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EXPANSIONS OF ARBITRARY ANALYTIC FUNCTIONS IN SERIES OF EXPONENTIALS

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1. Introduction. Let $\phi \neq 0$ be an entire function of one complex variable and of exponential type. Let B denote the set of all monomial exponentials of the form $z^h e^{\zeta z}$ where ζ is a zero of ϕ of order greater than h. If R is a simply connected plane region and $H(R)$ denotes the space of functions analytic in R with the topology of uniform convergence on compacta, then ϕ can be considered as an element of the topological dual $H'(R)$ if the Borel transform $\tilde{\phi}$ of ϕ is analytic on $\tilde{R}$, the complement of R. The duality is given by

$$\langle f, \phi \rangle = \frac{1}{2\pi i} \int_C f(w) \tilde{\phi}(w) dw,$$

where C is a simple closed curve in the common region of analyticity of f and $\tilde{\phi}$, and C winds once around the complement of a set in which $\tilde{\phi}$ is analytic. By the Polya representation

$$\phi(z) = \langle e^{zw}, \phi(w) \rangle.$$

B is then not total in $H(R)$ since ϕ annihilates B and $\phi \neq 0$. If $R + b$ is the translation of R by b, then B is not total in $H(R + b)$ since $e^{bz}\phi(z)$ is then in $H'(R + b)$ and annihilates B.

If P denotes the conjugate indicator diagram of ϕ, then $\tilde{\phi}$ is analytic on the complement of P and cannot be continued analytically to the extreme points of P. If a simply connected region R is a subset of the interior P^0 of P, then B might be total in $H(R)$. This is precisely the case when ϕ is of regular growth, that is, when there exists an increasing sequence $\{r_k\}$ of reals tending to infinity for which

$$r_k^{-1} \log|\phi(r_k e^{i\theta})| \rightarrow h(\theta)$$

uniformly in θ where h is the indicator function of ϕ. This follows from the Hahn-Banach Theorem and the examination of the indicator of a quotient when the denominator is of regular growth.

When B is total in $H(R)$, it is easily seen that B reduced by a finite number of monomial exponentials is still total in $H(R)$. It follows that each f in $H(R)$ can be written as a compactly convergent series whose terms are linear combinations of B. This may be accomplished by con-

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structing an exhaustive, increasing sequence of compacta in R and approximating within $1/n$ on the n-th compact the difference between f and its approximation on the preceding compact using monomial exponentials not previously used. It is evident that such series are not unique.

If ϕ is of regular growth, $R \subset P^0$, and f in $H(R)$ is analytic in a larger region S covering the complement of a region in which $\tilde{\phi}$ is analytic, then it is possible to associate with f a particular Fourier-type series that converges to f on certain compacta in $R \cap S$. That is the main content of Section 3. When $S \supset P$, the convergence in P^0 has been established by A. F. Leont'ev [**7**, **8**]. Our proof of that theorem generalizes to include functions f that are analytic in regions not necessarily covering P but necessarily (and imprecisely) covering the singularities of $\tilde{\phi}$, including the extreme points of P. Convergence to f is obtained on certain compacta. The generalization parallels the results of [**14**] established when ϕ is an exponential polynomial with constant coefficients. The method parallels those of [**1**, pp. 37, 45] when ϕ is an exponential polynomial with polynomial coefficients.

In Section 4 conditions on ϕ are increased so that when f is analytic on P, the sum of the series is determined at boundary points of P. This partially generalizes the results in [**1**] and is related to results of [**3**, **4**, **5**, **6**, **12**, **13**].

In Section 5 further restrictions on ϕ allow expansions to f on compacta in P^0 when f is analytic in P^0 and has continuous derivatives on P. Similar, but distinct, results have been obtained by Leont'ev [**9**, **11**].

The proof in Section 3 is written so as to permit modifications that give the results of the succeeding sections.

2. Preliminaries. Throughout, ϕ is of exponential type and of regular growth with

$$(1) \quad h(\theta) = \lim_{k\to\infty} \frac{\log |\phi(r_k e^{i\theta})|}{r_k},$$

the limit being uniform in θ for an increasing sequence $\{r_k\}^\infty$, of positive reals with $r_k \to \infty$. $\{\zeta_k\}$ is the sequence of zeros of ϕ arranged in an order of non-decreasing moduli with $m_k + 1$ the order of ζ_k. Taking a subsequence of $\{r_k\}$, if necessary, we may assume that $r_k > |\zeta_k|$; otherwise, zero terms are introduced in the series considered. P is the conjugate indicator diagram of ϕ with non-empty interior P^0. The Borel transform $\tilde{\phi}$ of ϕ is analytic in the region (open and connected) $\tilde{\Omega} \supset \tilde{P}$. The complement Ω of $\tilde{\Omega}$ is then contained in P. The interior Ω^0 of Ω is then simply connected and bounded. For positive δ, P_δ will denote the set of points of P that are at least a distance δ from the extreme points of P. Γ_k will denote the positively oriented circle $|z| = r_k$. In general, the complement, interior, closure, boundary, and difference of sets are denoted by $\tilde{A}$, A^0, $\bar{A}$, ∂A, and $A \sim B$, respectively. If A is a set and ϵ is positive, then $A \oplus \epsilon$ denotes the set of points of the form $a + e$ where $a \in A$ and $|e| < \epsilon$.

Definition 1. If γ is a complex plane curve from z to w parameterized by $\gamma(t)$, $a \leqq t \leqq b$, its *symmetric curve* is the curve parameterized by $z + w - \gamma(t)$, $a \leqq t \leqq b$. The symmetric curve is then the curve symmetric to the given curve with respect to the midpoint of the line segment $[z, w]$.

Definition 2. A compact set K in Ω^0 is *admissible* means that (1) each point z of K may be joined to each point w on $\partial\Omega$ with a curve $\gamma(z, w)$ in Ω whose symmetric curve is in P, and (2) the curves $\gamma(z, w)$ are uniformly bounded in length.

If $\Omega = P$, then each compact in P^0 is admissible with each γ taken as a straight line segment. When Ω is starlike with respect to each point in a compact K in Ω^0, then K is admissible taking curves γ as straight line segments. Suppose that Ω is a solid V-shaped region and T is a convex, triangular set cut from the bottom of the V by a horizontal line; then any compact in the interior of T is admissible.

Biorthogonality conditions such as those given in [**2**, p. 365] suggest coefficients for the terms in B to be used in a series associated with a function f. The curves Γ_k suggest a grouping of the sums of terms from B. These observations motivate the next definition, which will be extended in a special case in Section 5.

Definition 3. Let f be analytic in a simply connected region $S \supset \Omega$.

$$F(a, w, t) = \int_a^w e^{(w-s)t} f(s)ds$$

where the path of integration is in S. For k and h positive and non-negative integers, respectively,

$$L_{kh}(f) = \frac{1}{2\pi i} \int_{c_k} \frac{(t - \zeta_k)^h}{\phi(t)} \langle F(a, w, t), \phi(w) \rangle dt$$

where c_k is a positively oriented circle about ζ_k with no other zero of ϕ in or on c_k. The *ϕ-series* of f is then defined by

$$\sum_{p=1}^{\infty} \sum_{r_{p-1} < |\zeta_k| < r_p} \left(\sum_{h=0}^{m_k} \frac{1}{h!} L_{kh}(f) z^h \right) e^{\zeta_k z}$$

where $r_0 = -1$.

We note that $F(a, w, t)$ is an analytic function of w in S for each t and entire in t for each w in S. Also, $\langle F(a, w, t), \phi(w) \rangle$ is entire in t. $L_{kh}(f)$

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is easily seen to be independent of the choice of a in S. The abbreviated terminology does not indicate the dependence of the series on the choice of Ω, or $\tilde{\Omega}$, or the sequence $\{r_k\}$.

The p-th partial sum $S_p(z)$ of the series may be written as

$$\sum_{|\zeta_k|<r_p} \left(\frac{1}{2\pi i}\int_{c_k} \frac{\langle F(a, w, t), \phi(w)\rangle}{\phi(t)} \sum_{h=0}^{m_k} \frac{z^h(t-\zeta_k)^h}{h!}\,dt\right) e^{\zeta_k z}.$$

Replacing each m_k by infinity does not change the sum since the introduced terms are zero by Cauchy's Theorem. Writing the inner sum as an exponential enables us to write

$$(2)\quad S_p(z) = \frac{1}{2\pi i}\int_{\Gamma_p} \frac{e^{zt}}{\phi(t)} \langle F(a, w, t), \phi(w)\rangle dt.$$

3. Convergence on interior compacta.

THEOREM 1. *Assume that f is analytic in a simply connected region S containing Ω and that K is an admissible compact subset of Ω^0. Then the ϕ-series of f converges absolutely and uniformly to f in K.*

Proof. The pattern of the proof is to replace a with z in (2), integrate by parts producing $f(z)$ as one term, and then show that the remaining terms tend to zero as p tends to infinity. Since $S_p(z)$ is independent of the choice of a in S, we initially replace a by z for each z in S. Integrating $F(z, w, t)$ by parts k times gives for each z in S

$$(3)\quad S_p(z) = f(z) - Q_p(z) - T_p(z) + R_p(z)$$

where

$$Q_p(z) = \langle f, \phi\rangle \frac{1}{2\pi i}\int_{\Gamma_p} \frac{e^{zt}}{t\phi(t)}\,dt,$$

$$T_p(z) = \sum_{q=1}^{k-1} \langle f^{(q)}, \phi\rangle \frac{1}{2\pi i}\int_{\Gamma_p} \frac{e^{zt}}{t^{q+1}\phi(t)}\,dt, \quad \text{and}$$

$$R_p(z) = \frac{1}{2\pi i}\int_{\Gamma_p} \frac{1}{t^k\phi(t)} \left\langle \int_z^w e^{(z+w-s)t} f^{(k)}(s)ds, \phi(w)\right\rangle dt.$$

Since K is admissible and Ω is compact, $\epsilon > 0$ may be chosen sufficiently small so that both $K \oplus \epsilon \subset \Omega^0$ and $\Omega \oplus 5\epsilon \subset S$. Rectifiable simple closed curves C and D winding once about Ω are chosen in $\Omega \,\widetilde{\cap}\, S$ so that C is in $\Omega \oplus \epsilon \sim \Omega$ and D is in $\Omega \oplus 5\epsilon \sim \overline{\Omega \oplus 4\epsilon}$. Then each point of C is within ϵ of a point of Ω and so within ϵ of a point of $\partial\Omega$. Each point of D is at least 3ϵ from each point of C. For each p we choose $k_p = [3\epsilon r_p]$ and write $S_p(z)$ as in (3) after integrating by parts k_p times. Estimates on T_p and R_p are made by computing the derivatives of f using Cauchy integrals over D and computing the duality integrals with respect to w

using integrals over C. For notational simplicity we will temporarily write k for k_p.

Consider first $T_p(z)$ for z in K and $k \geqq 3$. Let M denote the maximum of $|f(\zeta)\tilde{\phi}(w)|$ for ζ on D and w on C; let L denote 2π times the maximum of the lengths of C and D. Writing $\langle f^{(q)}, \phi \rangle$ as an integral over C and $f^{(q)}(w)$ for w on C as a Cauchy integral over D, we obtain the estimate

$$|\langle f^{(q)}, \phi \rangle| \leqq q!\, ML^2/(3\epsilon)^{q+1}.$$

Also

$$\sum_{q=1}^{k-1} q!/k^{q-1} \leqq 1 + 2(k-2)/k < 3.$$

These estimates, together with the fact that $k = k_p \leqq 3\epsilon r_p$ give for t on Γ_p

$$\left| \sum_{q=1}^{k-1} \langle f^{(q)}, \phi \rangle / t^{q+1} \right| \leqq M\left(\frac{L}{\epsilon r_p}\right)^2.$$

The regularity of ϕ and the choice of ϵ imply that

(4) $\log|\phi(r_p e^{i\theta})| \geqq r_p(h(\theta) - \epsilon/2)$

for all p sufficiently large, and

(5) $\mathscr{R}(ze^{i\theta}) \leqq h(\theta) - \epsilon$

for all z in K.

Writing $T_p(z)$ with Γ_p parameterized by $t = r_p e^{i\theta}$, we conclude that

$$T_p(z) = O(r_p^{-1} \exp(-\epsilon r_p/2))$$

for z in K and all large p. Hence $T_p(z) \to 0$ uniformly on K as $p \to \infty$.

Now consider $R_p(z)$ for z in K (or even z in P when $\Omega = P$). We again use C and D as in the T_p estimate. Using the admissibility of K, we choose for the integral from z in K to w on C a path $\eta(z, w) = \gamma(z, w') + [w', w]$ where w' is on $\partial\Omega$ with $|w - w'| < \epsilon$ and where $\gamma(z, w')$ is chosen so its symmetric curve is in P. Then $\eta(z, w)$ is in $\Omega \oplus \epsilon$ and its symmetric curve is in $P \oplus \epsilon$. The lengths of the curves $\eta(z, w)$ are uniformly bounded by some $b > 0$ since the $\gamma(z, w')$ are uniformly bounded in length. Since $z + w - s$ is in $P \oplus \epsilon$ when s is on $\eta(z, w)$, we have for all θ

$$\mathscr{R}((z + w - s)e^{i\theta}) \leqq h(\theta) + \epsilon.$$

Coupling this inequality with (4) and using the fact that $k = k_p \leqq 3\epsilon r_p$, we have

$$|R_p(z)| \leqq \frac{ML^2 b}{3\epsilon} \frac{k!}{k^k} r_p \exp(2\epsilon r_p).$$

This inequality, together with the inequalities $k! < k^k e^{-k}(2k + 1)$ and

$k_p \leqq 3\epsilon r_p < k_p + 1$, implies that for z in K and p large

$$R_p(z) = O(r_p(6\epsilon r_p + 1) \exp(-\epsilon r_p)).$$

Hence $R_p(z) \to 0$ uniformly on K as $p \to \infty$.

$Q_p(z) \to 0$ uniformly on K since (4) and (5) insure that $Q_p(z) = O(\exp(-\epsilon r_p/2))$. Q_p was separated from T_p here for use in the theorem of the next section.

The absolute convergence of the series also follows from the above estimates. For $S_p(z) - f(z) = O(\exp(-\epsilon r_p/4))$ for z in K. Hence $S_{p+1}(z) - S_p(z)$ also admits such an estimate. Since ϕ is of exponential type, $|\zeta_k| > kc$ for some $c > 0$ and all $k > 1$. Our assumption that $r_k > |\zeta_k|$ gives $r_p > pc$ and $S_{p+1}(z) - S_p(z) = O(1/p^2)$, insuring absolute convergence.

4. Convergence on the boundary. By increasing the assumptions on the lower bound of ϕ on Γ_p, one can draw conclusions concerning the sum of the ϕ-series of f at points on ∂P when f is analytic on P. Uniform convergence to f in a neighborhood of P cannot be expected, for that would imply that f satisfies the convolution equation $\langle f(z + w), \phi(w) \rangle = 0$ in a neighborhood of the origin.

The second conclusion of the following theorem corresponds to the convergence of a Fourier series of f on $[-1, 1]$ to $(f(1) + f(-1))/2$ at $z = 1$ when $\phi(z) = e^z - e^{-z}$, a result that was generalized in [**1**].

THEOREM 2. *In addition to* (1), ϕ *satisfies*

$$(6) \quad |\phi(r_p e^{i\theta})| \geqq A \exp(h(\theta) r_p)$$

for some $A > 0$ *and all* p. f *is analytic on* P. *Then:*

(a) *The* ϕ*-series of* f *converges to* f *at each boundary point of* P *that lies on only one support line of* P, *in particular at each non-extreme point.*

(b) *If* z *on* ∂P *is on lines of support* $\mathscr{R}(ze^{i\theta}) = h(\theta)$ *for* θ *in* $[\theta_1, \theta_2]$ *only and*

$$\lim_{p\to\infty} \exp(zr_p e^{i\theta})/\phi(r_p e^{i\theta}) = L(z)$$

uniformly on compact subsets of (θ_1, θ_2), *then the* ϕ*-series of* f *converges at* z *to*

$$f(z) - \frac{(\theta_2 - \theta_1)}{2\pi} \langle f, \phi \rangle L(z).$$

$\theta_2 - \theta_1$ *is the supplement of the tangential interior angle of* P *at* z.

(c) *The* ϕ*-series of* f *converges uniformly to* f *on each* P_δ.

Proof. We will modify the proof of Theorem 1, treating P as K and taking P as Ω with $\eta(z, w) = [z, w]$. Conditions (4) and (5) are then replaced by (6) and

$$(7) \quad \mathscr{R}(ze^{i\theta}) \leqq h(\theta).$$

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Then for z in P, $T_p(z) = O(1/r_p)$ while the estimate on $R_p(z)$ is unchanged. Hence the convergence of the series depends on the behavior of $Q_p(z)$ as $p \to \infty$.

Let

$$I_p(z) = \frac{1}{2\pi i}\int_{\Gamma_p} \frac{e^{zt}}{t\phi(t)}\,dt$$

and let $I_p(z, \alpha, \beta)$ be that part of $I_p(z)$ obtained by integration over Γ_p from $r_p \exp(i\alpha)$ to $r_p \exp(i\beta)$ when $\alpha \leqq \beta$. The inequalities (6) and (7) imply that

$$|I_p(z, \alpha, \beta)| \leqq (\beta - \alpha)/(2\pi A).$$

(a) The hypothesis implies that if z is such a boundary point, then $\mathscr{R}(ze^{i\theta}) \leqq h(\theta)$ with equality holding for exactly one $\theta = \theta_0$. If ϵ is in $(0, \pi)$, then $I_p(z, \theta_0 - \epsilon, \theta_0 + \epsilon)$ is bounded in modulus by $\epsilon/(\pi A)$ and $I_p(z, \theta_0 + \epsilon, \theta_0 - \epsilon + 2\pi)$ is bounded by $A^{-1} \exp(-\eta r_p)$ where $\eta > 0$ is chosen so that

$$h(\theta) - \mathscr{R}(ze^{i\theta}) \geqq \eta$$

for θ in that closed interval. By choosing ϵ small and then p large, we see that $I_p(z) \to 0$ as $p \to \infty$.

(b) The fact that $I_p(z, \theta_2, \theta_1 + 2\pi) \to 0$ as $p \to \infty$ is established as in (a). Writing the difference between $I_p(z, \theta_1, \theta_2)$ and $L(z)(\theta_2 - \theta_1)/2\pi$ as the sum of integrals from θ_1 to $\theta_1 + \epsilon$, $\theta_1 + \epsilon$ to $\theta_2 - \epsilon$, and $\theta_2 - \epsilon$ to θ_2, the sum of the first and third integrals is bounded by $\epsilon(A^{-1} + L(z))/\pi$ while the second tends to zero as $p \to \infty$ as a result of the limit hypothesis. It follows that

$$I_p(z, \theta_1, \theta_2) \to L(z)(\theta_2 - \theta_1)/2\pi.$$

(c) Consider P_δ as the union of $P_\delta \cap P^0$ and $P_\delta \cap \partial P$. If z is in $P_\delta \cap P^0$, then $\mathscr{R}(ze^{i\theta}) < h(\theta)$ for all θ.

We assert that if z is in $P_\delta \cap \partial P$, then z is in a finite union of closed disjoint line segments I_j on which $\mathscr{R}(z \exp(i\theta_j)) = h(\theta_j)$ for unique θ_j. Let N denote the set of non-extreme points of ∂P. Then $P_\delta \cap \partial P \subset N$. Each point of N is an interior point of an open line segment of ∂P (in the topology relative to ∂P) with extreme points as endpoints. Hence N is the union of such intervals. Since $P^0 \neq \emptyset$ and P is convex, ∂P is a Jordan curve and N is a countable union of disjoint open intervals with endpoints extreme points; say $N = \cup J_j$. Since ∂P is rectifiable, for some $n > 0$ the length of the J_j is less than 2δ when $j > n$. For such j the interval J_j is not in P_δ. Denoting by I_j the interval J_j minus open end segments of length δ, it follows that $P_\delta \cap \partial P$ is contained in $\sum_1^n I_j$. Since

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each I_j is on a unique line of support of P,

$$\mathscr{R}(z \exp(i\theta_j)) = h(\theta_j)$$

for z on I_j for a unique θ_j.

Then for z in P_δ, $h(\theta) - \mathscr{R}(ze^{i\theta}) \geqq 0$ with equality only for some z when $\theta = \theta_j$, $j = 1, 2, \ldots, n$. $I_p(z)$ can then be written as the sum of an integral over small ϵ-neighborhoods of the θ_j and an integral over the remaining intervals. The first is bounded by $\epsilon n/\pi A$ while the second is bounded by $A^{-1} \exp(-\eta r_p)$ where $\eta > 0$ is chosen so that

$$h(\theta) - \mathscr{R}(ze^{i\theta}) \geqq \eta$$

when θ is in the compact intervals not containing the θ_j and z is in the compact P_δ. Again choosing ϵ small and p large, it follows that $I_p(z) \to 0$ uniformly on P_δ as $p \to \infty$ and the series converges uniformly to f.

5. Extensions. Additional conditions may be placed on ϕ that imply the continuity of $\tilde{\phi}$ on $\tilde{P} \cup \partial P$ and allow the "duality" integrals $\langle h, \phi \rangle$ to be taken on ∂P when h is continuous there. This is the case when there is a positive function $v(r)$ with $\int_{r'}^{\infty} v(r)dr < \infty$ for some $r' > 0$ and

(8) $\quad |\phi(re^{i\theta})| \leqq v(r)e^{h(\theta)r}$

for all $r > 0$.

When f is continuous on P and analytic in P^0, the ϕ-series of f may be defined as earlier with the paths of integration from a in P to w on ∂P taken as straight line segments. Then $F(a, w, t)$ is analytic in w in P^0 and continuous in P for each t. For each w in P, $F(a, w, t)$ is entire in t. $L_{kh}(f)$ is independent of the choice of a in P by the extended Cauchy Theorem for triangles.

Using this extended definition of a ϕ-series and placing appropriate lower bounds on $|\phi|$ on Γ_p, we can establish convergence of the series to f on compacta in P^0.

THEOREM 3. *In addition to* (1) *and* (8)

(9) $\quad |\phi(r_pe^{i\theta})| \geqq Ar_p^{-\mu} \exp(h(\theta)r_p)$

for some $A > 0$, $\mu \geqq 0$ *and all* p. f *is in* $C^k(P) \cap H(P^0)$ *and* $k > \mu$. *Then the* ϕ*-series of* f *converges to* f *uniformly on compacta in* P^0.

Proof. For compact K in P^0 choose $\epsilon > 0$ sufficiently small so that $K \oplus \epsilon \subset P$. For the proof it is sufficient to consider the case when $k - 1 \leqq \mu < k$ since $C^k(P) \subset C^{k-1}(P)$. As in the proof of Theorem 1, we replace a by z in $S_p(z)$, integrate by parts k times since f is in $C^k(P)$, and obtain (3). Consider first $Q_p(z)$ and $T_p(z)$ for z in K. Inequalities

(5) and (9) yield the estimate

$$|e^{zt}/t^q\phi(t)| \leqq r_p^{\mu-q}A^{-1}\exp(-\epsilon r_p)$$

for $q = 0, 1, \ldots, k-1$. Hence $Q_p(z)$ and $T_p(z) \to 0$ as $p \to \infty$.

We now show that $R_p(z) \to 0$ as $p \to \infty$. The crucial estimate here is on the modulus of

$$e^{(z+w-s)t}/t^{k-1}\phi(t)$$

for $t = r_p e^{i\theta}$, z in K, w on ∂P, and s on $[z, w]$. The numerator is of modulus near $h(\theta)r_p$ when s is near z. When s is bounded from z by a small $\eta > 0$, the numerator is of modulus less than $\exp(h(\theta) - \eta)r_p$. Suppose that $\delta \in (0, \epsilon)$. For each p let

$$\delta_p = \delta r_p^{k-\mu-1} \leqq \delta.$$

To estimate $R_p(z)$ we write it as the sum of two terms corresponding to integrations from z to $z + \delta_p(w - z)$ and from $z + \delta_p(w - z)$ to w. The first will be bounded in modulus by a constant multiple of δ using (9) and the fact that the integration with respect to s is over an interval of small length δ_p. The second integral is bounded in modulus by a constant multiple of

$$r_p^{\mu-k-1}\exp(-\delta r_p^{k-\mu})$$

since on that interval of integration with respect to s,

$$\mathscr{R}((z + w - s)e^{i\theta}) < (h(\theta) - \delta_p)r_p, \quad \text{and} \quad \delta_p r_p = \delta r_p^{k-\mu}.$$

Since $k > \mu$, the second integral is arbitrarily small when p is large. By choosing δ small and then p large, it is seen that $R_p(z) \to 0$ uniformly on K as $p \to \infty$.

It is evident that either stronger conditions on f or on $|\phi|$ on Γ_p will simplify the proof and can be made to yield absolute convergence.

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