \leqslant \|x\|^2 \times \begin{cases} \max\limits_{i=\overline{1,n}} |\alpha_i|^2 \sum\limits_{i=1}^{n} \|y_i\|^2 \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2p}\right)^{1/p} \left(\sum\limits_{i=1}^{n} \|y_i\|^{2q}\right)^{1/q} & \text{if } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \sum\limits_{i=1}^{n} |\alpha_i|^2 \max\limits_{i=\overline{1,n}} \|y_i\|^2 \end{cases}$$

$$+\|x\|^2 \times \begin{cases} \max\limits_{1\leqslant i\neq j\leqslant n}\{|\alpha_i|\,|\alpha_j|\} \sum\limits_{1\leqslant i\neq j\leqslant n} |(y_i,y_j)| \\ \left[\left(\sum\limits_{i=1}^n |\alpha_i|^r\right)^2 - \sum\limits_{i=1}^n |\alpha_i|^{2r}\right]^{1/r} \left(\sum\limits_{1\leqslant i\neq j\leqslant n} |(y_i,y_j)|^s\right)^{1/s} \\ \textit{if } r>1,\ \dfrac{1}{r}+\dfrac{1}{s}=1; \\ \left[\left(\sum\limits_{i=1}^n |\alpha_i|\right)^2 - \sum\limits_{i=1}^n |\alpha_i|^2\right] \max\limits_{1\leqslant i\neq j\leqslant n} |(y_i,y_j)|. \end{cases}$$

PROOF: Follows by Theorem 2 and Proposition 2, (i) on choosing $A_i = ((\cdot, y_i)/\|y_i\|)y_i$ and taking into account that $\|A_i\| = \|y_i\|$,

$$\|A_iA_j^*\| = |(y_i,y_j)|, \qquad i,j \in \{1,\dots,n\}.$$

We omit the details. ∎

Using Corollaries 2–5 and Remark 1, we may state the following particular inequalities:

$$\left\|\sum_{i=1}^n \alpha_i \frac{(x,y_i)}{\|y_i\|} y_i\right\| \leqslant \|x\| \max_{i=\overline{1,n}} |\alpha_i| \left(\sum_{i,j=1}^n |(y_i,y_j)|\right)^{1/2}; \tag{3.6}$$

$$\left\|\sum_{i=1}^n \alpha_i \frac{(x,y_i)}{\|y_i\|} y_i\right\|^2 \leqslant \|x\|^2 \left[\left(\sum_{i=1}^n |\alpha_i|^{2p}\right)^{1/p} \left[\left(\sum_{i=1}^n \|y_i\|^{2q}\right)^{1/q} + (n-1)^{1/p}\left(\sum_{1\leqslant i\neq j\leqslant n} |(y_i,y_j)|^q\right)^{1/q}\right]\right], \tag{3.7}$$

where $p>1$, $1/p+1/q=1$;

$$\left\|\sum_{i=1}^n \alpha_i \frac{(x,y_i)}{\|y_i\|} y_i\right\|^2 \leqslant \|x\|^2 \sum_{i=1}^n |\alpha_i|^2 \left[\max_{i=\overline{1,n}} \|y_i\|^2 + (n-1) \max_{1\leqslant i\neq j\leqslant n} |(y_i,y_j)|\right]; \tag{3.8}$$

$$\left\|\sum_{i=1}^n \alpha_i \frac{(x,y_i)}{\|y_i\|} y_i\right\|^2 \leqslant \|x\|^2 \sum_{i=1}^n |\alpha_i|^2 \left[\max_{i=\overline{1,n}} \|y_i\|^2 + \left(\sum_{1\leqslant i\neq j\leqslant n} |(y_i,y_j)|^2\right)^{1/2}\right]; \tag{3.9}$$

$$\left\|\sum_{i=1}^n \alpha_i \frac{(x,y_i)}{\|y_i\|} y_i\right\|^2 \leqslant \|x\|^2 \sum_{i=1}^n |\alpha_i|^2 \left[\max_{i=\overline{1,n}} \|y_i\|^2 + \sum_{1\leqslant i\neq j\leqslant n} |(y_i,y_j)|\right]; \tag{3.10}$$

$$\left\|\sum_{i=1}^n \alpha_i \frac{(x,y_i)}{\|y_i\|} y_i\right\|^2 \leqslant \|x\|^2 \sum_{i=1}^n |\alpha_i|^2 \left[\max_{i=\overline{1,n}} \|y_i\|^2 + n^{(2/r)-1}\left(\sum_{1\leqslant i\neq j\leqslant n} |(y_i,y_j)|^s\right)^{1/s}\right], \tag{3.11}$$

where $1<r\leqslant 2$, $1/s+1/r=1$.

REMARK 2. The choice $\alpha_i = \|y_i\|$ $(i=1,\dots,n)$ will produce some interesting bounds for

$$\left\|\sum_{i=1}^n (x,y_i)y_i\right\|^2.$$

We omit the details.

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