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TWISTED GROUP RINGS WHOSE UNITS FORM AN FC-GROUP

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ABSTRACT. Let $U(K_\lambda G)$ be the group of units of the infinite twisted group algebra $K_\lambda G$ over a field K. We describe the FC-centre ΔU of $U(K_\lambda G)$ and give a characterization of the groups G and fields K for which $U(K_\lambda G) = \Delta U$. In the case of group algebras we obtain the Cliff-Sehgal-Zassenhaus theorem.

1. **Introduction.** Let G be a group, K a field and $\lambda\colon G \times G \longmapsto U(K)$ a 2-cocycle of G with respect to the trivial action of G. Then the twisted group algebra $K_\lambda G$ of G over the field K is an associative K-algebra with basis $\{u_g \mid g \in G\}$ and with multiplication defined for all $g, h \in G$ by

$$u_g u_h = \lambda_{g,h} u_{gh}, \quad (\lambda_{g,h} \in \lambda)$$

and using distributivity.

Let $U(K_\lambda G)$ be the group of units of $K_\lambda G$ and let ΔU be its subgroup consisting of all elements with finitely many conjugates in $U(K_\lambda G)$. This subgroup ΔU is called the FC-*centre* of $U(K_\lambda G)$. Clearly, if $\Delta U = U(K_\lambda G)$, then $U(K_\lambda G)$ is an FC-group (group with finite conjugacy classes).

The problem to study the group of units of group rings with FC property was posed by S. K. Sehgal and H. J. Zassenhaus [1]. For a field K of characteristic 0 they described all groups G without subgroups of type p^∞ for which the group of units of the group algebra of G over K is an FC-group. This was spelling for arbitrary groups by H. Cliff and S. K. Sehgal [2].

In this paper we describe the subgroup ΔU when $K_\lambda G$ is infinite. Let $t(\Delta U)$ be the group of all elements of finite order of ΔU. Then ΔU is a solvable group of length at most 3 and the subgroup $t(\Delta U)$ is nilpotent of class at most 2. This is new even for group algebras. We use this result for the characterization of those cases when $U(K_\lambda G)$ has FC property, and obtain a generalization of the Cliff-Sehgal-Zassenhaus theorem for twisted group algebras.

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2. **The** FC**-centre of** $U(K_\lambda G)$**.** By a theorem of B. H. Neumann [3] the elements of finite order in ΔU form a normal subgroup which we denote by $t(\Delta U)$, and the factorgroup $\Delta U/t(\Delta U)$ is a torsion free abelian group. Evidently, $\bar{G} = \{\kappa u_a \mid \kappa \in U(K), a \in G\}$ is a subgroup in $U(K_\lambda G)$, while $U(K)$ is a normal subgroup in $\bar{G}$, with factorgroup $\bar{G}/U(K)$ isomorphic to G.

If x is a nilpotent element of the ring $K_\lambda G$ then the element $y = 1 + x$ is a unit in $K_\lambda G$ and is referred to as a unipotent element of $U(K_\lambda G)$.

Let $\zeta(G)$ be the centre of the group G and $[g,h] = g^{-1}h^{-1}gh$ $(g, h \in G)$.

LEMMA 1. *Let $K_\lambda G$ be an infinite twisted group algebra. Then all unipotent elements of the subgroup ΔU are central in ΔU.*

PROOF. Let $y = 1 + x$ be a unipotent element of ΔU and $v \in \Delta U$. Then for a positive integer k we have $x^k = 0$ and by induction on k we will prove $vx = xv$.

The subgroup $\bar{G} = \{\kappa u_a \mid \kappa \in U(K), a \in G\}$ is infinite and by Poincaré's theorem the centralizer S of the subset $\{v, y\}$ of $\bar{G}$ is a subgroup of finite index in $\bar{G}$. Since $\bar{G}$ is infinite, S is infinite and $fy = yf$ for all $f \in S$. Then xf is nilpotent and $1 + xf$ is a unit in $K_\lambda G$. We can see easily that the set $\{(1 + xf)^{-1}v(1 + xf) \mid f \in S\}$ is finite. Let $v_1, \ldots, v_t$ be the elements of this set and $W_i = \{f \in S \mid (1 + xf)^{-1}v(1 + xf) = v_i\}$. Then $S = \cup W_i$ and there exists an index j such that W_j is infinite. Fix an element $f \in W_j$. Any element $q \in W_j$, $q \neq f$ satisfies

$$(1 + xf)^{-1}v(1 + xf) = (1 + xq)^{-1}v(1 + xq)$$

and

$$v(1 + xf)(1 + xq)^{-1} = (1 + xf)(1 + xq)^{-1}v.$$

Then

$$v\{(1 + xq) + (xf - xq)\}(1 + xq)^{-1} = \{(1 + xq) + (xf - xq)\}(1 + xq)^{-1}v,$$
$$v\big(1 + x(f - q)(1 + xq)^{-1}\big) = \big(1 + x(f - q)(1 + xq)^{-1}\big)v$$

and

$$vx(f - q)(1 + xq)^{-1} = x(f - q)(1 + xq)^{-1}v. \tag{1}$$

Now we use the induction mentioned above. For $k = 1$ the statement is trivial; so we suppose it is true for all $1 \leq n < k$, where $k \geq 2$ is any given integer.

If $m \geq 2$, then by induction hypothesis $x^m v = vx^m$ for all $v \in \Delta U$. Clearly, if $i \geq 1$ then

$$x(f - q)x^i q^i v = (f - q)x^{i+1}q^i v = (f - q)vx^{i+1}q^i = vx(f - q)x^i q^i.$$

From (1) we have

$$vx(f - q)\big(1 - xq + x^2q^2 + \cdots + (-1)^{k-1}x^{k-1}q^{k-1}\big)$$
$$= x(f - q)\big(1 - xq + x^2q^2 + \cdots + (-1)^{k-1}x^{k-1}q^{k-1}\big)v.$$

So $(f-q)(vx-xv)=0$.

Now suppose $xv \neq vx$. The element $q^{-1}f \in \bar{G}$ can be written as λu_h $\big(\lambda \in U(K), h \in G\big)$. By $vx - xv = \sum_{i=1}^{s} \alpha_i u_{g_i} \neq 0$ we have

$$\sum_{i=1}^{s} \lambda \alpha_i u_h u_{g_i} - \sum_{i=1}^{s} \alpha_i u_{g_i} = 0.$$

If $h \in G$ satisfies this equation, then $g_1 = hg_j$ for some j, and the number of such elements h is finite. Since $W_j = \{\lambda u_h \mid \lambda \in U(K)\}$ is an infinite set, there exist h and different elements $\lambda_1, \lambda_2 \in K$ such that $\lambda_1 u_h, \lambda_2 u_h \in W_j$. Then $(\lambda_i u_h - 1)(vx-xv) = 0$, $(i = 1,2)$ and we obtain $(\lambda_1 u_h - \lambda_2 u_h)(vx-xv) = 0$. This condition is satisfied only if $vx = xv$ but does not hold. ∎

LEMMA 2. *Let $K_\lambda G$ be an infinite twisted group algebra, H a finite subgroup of ΔU and L_H the subalgebra of $K_\lambda G$ generated by H. Then the group of units $U(L_H)$ of the algebra L_H is contained in ΔU, and the factorgroup $U(L_H)/\big(1+J(L_H)\big)$ is abelian.*

PROOF. If H is a finite subgroup of ΔU and L_H is the subalgebra of $K_\lambda G$ generated by H, then L_H is an algebra of finite rank over K and its radical $J(L_H)$ is nilpotent. Then $U(L_H)$ is a subgroup of ΔU and by Lemma 1 all unipotent elements of $U(L_H)$ are central in ΔU. Therefore $1 + J(L_H)$ is a central subgroup of ΔU and $J(L_H) \subset \zeta(L_H)$, where $\zeta(L_H)$ is the centre of L_H. Then by Theorem 48.3 in [4] (p. 209)

$$L_H = L_H e_1 \oplus \cdots \oplus L_H e_n \oplus N, \tag{2}$$

where $L_H e_i$ is a semiprime algebra (*i.e.* $L_H e_i / J(L_H e_i)$ is a division ring), N is a commutative artinian radical algebra, $e_1, \ldots, e_n$ are pairwise orthogonal idempotents. By Lemma 13.2 in [4] (p. 57) any idempotent e_i is central in L_H and $U(L_H e_i)$ is isomorphic to the subgroup $\langle 1 - e_i + z e_i \mid z \in U(L_H) \rangle$ of $U(L_H)$.

Since $U(L_H e_i)$ is a subgroup of the FC-group ΔU it is an FC-group, too. As $J(L_H e_i)$ is nilpotent (see [5]),

$$U(L_H e_i)/\big(1 + J(L_H e_i)\big) \cong U\big(L_H e_i / J(L_H e_i)\big). \tag{3}$$

By Scott's theorem [7], in the skewfield $L_H e_i / J(L_H e_i)$ every nonzero element is either central or its conjugacy class is infinite. Thus the FC-group $U(L_H e_i)/\big(1 + J(L_H e_i)\big)$ is abelian.

Decomposition (2) implies

$$L_H / J(L_H) \cong L_H e_1 / J(L_H e_1) \oplus \cdots \oplus L_H e_n / J(L_H e_n)$$

and

$$U(L_H)/\big(1 + J(L_H)\big) \cong U\big(L_H / J(L_H)\big) \cong U\big(L_H e_1 / J(L_H e_1)\big) \times \cdots \times U\big(L_H e_n / J(L_H e_n)\big).$$

Therefore $U(L_H)/\big(1 + J(L_H)\big)$ is abelian. ∎

THEOREM 1. *Let $K_\lambda G$ be an infinite twisted group algebra and $t(\Delta U)$ the subgroup of ΔU consisting of all elements of finite order in ΔU. Then all elements of the commutator subgroup of $t(\Delta U)$ are unipotent and central in ΔU.*

PROOF. Let H be a finite subgroup of $t(\Delta U)$ and L_H the subalgebra of $K_\lambda G$, generated by H. Then the elements of the subgroup $H_1 = H \cap \big(1 + J(L_H)\big)$ are unipotent and (by Lemma 1) central in ΔU. The subgroup $H\big(1 + J(L_H)\big)$ is contained in $U(L_H)$ and

$$H/H_1 = H/\Big(H \cap \big(1 + J(L_H)\big)\Big) \cong \Big(H\big(1 + J(L_H)\big)\Big)/\big(1 + J(L_H)\big).$$

By Lemma 2 the factorgroup $U(L_H)/\big(1 + J(L_H)\big)$ is abelian. So H/H_1 is abelian and the commutator subgroup of H is contained in H_1 and consists of unipotent elements.

Since the commutator subgroup of $t(\Delta U)$ is the union of the commutator subgroups of the finite subgroups of $t(\Delta U)$, all elements of the commutator subgroup of $t(\Delta U)$ are unipotent and central in ΔU. ∎

THEOREM 2. *Let $K_\lambda G$ be an infinite twisted group algebra where* char(K) *does not divide the order of any element of the subgroup ΔG. Then $t(\Delta U)$ is abelian.*

PROOF. Let H be a finite subgroup of the commutator subgroup of $t(\Delta U)$. Then (by Theorem 1) H is contained in the centre of ΔU. The set $\{u_g^{-1} H u_g \mid g \in \Delta G\}$ contains only a finite number of subgroups $H_1, H_2, \ldots, H_t$. The subgroup $L = H_1 H_2 \cdots H_t$ is finite and is invariant under inner automorphism $f_g(x) = u_g^{-1} x u_g$ of the ring $K_\lambda \Delta G$, where $g \in \Delta G$. Let $x_1, \ldots, x_s$ be all elements of L. Then $y_i = x_i - 1$ is a nilpotent element, and in the commutative ring L the elements $y_1, \ldots, y_s$ commute. Therefore

$$J \cong \Big\{\sum_{i=1}^{s} \alpha_i y_i \mid \alpha_i \in K, x_i = y_i + 1 \in L\Big\}$$

is a nilpotent subring. Let

$$F = \Big\{\sum_{i=1}^{s} \alpha_i y_i z_i \mid \alpha_i \in K, x_i = y_i + 1 \in L, z_i \in K_\lambda \Delta G\Big\}.$$

Let us prove that F is a nilpotent right ideal of $K_\lambda \Delta G$. If $z = \sum_j \beta_j u_{g_j} \in K_\lambda \Delta G$ then $y_i z = \sum_j \beta_j u_{g_j} u_{g_j}^{-1} y_i u_{g_j}$, and $u_{g_j}^{-1} y_i u_{g_j}$ equals one of the elements $y_1, \ldots, y_s$. This and the nilpotency of the ring J imply that F is a nilpotent ring. By Passman's theorem [6], if char(K) does not divide the order of any element of ΔG then $K_\lambda \Delta G$ does not contain nilideals. Therefore $F = 0, L = 1$ and the commutator subgroup $t(\Delta U)$ is trivial so $t(\Delta U)$ is abelian. ∎

COROLLARY. *Let $K_\lambda \Delta G$ be an infinite twisted group algebra. Then ΔU is a solvable group of length at most* 3, *and the subgroup $t(\Delta U)$ is nilpotent of class at most* 2.

3. **The** FC **property of** $U(K_\lambda G)$.

LEMMA 3. *Let L be a subfield of the twisted group algebra $K_\lambda G$, where K is a subfield of L, $g \in G$ an element of order n and*

$$\lambda_g = u_g^n = \lambda_{g,g}\lambda_{g,g^2}\cdots\lambda_{g,g^{n-1}}.$$

If $\alpha^n \neq \lambda_g$ for some $\alpha \in L$ and $\alpha u_g = u_g\alpha$ then $u_g - \alpha$ is a unit in $K_\lambda G$. Furthermore, if L is an infinite field then the number of such units is infinite.

PROOF. Let $\alpha \in L, \alpha^n \neq \lambda_g$ and $u_g\alpha = \alpha u_g$. Then $\lambda_g - \alpha^n$ is a nonzero element of L and

$$(\alpha^{n-1} + \alpha^{n-2}u_g + \cdots + \alpha u_g^{n-2} + u_g^{n-1})(\lambda_g - \alpha^n)^{-1}$$

is the inverse of $u_g - \alpha$. We know that the number of solutions of the equation $x^n - \lambda_g = 0$ in L does not exceed n. Thus in an infinite field L there are infinitely many elements not satisfying the equation $x^n - \lambda_g = 0$. ∎

LEMMA 4. *Let G be an infinite locally finite group where* char(K) *does not divide the order of any element of G. If $U(K_\lambda G)$ is an* FC*-group then G is abelian and $K_\lambda G$ is commutative.*

PROOF. Let W be a finite subgroup of G. Then the subalgebra $K_\lambda W$ is a semiprime artinian ring and by the Wedderburn-Artin theorem

$$K_\lambda W = M(n_1, D_1) \oplus \cdots \oplus M(n_t, D_t),$$

where each D_k is a skewfield and $M(n_k, D_k)$ is a full matrix algebra. Let $e_{i,j}, e_{j,i}$ be matrix units in $M(n_k, D_k)$ and $i \neq j$. Then the unipotent elements $1 + e_{i,j}$, $1 + e_{j,i}$ are central in $K_\lambda G$ (see Theorem 1) which is impossible if $i \neq j$. Thus $n_k = 1$ and $K_\lambda W$ is a direct sum of skewfields, $K_\lambda W = D_1 \oplus D_2 \oplus \cdots \oplus D_t$ and

$$U(K_\lambda W) = U(D_1) \times U(D_2) \times \cdots \times U(D_t).$$

By Scott's theorem [7] any nonzero element of a skewfield is either central or has an infinite number of conjugates. Therefore $K_\lambda W$ is a direct sum of fields and W is abelian. Since G is a locally finite group, G is abelian and $K_\lambda G$ is a commutative algebra. ∎

LEMMA 5. *Let $K_\lambda G$ be infinite and* char(K) *does not divide the order of any element of the normal torsion subgroup L of G. If $U(K_\lambda G)$ is an* FC*-group then all idempotents of $K_\lambda L$ are central in $K_\lambda G$.*

PROOF. Let the idempotent $e \in K_\lambda L$ be noncentral in $K_\lambda G$. Then there exists $g \in G$ such that $eu_g \neq u_g e$. The subgroup $H = \langle g^{-i}\,\mathrm{supp}(e) g^i \mid i \in \mathbb{Z}\rangle$ is finite and for any $a \in G$ the subalgebra $K_\lambda H$ of $K_\lambda L$ is invariant under the inner automorphism $\phi(x) = u_a^{-1}xu_a$. It is easy to see (by Lemma 4) that $K_\lambda H$ is a commutative semisimple K-algebra of finite rank and the idempotent $e \in K_\lambda H$ is a sum of primitive idempotents. Consequently, there exists a primitive idempotent f of $K_\lambda H$ which does not commute with u_g. Then the idempotents f and $u_g^{-1}fu_g$ are orthogonal and $(u_g f)^2 = u_g f u_g f = u_g^2(u_g^{-1}fu_g)f = 0$. By Theorem 1 the unipotent element $1 + u_g f$ commutes with u_g and $(1 + u_g f)u_g = u_g(1 + u_g f)$ implies $u_g f = fu_g$, which is impossible. Thus, all idempotents of $K_\lambda L$ are central in $K_\lambda G$. ∎

LEMMA 6. *Let $U(K_\lambda G)$ be an* FC*-group and $t(G)$ the set of all elements of finite order in G. Then*

1. *G is an* FC*-group;*
2. *if there exists an infinite subfield L in the centre of $K_\lambda G$ such that $L \supseteq K$ then $t(G)$ is central in G and $\lambda_{g,h} = \lambda_{h,g}$ $\big(h \in t(G), g \in G\big)$.*

PROOF. If $U(K_\lambda G)$ is an FC-group then $\bar{G} = \{\lambda u_g \mid \lambda \in U(K), g \in G\}$ is an FC-group. Clearly $U(K)$ is normal in $\bar{G}$ and $G \cong \bar{G}/U(K)$. We conclude that G is an FC-group as it is a homomorphic image of the FC-group $\bar{G}$.

Let L be an infinite field which satisfies condition 2 of the lemma. Then by Lemma 3 for any $h \in t(G)$ there exists a countable set $S = \{\alpha_i \in L \mid i \in \mathbb{Z}\}$ such that $u_h - \alpha_i$ is a unit for all $i \in \mathbb{Z}$. Suppose that $u_g u_h \neq u_h u_g$ for some $g \in G$. Next we observe that the equality

$$(u_h - \alpha_i)u_g(u_h - \alpha_i)^{-1} = (u_h - \alpha_j)u_g(u_g - \alpha_j)^{-1}$$

holds only in case $\alpha_i = \alpha_j$. Since

$$(u_h - \alpha_i)(u_h - \alpha_j)^{-1} = 1 + (\alpha_j - \alpha_i)(u_h - \alpha_j)^{-1},$$

we obtain $(\alpha_i - \alpha_j)(u_g u_h - u_h u_g) = 0$ and $\alpha_i = \alpha_j$. It follows that the set

$$\{(u_h - \alpha_j)u_g(u_h - \alpha_j)^{-1} \mid i \in \mathbb{Z}\}$$

is infinite which contradicts the condition that $U(K_\lambda G)$ is an FC-group. Then $u_g u_h = u_h u_g$, therefore $[g,h] = 1$, $t(G) \subseteq \zeta(G)$ and $\lambda_{g,h} = \lambda_{h,g}$ $\big(h \in t(G), g \in G\big)$. ∎

LEMMA 7. *Let G be an abelian torsion group, $K_\lambda G$ a commutative semisimple algebra and v an idempotent of $K_\lambda G$. If $K_\lambda Gv$ contains a finite number of idempotents then $K_\lambda Gv$ is a direct sum of finitely many fields.*

PROOF. If $e_1, \ldots, e_s$ are all the idempotents of $K_\lambda Gv$, then

$$L = \langle \operatorname{supp}(e_1), \ldots, \operatorname{supp}(e_t) \rangle$$

is a finite subgroup in G and $K_\lambda Lv$ is a direct sum of finitely many fields,

$$K_\lambda Lv = (K_\lambda Lv)f_1 \oplus \cdots \oplus (K_\lambda Lv)f_t,$$

where $f_1, \ldots, f_t$ are orthogonal primitive idempotents of $K_\lambda Lv$. The corresponding direct sum in $K_\lambda Gv$ is

$$K_\lambda Gv = (K_\lambda Gv)f_1 \oplus \cdots \oplus (K_\lambda Gv)f_t.$$

We show that the element $0 \neq x \in (K_\lambda Gv)f_i$ is a unit. $R = \langle L, \operatorname{supp}(x) \rangle$ is a finite subgroup and $K_\lambda Rv$ is a direct sum of finitely many fields,

$$K_\lambda Rv = (K_\lambda Rv)l_1 \oplus \cdots \oplus (K_\lambda Rv)l_t,$$

and each idempotent f_i is either equal to an idempotent l_j or is a sum of these idempotents. If $f_i = l_j$ then $xf_i \in (K_\lambda Rv)l_j$ and x is a unit in $(K_\lambda Lv)f_i$. If $f_i = l_{i_1} + l_{i_2}$ $(l_{i_1}, l_{i_2} \in K_\lambda Lv)$ then $(K_\lambda Lv)f_i = (K_\lambda Lv)l_{i_1} \oplus (K_\lambda Lv)l_{i_2}$, but this does not hold. ∎

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THEOREM 3. *Let $K_\lambda G$ be an infinite twisted group algebra of* $\mathrm{char}(K_\lambda G) = p$, *such that $t(G)$ contains a p-element and either the field K is perfect or for any element $g \in G$ of order p^k, the element $u_g^{p^k}$ is algebraic over the prime subfield of K. Then $U(K_\lambda G)$ is an* FC*-group if and only if G is an* FC*-group and satisfies the following conditions:*

1. *$p = 2$ and $|G'| = 2$;*
2. *$t(G)$ is central in G and $t(G) = G' \times H$, where H is abelian, and has no 2-elements;*
3. *$K_\lambda H$ is a direct sum of a finite number of fields;*[1]
4. *$\{\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},g}\lambda_{h^{-1}g,h} \mid h \in H\}$ is a finite set for all $g \in G$.*

PROOF (NECESSITY). By Lemma 6 G is an FC-group. Let g be an element of order p^k. Then $u_g^{p^k} = \lambda_g \in U(K)$ and in the perfect field K we can take a p^k-th root of λ_g which we denote by μ. If K_0 is the prime subfield of K and λ_g is algebraic over K_0 then $K_0(\lambda_g)$ is a finite field and so it is perfect. Thus $u_g - \mu$ is nilpotent and $1 + \mu - u_g$ and (by Theorem 1) $1 - (u_g - \mu)u_a$ is central in $U(K_\lambda G)$. Then for any $b \in G$ by

$$u_b\big(1 - (u_g - \mu)u_a\big) = \big(1 - (u_g - \mu)u_a\big)u_b$$

implies

$$u_b u_g u_a - \mu u_b u_a - u_g u_a u_b + \mu u_a u_b = 0. \tag{4}$$

Each u_g can be written in the form $\mu + (u_g - \mu)$ and so $\mu^{-1}u_g = 1 + \mu^{-1}(u_g - \mu)$. Thus $\mu^{-1}u_g$ is an unipotent element and it commutes with u_b and u_a. Then (4) can be written as

$$u_g u_b u_a - u_g u_a u_b - \mu u_b u_a + \mu u_a u_b = 0. \tag{5}$$

If $[a,b] = 1$ then, by (5), we have $(\lambda_{a,b} - \lambda_{b,a})(u_g - \mu) = 0$. From this equation we get that the coefficient of u_g must be zero and $\lambda_{a,b} = \lambda_{b,a}$. Thus, $u_b u_a = u_a u_b$.

Let $[a,b] \neq 1$. Then by (5), $u_g u_b u_a = -\mu u_a u_b$ and $u_g u_a u_b = -\mu u_b u_a$. So

$$\begin{cases} u_g = -\mu[u_a^{-1}, u_b^{-1}]^{-1}, \\ u_g = -\mu[u_a^{-1}, u_b^{-1}]. \end{cases} \tag{6}$$

Consequently $u_g^2 = \mu^2$ and $(u_g\mu^{-1})^2 = 1$. Note that in (6) g may be any p-element, further a and b may be any noncommuting elements of G. This is possible only if $p = 2$. Then the commutator subgroup $\bar{G}'$ of group $\bar{G} = \{\kappa u_a \mid \kappa \in U(K), a \in G\}$ is of order 2 and coincides with the Sylow 2-subgroup of $\bar{G}$. Thus $\bar{G}' \subseteq \zeta(\bar{G})$ and $\bar{G}$ is a nilpotent group of class at most 2. Let

$$L = \langle \mu u_h \mid \mu \in U(K), h \in t(G) \rangle.$$

Then $L/U(K)$ is nilpotent torsion group and its 2-Sylow subgroup is of order 2. Here L is abelian because $\bar{G}'$ is of order 2 and it is a subgroup in L. Therefore $t(G)$ is abelian and

[1] If $K_\lambda H$ is a group ring then H is a finite abelian group.

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$t(G) = S \times H$, where $S = \langle g \mid g^2 = 1 \rangle$ is the Sylow 2-subgroup of $t(G)$ and all elements of H are of odd order.

We show that $K_\lambda H$ is central in $K_\lambda G$. Let $h \in H, a \in G$ and $[u_a, u_h] \neq 1$. Then $[u_a, u_h] = \mu u_g$ and

$$\lambda u_{a^{-1}h^{-1}ah} = \mu u_g. \tag{7}$$

It is clear that $[a, h] \in H$ and the order of $[a, h]$ is odd because H is normal in G. Since g is a 2-element, (7) does not hold. Thus $K_\lambda H$ is central in $K_\lambda G$ and $t(G) \subseteq \zeta(G)$.

Let us prove that $K_\lambda H$ contains only a finite number of idempotents. Suppose $K_\lambda H$ contains an infinite number of idempotents $e_1, e_2, \ldots$. If $d, b \in G$ and $[b, d] = g \neq 1$ then $g^2 = 1$ and (by Lemma 5) $1 - e_i + u_d e_i$ is a unit. Clearly,

$$(1 - e_i + u_d e_i)^{-1} u_b (1 - e_i + u_d e_i) = u_b (1 - e_i + \mu u_g e_i),$$

where $\mu = \lambda^{-1}_{d,d^{-1}} \lambda^{-1}_{b,b^{-1}} \lambda_{d^{-1},b} \lambda_{d^{-1}b,d} \lambda_{d^{-1}bd,b^{-1}}$.

If $i \neq j$ then $1 - e_i + \mu u_g e_i \neq 1 - e_j + \mu u_g e_j$. Indeed, if $1 - e_i + \mu u_g e_i = 1 - e_j + \mu u_g e_j$, then $(e_i - e_j)(\mu u_g - 1) = 0$. Since $e_i - e_j \in K_\lambda H$ and $u_g \notin K_\lambda H$, the last equality is true only in case $i = j$. Therefore if $i \neq j$ then $1 - e_i + \mu u_g e_i \neq 1 - e_j + \mu u_g e_j$ and u_b has an infinite number of conjugates, which does not hold. Thus $K_\lambda H$ contains a finite number of idempotents $e_1, e_2, \ldots, e_t$, and (by Lemma 7) $K_\lambda H$ is a direct sum of a finite number of fields.

Since $\{u_g^{-1} u_h u_g \mid g \in G\}$ is a finite set, we obtain condition 4 of the theorem.

SUFFICIENCY. Let the conditions of the theorem be satisfied. We prove that $U(K_\lambda G)$ is an FC-group.

Let $G' = \langle a \mid a^2 = 1 \rangle$ be the commutator subgroup of G and $\mu^2 = \lambda_{a,a}$. Thus the ideal $\Im = K_\lambda G(u_a - \mu)$ is nilpotent.

In $K_\lambda G$ we choose a new basis $\{w_g \mid g \in G\}$,

$$w_g = \begin{cases} u_g, & \text{if } g \in G \setminus \langle a \rangle, \\ \mu^{-1} u_g, & \text{if } g \in \langle a \rangle. \end{cases}$$

Let $G = \cup b_j \langle a \rangle$ be the decomposition of the group G by the cosets of $\langle a \rangle$. Any element $x + \Im \in K_\lambda G / \Im$ can be written as

$$\begin{aligned} x + \Im &= \sum_i \alpha_i w_{b_i} + \sum_i \beta_i w_{b_i} w_a + \Im \\ &= \sum_i \alpha_i w_{b_i} + \sum_i \beta_i w_b (w_a - 1) + \sum_i \beta_i w_{b_i} + \Im = \sum_i (\alpha_i + \beta_i) w_{b_i} + \Im. \end{aligned}$$

We show that $K_\lambda G / \Im$ is commutative. Indeed

$$(w_g + \Im)(w_h + \Im) = w_g w_h + \Im = w_h w_g [w_g, w_h] + \Im,$$

and the commutator $[w_g, w_h]$ is either 1 or w_a. If $[w_g, w_h] = w_a$ then

$$w_g w_h + \Im = w_h w_g w_a + \Im = w_h w_g (w_a - 1) + w_h w_g + \Im = w_h w_g + \Im.$$

We will construct the twisted group algebra $K_\mu H$ of the group $H = G/\langle a\rangle$ over the field K with the system of factors μ.

Let $R_l(G/\langle a\rangle)$ be a fixed set of representatives of all left cosets of the subgroup $\langle a\rangle$ in G and $H = \big\langle h_i = b_i\langle a\rangle \mid b_i \in R_l(G/\langle a\rangle)\big\rangle$. Let t_{h_i} denote element $w_{b_i} + \Im$. If $h_ih_j = h_k$, then $b_ib_j = b_ka^s$ ($s = \{0,1\}$), and

$$\begin{aligned} t_{h_i}t_{h_j} &= w_{b_i}w_{b_j} + \Im = \lambda_{b_i,b_j}w_{b_ka^s} + \Im = \lambda_{b_i,b_j}\lambda_{b_k,a^s}^{-1}w_{b_k}w_{a^s} + \Im \\ &= \lambda_{b_i,b_j}\lambda_{b_k,a^s}^{-1}w_{b_k} + \lambda_{b_i,b_j}\lambda_{b_k,a^s}^{-1}w_{b_k}(w_{a^s} - 1) + \Im = \lambda_{b_i,b_j}\lambda_{b_k,a^s}^{-1}w_{b_k} + \Im. \end{aligned}$$

Let $\mu_{h_i,h_j} = \lambda_{b_i,b_j}\lambda_{b_k,a^s}^{-1}$ and $\mu = \{\mu_{a,b} \mid a,b \in H\}$. Let $\{t_h \mid h \in H\}$ be a basis of the twisted group algebra $K\mu H$ with the system of factors μ. Clearly $t_{h_i}t_{h_j} = \mu_{b_i,b_j}t_{h_k}$.

Let $t(H)$ be the set of elements of finite order of H and $H = \cup c_it(H)$ the decomposition of the group H by the cosets of the subgroup $t(H)$. Then $x, x^{-1} \in U(K_\mu H)$ can be written as

$$x = \sum_{i=1}^{t} \alpha_i t_{c_i} \quad \text{and} \quad x^{-1} = \sum_{i=1}^{s} \beta_i t_{d_i},$$

where α_i, β_j are nonzero elements of $K_\mu t(H)$. The subgroup

$$L = \langle \mathrm{supp}(\alpha_1), \ldots, \mathrm{supp}(\alpha_t), \mathrm{supp}(\beta_1), \ldots, \mathrm{supp}(\beta_s)\rangle$$

is finite and $K_\mu L$ is a direct sum of fields

$$K_\mu L = e_1K_\mu L \oplus \cdots \oplus e_nK_\mu L. \tag{8}$$

Let $xe_k = \sum_{i=1}^{n} \gamma_i t_{c_i}$ and $x^{-1}e_k = \sum_{i=1}^{m} \delta_i t_{d_i}$, where γ_i, δ_j are nonzero elements of the field $K_\mu Le_k$.

We know [8], that a torsion free abelian group is orderable. Therefore we can assume that

$$c_{i_1}t(H) < c_{i_2}t(H) < \cdots < c_{i_n}t(H)$$

and

$$d_{j_1}t(H) < d_{j_2}t(H) < \cdots < d_{j_m}t(H).$$

Then $c_{i_1}d_{j_1}t(H)$ is called the *least* and $c_{i_n}d_{j_m}t(H)$ is called the *greatest* among the elements of the form $c_{i_s}d_{j_q}t(H)$. It is easy to see that $c_{i_1}d_{j_1}t(H) < c_{i_n}d_{j_m}t(H)$ if $n > 1$ or $m > 1$. Therefore $\gamma\delta_1 t_{c_{i_1}}t_{d_{j_1}} \neq \gamma_n\delta_m t_{c_{i_n}}t_{d_{j_m}}$. Since $x^{-1}e_kxe_k = e_k$, we have $n = m = 1$, $xe_k = \gamma t_{c_r}$ and $x^{-1}e_k = \gamma^{-1}t_{c_r}^{-1}$. Thus x and x^{-1} can be written as

$$x = \sum_{i=1}^{t} \gamma_i t_{c_i} \quad \text{and} \quad x^{-1} = \sum_{i=1}^{t} \gamma_i^{-1} t_{c_i}^{-1},$$

where $\gamma_1, \ldots, \gamma_t$ are orthogonal elements.

Let $\phi: K_\lambda G/\Im \mapsto K_\mu H$ be an isomorphism of these algebras. If $x \in U(K_\lambda G)$ then $\phi(x + \Im) = \sum_{i=1}^{t} \gamma_i t_{c_i}$ where $\gamma_i \in K_\mu Le_i$. It is easy to see that there exists an abelian subgroup $\bar{L}$ of G such that $L = \bar{L}/\langle a\rangle$. The algebra $K_\lambda\bar{L}$ is commutative and its radical is a nilpotent ideal equal to $\Im \cap K_\lambda\bar{L}$. Since $K_\mu\bar{L}/(\Im \cap K_\lambda\bar{L}) \cong K_\lambda L$, the classical method of lifting idempotents yields idempotents $f_1,\dots,f_t$ in $K_\mu\bar{L}$ such that $f_1+\cdots+f_t=1$ and $f_i+\Im=e_i$. Then $x=xf_1+\cdots+xf_t$ and $\phi(xf_i+\Im)=\gamma_i t_{c_i}$, where $h_i=b_i\langle a\rangle$, $b_i\in G$. There exists an element $v_i\in K_\lambda\bar{L}f_i$ such that $\phi(v_i+\Im)=\gamma_i$ and $\phi(v_iw_{g_i}+\Im)=\gamma_i t_{h_i}$. We can find an element $r\in\Im$ such that $xf_i=(v_i+rf_i)w_{g_i}$.

Clearly $s_i=v_i+rf_i$ is a unit in $K_\mu\bar{L}f_i$ and is central in $K_\lambda G$. So $s_1,\dots,s_t$ are orthogonal and $x=\sum_{i=1}^t s_iw_{g_i}$, $x^{-1}=\sum_{i=1}^t s_i^{-1}w_{g_i}^{-1}$. Since $s_i\in\zeta(K_\lambda G)$, $x^{-1}w_gx=\sum_{i=1}^t w_{g_i}^{-1}w_gw_{g_i}$ for any $g\in G$. By condition 4 our theorem w_g has a finite number of conjugates, because G is an FC-group. Thus $U(K_\lambda G)$ is an FC-group. ∎

LEMMA 8. *Let K be a field such that* $\mathrm{char}(K)$ *does not divide the order of any element of $t(G)$, $K_\lambda t(G)$ a commutative algebra that does not contain a minimal idempotent. Then for any idempotent $e\in K_\lambda t(G)$ there exists an infinite set of idempotents $e_1=e,e_2,\dots$ such that*

$$e_ke_{k+1}=e_{k+1}\quad (k\in\mathbb{N}). \tag{9}$$

PROOF. Suppose $K_\lambda t(G)$ does not contain a minimal idempotent. First we prove that for any idempotent there exists an infinite set of idempotents $e_1,e_2,\dots$ in $K_\lambda t(G)$ satisfying condition (9).

Let e_1 be an idempotent of $K_\lambda t(G)$ and $H_1=\langle\mathrm{supp}(e_1)\rangle$. Then the ideal $K_\lambda t(G)e_1$ is not minimal and so contains a proper ideal $\Im_1$ of $K_\lambda t(G)$. Let $0\neq x_1\in\Im_1$ and $H_2=\langle H_1,\mathrm{supp}(x_1)\rangle$. Then $\overline{\Im_1}=\Im_1\cap K_\lambda H_2$ is an ideal of $K_\lambda H_2$ and $\overline{\Im_1}$ is generated by the idempotent e_2 because H_2 is a finite subgroup of $t(G)$ and the commutative algebra $K_\lambda H_2$ is semiprime. It is easy to see that $e_1=e_2+f$, $f\neq 0$ and $e_1e_2=e_2$. Indeed, if $f=0$, then $e_1=e_2$ and $K_\lambda t(G)e_1=K_\lambda t(G)e_2\subset\Im_1$, which does not hold. The ideal $K_\lambda t(G)e_2$ contains a proper ideal $\Im_2$ of $K_\lambda t(G)$. We choose a nonzero element $0\neq x_2\in\Im_2$ and consider the subgroup $H_3=\langle H_2,\mathrm{supp}(x_2)\rangle$. The ideal $\overline{\Im_2}=\Im_2\cap K_\lambda H_3$ is generated by the idempotent e_3 and $e_2e_3=e_3\neq e_2$. This method enables us to construct an infinite number of idempotents $e_1,e_2,\dots$, satisfying condition (9), which completes the proof. ∎

LEMMA 9. *Let K be a field such that* $\mathrm{char}(K)$ *does not divide the order of any element of $t(G)$, and $U(K_\lambda G)$ an* FC-*group. If the commutative algebra $K_\lambda t(G)$ contains an infinite number of central idempotents $f_1,f_2,\dots$ and $g=[a,b]$ $(a,b\in G)$ is an element of order n then the commutators $[u_a,u_b]$ and $[a,b]$ have the same order and*

$$(f_i-f_j)(1-[u_a,u_b])=0 \tag{10}$$

for some $i\neq j$.

PROOF. Let $g=[a,b]\neq 1$ where $a,b\in G$. By B. H. Neumann's theorem $G/t(G)$ is abelian, thus $g\in t(G)$ and $1-f_i+u_bf_i$ is a unit in $K_\lambda G$. The element u_a has a finite number of conjugates in $U(K_\lambda G)$ and

$$(1-f_i+u_b^{-1}f_i)u_a(1-f_i+u_bf_i)=u_a(1-f_i+[u_a,u_b]f_i).$$

Consequently there exist i and j $(i < j)$, such that

$$1 - f_i + [u_a, u_b]f_i = 1 - f_j + [u_a, u_b]f_j$$

and

$$(f_i - f_j)(1 - [u_a, u_b]) = 0. \tag{11}$$

If n is the order of $g = [a, b]$ then

$$[u_a, u_b] = \lambda^{-1}_{a^{-1},a}\lambda^{-1}_{b^{-1},b}\lambda_{a^{-1},b^{-1}}\lambda_{a^{-1}b^{-1},a}\lambda_{a^{-1}b^{-1}a,b}u_g$$

and $[u_a, u_b]^n = \gamma \in U(K)$. Then by (11) we have $\gamma(f_i - f_j) = f_i - f_j$. So $\gamma = 1$ and $[u_a, u_b]^n = 1$. ∎

THEOREM 4. *Let $K_\lambda G$ be an infinite twisted group algebra, and* char(K) *does not divide the order of any element of $t(G)$. If $K_\lambda t(G)$ contains only a finite number of idempotents then $U(K_\lambda G)$ is an* FC-*group if and only if G is an* FC-*group and the following conditions are satisfied:*

1. *all idempotents of $K_\lambda t(G)$ are central in $K_\lambda G$;*
2. $\{\lambda^{-1}_{h,h^{-1}}\lambda_{h^{-1},g}\lambda_{h^{-1}g,h} \mid h \in H\}$ *is a finite set for every $g \in G$;*
3. *$K_\lambda t(G)$ is a direct sum of a finite number of fields;*
4. *if $K_\lambda t(G)$ is infinite then it is central in $K_\lambda G$.*

PROOF (NECESSITY). By Lemmas 4, 6 and 7 $K_\lambda t(G)$ is commutative, G is an FC-group and all idempotents of $K_\lambda t(G)$ are central in $K_\lambda G$. Since $\{u_g^{-1}u_hu_g \mid g \in G\}$ is a finite set, condition 2 of the theorem is satisfied.

Since $K_\lambda t(G)$ contains only a finite number of idempotents (by Lemma 7) $K_\lambda t(G)$ is a direct sum of a finite number of fields. Let $K_\lambda t(G)$ be infinite and $K_\lambda t(G)e_i$ a field in this direct decomposition of $K_\lambda t(G)$. Lemma 5 implies that $K_\lambda t(G)e_i$ is invariant under the inner automorphism $\psi(x) = u_g^{-1}xu_g$ for any $g \in G$. Since $\langle u_g, K_\lambda t(G)e_i \setminus \{0\}\rangle$ is an FC-group there exists a infinite subfield L_g of $K_\lambda t(G)e_i$ such that $yu_g = u_gy$ for every $y \in L$. Let $H = \langle g, t(G)\rangle$. Then $K_\lambda H$ is subalgebra of $K_\lambda G$ and (by Lemma 6) $K_\lambda t(G)$ is central in $K_\lambda H$.

SUFFICIENCY. Let $K_\lambda t(G)$ be a direct sum of fields,

$$K_\lambda t(G) = F_1 \oplus F_2 \oplus \cdots \oplus F_t.$$

Then $F_i = K_\lambda t(G)e_i$, where e_i is a central idempotent in $K_\lambda G$. It is easy to see that $K_\lambda G$ is a direct sum of ideals

$$K_\lambda G = K_\lambda Ge_1 \oplus \cdots \oplus K_\lambda Ge_t. \tag{12}$$

Let us prove that $K_\lambda Ge_q$ is isomorphic to a crossed product $F_q * H$ of the group $H = G/t(G)$ and the field F_q.

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Let $R_l\big(G/t(G)\big)$ be a fixed set of representatives of all left cosets of the subgroup $t(G)$ in G. Any element $x \in K_\lambda Ge_q$ can be written as

$$x = e_q u_{c_1}\gamma_1 + \cdots + e_q u_{c_s}\gamma_s,$$

where $\gamma_k \in K_\lambda t(G), c_k \in R_l\big(G/t(G)\big)$. If $c_i c_j = c_k h$ $\big(h \in t(G)\big)$ then

$$u_{c_i}u_{c_j} = u_{c_ic_j}\lambda_{c_i,c_j} = u_{c_kh}\lambda_{c_i,c_j} = u_{c_k}u_h\lambda^{-1}_{c_k,h}\lambda_{c_i,c_j}.$$

We will construct the crossed product $F_q * H$, where

$$H = \big\{h_i = c_i t(G) \mid c_i \in R_l\big(G/t(G)\big)\big\}.$$

Let $\alpha \in F_q$ and σ be a map from H to the group of automorphism $\mathrm{Aut}(F_q)$ of the field F_q such that $\sigma(h_i)(\alpha) = u_{c_i}^{-1}\alpha u_{c_i}$ and let $\mu_{h_i,h_j} = u_h\lambda^{-1}_{c_k,h}\lambda_{c_i,c_j}$.

Clearly, the set $\mu = \{\mu_{a,b} \in U(F_q) \mid a,b \in H\}$ of nonzero elements of the field F_q satisfies

$$\mu_{a,bc}\mu_{b,c} = \mu_{ab,c}\mu_{a,b}^{\sigma(c)}$$

and

$$\alpha^{\sigma(a)\sigma(b)} = \mu^{-1}_{a,b}\alpha^{\sigma(ab)}\mu_{a,b},$$

where $\alpha \in F_q$ and $a,b,c \in H$.

Then $F_q * H = \{\sum_{h\in H} w_h\alpha_h \mid \alpha_h \in F_q\}$ is a crossed product of the group H and the field F_q and we have $w_{d_i}w_{d_j} = w_{d_k}\mu_{d_i,d_j}$ and $\alpha w_{d_i} = w_{d_i}\alpha^{\sigma(d_i)}$.

Clearly, $F_q * H$ and $K_\lambda Ge_q$ are isomorphic because

$$u_{c_i}\alpha u_{c_j} = u_{c_i}u_{c_j}(u_{c_j}^{-1}\alpha u_{c_j}) = u_{c_k}\mu_{c_i,c_j}\alpha^{\sigma(c_j)}.$$

We know [5] that the group of units of the crossed product $K * H$ of the torsion free abelian group H and the field K consists of the elements $w_h\alpha$, where $\alpha \in U(K)$, $h \in H$.

By (12), for every $y \in U(K_\lambda G)$,

$$y = u_{c_1}\gamma_1 + \cdots + u_{c_t}\gamma_t$$

and

$$y^{-1} = u_{c_1}^{-1}\gamma_1^{-1} + \cdots + u_{c_t}^{-1}\gamma_t^{-1},$$

where $\gamma_1, \ldots, \gamma_t$ are orthogonal elements.

Let $x = \delta_1 u_{d_1} + \cdots + \delta_t u_{d_t} \in U(K_\lambda G)$. Then

$$yxy^{-1} = u_{c_1}\gamma_1\delta_1 u_{d_1}u_{c_1}^{-1}\gamma_1^{-1} + \cdots + u_{c_t}\gamma_1\delta_t u_{d_t}u_{c_t}^{-1}\gamma_t^{-1}.$$

If $K_\lambda t(G)$ is infinite then $K_\lambda t(G) \subseteq \zeta(K_\lambda G)$ and

$$yxy^{-1} = \sum_{i=1}^{t}\delta_i u_{c_i}u_{d_i}u_{c_i}^{-1} = \sum_{i=1}^{t}\delta_i\lambda^{-1}_{c_i,c_i^{-1}}\lambda_{c_i,d_i}\lambda_{c_id_i,c_i^{-1}}u_{c_id_ic_i^{-1}}.$$

Since G is an FC-group, by condition 2 of the theorem, x has a finite number of conjugates, so $U(K_\lambda G)$ is an FC-group.

If $K_\lambda t(G)$ is finite then F_q is a finite field and

$$y^{-1}xy = \sum_{i=1}^{t}\gamma_i^{-1}u_{c_i}^{-1}\delta_i u_{d_i}u_{c_i}\gamma_i = \sum_{i=1}^{t}\lambda^{-1}_{c_i,c_i^{-1}}\lambda_{c_i^{-1},d_i}\lambda_{c_i^{-1}d_i,c_i}\gamma_1^{-1}\delta_i^{\sigma(c_i^{-1})}\gamma_i^{\sigma(c_i^{-1}d_i^{-1}c_i)}u_{c_i^{-1}d_ic_i}.$$

Since G is an FC-group and F_q is a finite field, x has a finite number of conjugates, so $U(K_\lambda G)$ is an FC-group. ∎

THEOREM 5. *Let $K_\lambda G$ be infinite and* char(K) *does not divide the order of any element of* $t(G)$. *If $K_\lambda t(G)$ contains an infinite number of idempotents then $U(K_\lambda G)$ is an* FC-*group if and only if G is an* FC-*group and the following conditions are satisfied:*

1. *$K_\lambda t(G)$ is central in $K_\lambda G$ and contains a minimal idempotent;*
2. *$\{\lambda^{-1}_{h,h^{-1}}\lambda_{h^{-1},g}\lambda_{h^{-1}g,h} \mid h \in H\}$ is a finite set for any $g \in G$;*
3. *the commutator subgroups of G and of $\bar{G} = \{\kappa u_a \mid \kappa \in U(K), a \in G\}$ are isomorphic and G' is either a finite group or isomorphic to the group $\mathbb{Z}(q^\infty)$ $(q \neq p)$, and there exists an $n \in \mathbb{N}$, such that the field K does not contain the primitive q^n-th root of* 1;
4. *for every finite subgroup H of the commutator subgroup of $\bar{G}$ the element $e_H = \frac{1}{|H|}\sum_{h\in H} h$ is a idempotent of $K_\lambda t(G)$, and $K_\lambda t(G)(1-e_H)$ is a direct sum of a finite number of fields;*[2]

PROOF (NECESSITY). By Lemmas 4, 6 and 7 $K_\lambda t(G)$ is commutative, G is an FC-group and all idempotents of $K_\lambda t(G)$ are central in $K_\lambda G$.

Let us prove that $K_\lambda t(G)$ contains a minimal idempotent. Suppose the contrary. Let $a, b \in G$ and $1 \neq [a,b] = g$. Since g is an element of finite order n, by Lemma 9, $[u_a,u_b]^n = 1$ and

$$f = \frac{1}{n}(1 + [u_a,u_b] + [u_a,u_b]^2 + \cdots + [u_a,u_b]^{n-1})$$

is an idempotent. By Lemma 11, for $1-f$ one can construct an infinite sequence of idempotents $e_1 = 1-f, e_2, \ldots,$ satisfying (9). By Lemma 9,

$$(1-[u_a,u_b])(e_i - e_j) = 0,$$

where $i<j$. Consequently $([u_a,u_b])^k(e_i-e_j) = (e_i-e_j)$ for all k and $f(e_i-e_j) = e_i - e_j$. This implies $(1-f)(e_i-e_j) = 0$. Since $e_1 = 1-f$, $e_1(e_i-e_j) = 0$. If we multiply this equality from the right by the elements $e_2, \ldots, e_{j-1}$, by (9) we obtain $e_{j-1} - e_j = 0$. Now we arrived at a contradiction, which proves that $K_\lambda t(G)$ contains a minimal idempotent.

It is easy to see that $t(G)$ is infinite, otherwise $K_\lambda t(G)$ would contain a finite number of idempotents. $K_\lambda t(G)$ contains a minimal idempotent e, and so there exist only a finite number of elements $g \in t(G)$, such that $eu_g = e$. Consequently $K_\lambda t(G)e$ is an infinite field and contains K as a subfield. Then as in the proof of Theorem 4, $K_\lambda t(G)$ is central in $K_\lambda G$.

Since $\{u_g^{-1}u_hu_g \mid g \in G\}$ is a finite set, we obtain condition 2 of the theorem.

Suppose $c \in G'$ and

$$c = [a_1,b_1][a_2,b_2]\cdots[a_n,b_n].$$

Since $K_\lambda t(G)$ is central in $K_\lambda G$ and $1 - e_i + e_iu_{b_k} \in U\big(K_\lambda t(G)\big)$ we have

$$\begin{aligned}\prod_{k=1}^{n}(1-e_i+e_iu_{b_k}^{-1})u_{a_k}(1-e_i+e_iu_{b_k}) &= \prod_{k=1}^{n}\Big(u_{a_k}(1-e_i+e_i[u_{a_k},u_{b_k}])\Big)\\ &= \prod_{k=1}^{n}(u_{a_k})\Big(\prod_{k=1}^{n}(1-e_i+e_i[u_{a_k},u_{b_k}])\Big)\end{aligned}$$

[2] If $K_\lambda G$ is a group ring, then 1 and 3 imply 4 (see [6] p. 690, Lemma 4.3, also [10]).

for all $i \in \mathbb{N}$. Since each $u_{a_1}, u_{a_2}, \ldots, u_{a_k}$ has a finite number of conjugates, there are only a finite number of different elements of the form $\prod_{k=1}^{n}(1 - e_i + e_i[u_{a_k}, u_{b_k}])$. These elements will be denoted by $w_1, \ldots, w_t$. Let

$$W_r(c) = \Big\{ i \in \mathbb{N} \;\Big|\; \prod_{k=1}^{n}(1 - e_i + e_i[u_{a_k}, u_{b_k}]) = w_r \Big\}.$$

It is easy to see that the set of natural numbers $\mathbb{N}$ is the union of the subsets $W_i(c)$ ($i = 1, \ldots, t$), of which at least one is infinite. If $W_1(c)$ is infinite and $i,j \in W_1(c)$ then

$$(e_i - e_j)\Big(1 - \prod_{k=1}^{n}[u_{a_k}, u_{b_k}]\Big) = 0. \tag{13}$$

This implies that if

$$\prod_{k=1}^{n}[u_{a_k}, u_{b_k}] = \gamma \in U(K)$$

then $\gamma = 1$.

Now we prove that the commutator subgroups of G and of $\bar{G} = \{\kappa u_a \mid \kappa \in U(K), a \in G\}$ are isomorphic. It is easy to see that the map $\tau(\lambda u_g) = g$ $\big(\lambda \in U(K), g \in G\big)$ is a homomorphism from $\bar{G}$ to G. Every element $h \in \bar{G}'$ can be written as

$$h = [u_{a_1}, u_{b_1}][u_{a_2}, u_{b_2}] \cdots [u_{a_n}, u_{b_n}].$$

As we have shown above, if $h = \lambda \in U(K)$ then $\lambda = 1$. Thus, τ is an isomorphism from $\bar{G}'$ to G'.

Let H be a finite subgroup of $\bar{G}'$. Then $e_H = \frac{1}{|H|}\sum_{h\in H} h$ is an idempotent of $K_\lambda t(G)$. Suppose that $K_\lambda t(G)(1 - e_H)$ contains an infinite number of idempotents $e_1, e_2, \ldots$. If $H = \{h_1, h_2, \ldots, h_s\}$, then, as it is shown above, for every $h_j \in H$,

$$\mathbb{N} = W_1(h_j) \cup \cdots \cup W_{r(j)}(h_j),$$

where $j = 1, 2, \ldots, s$, and for every $k \neq l$, $W_k(h_j)$ and $W_l(h_j)$ have empty intersection.

It is clear that there exists an infinite subset $M = W_{i_1}(h_1) \cap \cdots \cap W_{i_s}(h_s)$. If $i,j \in M$, then by (13), we have $(e_i - e_j)(1 - h_r) = 0$ for any r. Then

$$e_i - e_j = \frac{1}{|H|}\sum_{r=1}^{s}\big(h_r(e_i - e_j)\big) = e_H(e_i - e_j). \tag{14}$$

Since $e_i - e_j \in K_\lambda t(G)(1 - e_H)$, by (14),

$$e_i - e_j = (1 - e_H)(e_i - e_j) = (e_i - e_j) - e_H(e_i - e_j) = 0.$$

Thus, $K_\lambda t(G)(1 - e_H)$ contains a finite number of idempotents, and by Lemma 7, it can be given as a direct sum of a finite number of fields.

Let us prove that there exists only finitely many elements of prime order in G'.

Suppose the contrary. If $a, b \in G$ then $1 \neq [a, b] = g \in t(G)$. As we have seen above, if $h \in G'$, then there exists $\mu \in U(K)$ such that the order of the element μu_h equals the

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order of h. Then there exists a countably infinite subgroup S, generated by elements of prime order, such that $\langle g\rangle \cap S = 1$. By Prüfer's theorem [9] S is a direct product of cyclic subgroups $S = \prod_j \langle a_j\rangle$. If q_j is the order of the element a_j, then

$$e_j = \frac{1}{q_j}\big(1 + \mu u_{a_j} + (\mu u_{a_j})^2 + \cdots + (\mu u_{a_j})^{q_j-1}\big)$$

is a central idempotent and $x_i = 1-e_i+e_iu_a \in U(K_\lambda G)$. By Lemma 9, $(e_i-e_j)(1-\mu u_g) = 0$. Since $g \notin S$, we have $i = j$, which does not hold. Consequently G' contains only a finite number of elements of prime order and satisfies the minimum condition for subgroups (see [8]). Then

$$G' \cong P_1 \times P_2 \times \cdots \times P_t \times H,$$

where $P_i = \mathbb{Z}(q^\infty)$ and $|H| < \infty$. Let us prove that either $G' = \mathbb{Z}(q^\infty)$ or $|G'|$ is finite.

Let $a, b \in G$ and $1 \neq [a,b] = g \in t(G)$. Suppose there exists l such that $g \notin P_l = \langle a_1, a_2, \ldots \mid a_1^q = 1, a_{j+1}^q = a_j\rangle$. Then

$$e_k = \frac{1}{q^k}\big(1 + \mu u_{a_k} + (\mu u_{a_k})^2 + \cdots + (\mu u_{a_k})^{q^k-1}\big)$$

is an idempotent, and $(e_i - e_j)(1 - \mu u_g) = 0$. This is true only for $i = j$, if $g \notin P_l$, which is impossible. Thus, $G' \cong \mathbb{Z}(q^\infty)$ or G' is a finite subgroup.

Let K be a field which contains a primitive q^n-th root ϵ_n of 1 for all n and

$$P_1 = \langle a_1, a_2, \ldots \mid a_1^q = 1, a_{j+1}^q = a_j\rangle.$$

Put

$$e_j = \frac{1}{q^j}\big(1 + \epsilon_j\mu u_{a_j} + (\epsilon_j\mu u_{a_j})^2 + \cdots + (\epsilon_j\mu u_{a_j})^{q^j-1}\big).$$

If $i \neq j$ then the element $(e_i - e_j)(1 - \mu u_g) \neq 0$ and by Lemma 9 this is impossible. Thus there exists $n \in \mathbb{N}$ such that K does not contain a primitive q^n-th root ϵ_n of 1.

SUFFICIENCY. Let us prove that any element $u_g (g \in G)$ has a finite number of conjugates in $U(K_\lambda G)$.

Let $\bar{G} = \{\kappa u_a \mid \kappa \in U(K), a \in G\}$.We prove that $H = \langle [u_g, \bar{G}]\rangle$ is a finite subgroup in $\bar{G}'$. If $\bar{G}'$ is finite, it is obvious. If $\bar{G}'$ is infinite then it is isomorphic to a subgroup of the group $\mathbb{Z}(q^\infty)$. Any element of $\bar{G}$ is of the form μu_h $\big(\mu \in U(K), h \in G\big)$ and

$$[u_g, \mu u_h] = \lambda_{g,g^{-1}}^{-1}\lambda_{h,h^{-1}}^{-1}\lambda_{g^{-1},h^{-1}}\lambda_{g^{-1}h^{-1},g}\lambda_{g^{-1}h^{-1}g,h}u_{g^{-1}h^{-1}gh}.$$

Since G is an FC-group, and for a fixed element g the set $\{\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},g}\lambda_{h^{-1}g,h} \mid h \in H\}$ is finite, the number of commutators $[u_g, \mu u_h]$ is finite. These commutators generate a finite cyclic subgroup H of $\mathbb{Z}(q^\infty)$. The element $e_H = \frac{1}{|H|}\sum_{h\in H} h$ is a idempotent in $K_\lambda t(G)$ and by condition 4 of the theorem, $K_\lambda t(G)(1 - e_H)$ is a direct sum of a finite number of fields $K_\lambda t(G)(1 - e_H)f_i$ $(i = 1, \ldots, s)$.

In $K_\lambda t(G)$ we have the decomposition

$$K_\lambda t(G) = K_\lambda t(G)e_H \oplus K_\lambda t(G)f_1 \oplus \cdots \oplus K_\lambda t(G)f_t.$$

Then

$$K_\lambda G = K_\lambda G e_H \oplus K_\lambda G f_1 \oplus \cdots \oplus K_\lambda G f_t.$$

If $x \in U(K_\lambda G)$ then

$$x = xe_H + xf_1 + \cdots + xf_t$$

and

$$x^{-1} = x^{-1}e_H + x^{-1}f_1 + \cdots + x^{-1}f_t.$$

Consequently

$$x^{-1}u_g x = x^{-1}e_H u_g x e_H + x^{-1}f_1 u_g x f_1 + \cdots + x^{-1}f_t u_g x f_t.$$

We show that the element xe_H is central in $U(K_\lambda G)$. If $x = \alpha_1 u_{h_1} + \cdots + \alpha_t u_{h_t}$, then

$$u_g x e_H = \alpha_1 u_g u_{h_1} e_H + \cdots + \alpha_t u_g u_{h_t} e_H = \alpha_1 u_{h_1} u_g [u_g, u_{h_1}] e_H + \cdots + \alpha_t u_{h_t} u_g [u_g, u_{h_t}] e_H$$

and $[u_g, u_h] \in H$. Clearly, $[u_g, u_{h_k}]e_H = e_H$ and

$$u_g x e_H = \alpha_1 u_{h_1} u_g e_H + \cdots + \alpha_t u_{h_t} e_H = xe_H u_g.$$

$K_\lambda G f_i$ is a crossed product $F * H$ of the group $H = G/t(G)$ and the field $F = K_\lambda t(G) f_i$. We know (see [5]) that the group of units of the crossed product $F * H$ of a torsion free abelian group H and a field F consists of the elements αu_h $\big(\alpha \in U(F), h \in H\big)$. The unit xf_i can be given as $\alpha_i u_{h_i}$ $(h_i \in G)$, where α_i is central in $U(K_\lambda G f_i)$. Thus

$$x^{-1} f_i u_g x f_i = u_{h_i}^{-1} \alpha_i^{-1} u_g \alpha_i u_{h_i} = u_{h_i}^{-1} u_g u_{h_i} = \lambda^{-1}_{h_i^{-1}, h_i} \lambda_{h_i^{-1}, g} \lambda_{h_i^{-1} g, h_i} u_{h_i^{-1} g h_i}.$$

Therefore

$$x^{-1} u_g x = u_g + \sum_{i=1}^{t} \lambda^{-1}_{h_i^{-1}, h_i} \lambda_{h_i^{-1}, g} \lambda_{h_i^{-1} g, h_i} u_{h_i^{-1} g h_i}.$$

Since G is an FC-group, by condition 2 of the theorem, u_g has a finite number of conjugates in $U(K_\lambda G)$. ∎

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