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RESEARCH ARTICLE

Bounding geometrically integral del Pezzo surfaces

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Abstract
We prove several boundedness statements for geometrically integral normal del Pezzo surfaces X over arbitrary fields. We give an explicit sharp bound on the irregularity if X is canonical or regular. In particular, we show that wild canonical del Pezzo surfaces exist only in characteristic 2. As an application, we deduce that canonical del Pezzo surfaces form a bounded family over $\mathbb{Z}$, generalising work of Tanaka. More generally, we prove the BAB conjecture on the boundedness of ε-klt del Pezzo surfaces over arbitrary fields of characteristic different from 2, 3 and 5.

Contents

1. Introduction

We work over a field k of prime characteristic $p > 0$. When running the Minimal Model Program (MMP for short) for klt projective varieties Z with canonical divisor K_Z not pseudo-effective, the outcomes are Mori fibre spaces (i.e., projective fibrations $f: X \to B$ of relative Picard rank 1 where X has klt singularities, $\dim B < \dim X$ and the anti-canonical divisor $-K_X$ is f-ample). It is then natural to study the geometry of X in terms of the base B and the general fibre. In characteristic $p > 0$, the theorem on generic smoothness on general fibres does not always hold, and there are examples of Mori fibre spaces where the general fibre might fail to be normal or even reduced [MS03]. In this case, it is natural to study the *generic* fibre $X_{k(B)}$, which is a klt Fano variety defined over the fraction field $k(B)$, which is *imperfect* as soon as $\dim(B) \geq 1$.

Thanks to the recent development of the 3-dimensional MMP [HX15, CTX15, BW17, HW22, Wal23], Mori fibre spaces are known to exist for 3-folds over fields of characteristic $p > 5$. The next step in the classification problem consists in understanding the generic fibre of a 3-fold Mori fibre space. This work is motivated by the following general question:

Question 1.1. Do the generic fibers of Mori fibre spaces form a bounded family? Can we give explicit bounds on their cohomological invariants?

The main invariant we are interested in is the *irregularity* of the generic fibre. Recall that the irregularity of $X_{k(B)}$ is defined as $h^1(X_{k(B)}, \mathcal{O}_{X_{k(B)}}) := \dim_{k(B)} H^1(X_{k(B)}, \mathcal{O}_{X_{k(B)}})$. The case of relative dimension 1 is easy to treat: regular Fano curves are conics, they have vanishing irregularity and they fail to be geometrically regular only in characteristic $p = 2$. The case of relative dimension 2 (i.e., the geometry of del Pezzo surfaces over imperfect fields) has turned out to be more difficult to handle. There are two known series of examples of canonical del Pezzo surfaces with positive irregularity:

1. In [Sch07], Schröer constructs a canonical del Pezzo surface X with a unique singular factorial point of type A_1, $h^1(X, \mathcal{O}_X) = 1, \rho(X) = 1$ and $K_X^2 = 1$ over an arbitrary imperfect field of characteristic 2.
2. In [Mad16], Maddock constructs regular del Pezzo surfaces X_1 and X_2 defined over an imperfect field of p-degree 3 (resp. 4) with $K_{X_d}^2 = d$ and $h^1(X_d, \mathcal{O}_{X_d}) = 1$. Moreover, X_1 is geometrically integral, and X_2 is not.

On the positive side, the recent works [PW22, FS20, JW21, BT22] indicate that the pathological behaviour of del Pezzo fibrations is particular to small characteristics. In this article, we further restrict the possibilities for the irregularity of geometrically integral canonical del Pezzo surfaces defined over imperfect fields. Our first main result is the following:

Theorem 1.2. *Let X be a geometrically integral normal locally complete intersection del Pezzo surface over a field k of characteristic p. If* $h^1(X, \mathcal{O}_X) \neq 0$*, then k is an imperfect field,* $\rho(X) = 1$*, and either*

1. $p = 3$, $h^1(X, \mathcal{O}_X) = 2$, $K_X^2 = 1$*, and X is not canonical, or*
2. $p = 2$, $h^1(X, \mathcal{O}_X) = 1$*, and* $K_X^2 \leq 2$.

We note that our bound on the irregularity in the regular case is sharp, as Maddock's example shows. In Proposition 4.11, we describe torsors over the regular wild del Pezzo surfaces in characteristic $p = 2$. This is a first step towards a classification of wild regular del Pezzo surfaces. In particular, it may be useful for the construction of explicit examples in the style of Maddock. Note that the hypothesis on geometric integrality is automatically satisfied for normal del Pezzo surfaces appearing as generic fibres of 3-folds by [Sch10, Theorem 2.3].

In the second part of this article, we prove boundedness results for del Pezzo surfaces over imperfect fields. The Borisov–Alexeev–Borisov (BAB) conjecture (see Conjecture 5.2) states that mildly singular (ε-klt) Fano varieties of dimension d form a bounded family over Spec$\mathbb{Z}$. While the conjecture has been proven over fields of characteristic 0 by Birkar [Bir21], it is still open over fields of characteristic p.

More precisely, while the case of del Pezzo surfaces over perfect fields has been known for a long time (see [Ale94, AM04] and [CTW17, Lemma 3.1]), already the boundedness of 3-dimensional Fano varieties is open. In this direction, the BAB conjecture for generic fibres of Mori fibre spaces would be desirable. In [Tan24], Tanaka showed that geometrically integral regular del Pezzo surfaces form a bounded family. Using Theorem 1.2 and the results on the irregularity of klt del Pezzo surfaces of [BT22], we are able to prove various instances of the BAB conjecture, following the strategy of Alexeev–Mori [AM04]:

Theorem 1.3. *The following classes of del Pezzo surfaces are bounded over* $\operatorname{Spec}\mathbb{Z}$*:*

$$\mathcal{X}_{dP,can} = \{X \mid X \text{ is a geometrically integral canonical del Pezzo surface}\},$$
$$\mathcal{X}^{tame}_{dP,\varepsilon} = \{X \mid X \text{ is a geometrically integral tame } \varepsilon\text{-klt del Pezzo surface}\}, \text{ and}$$
$$\mathcal{X}^{>5}_{dP,\varepsilon} = \left\{X \mid X \text{ is an } \varepsilon\text{-klt del Pezzo surface s.t. } char(H^0(X,\mathcal{O}_X)) \neq 2,3,5\right\}.$$

We briefly explain the organisation of the article. In section 2, we collect various results on geometry over imperfect fields and del Pezzo surfaces. In section 3, we generalise the main results of Tanaka [Tan24] to the canonical case. We use Ekedahl's technique [Eke88] on the construction of α-torsor to show an effective Kodaira vanishing theorem (Proposition 3.6) from which we deduce that ω_X^{-12} is very ample (Theorem 3.10). Starting from section 4, we specialise to the study of geometrically integral del Pezzo surfaces. We show that the Frobenius length of geometric non-normality (an invariant introduced by Tanaka [Tan21]) is at most 1 (Corollary 4.4) on normal Gorenstein del Pezzo surfaces, a result we use to find lower bounds on the dimension of the space of anti-pluricanonical sections. We combine these estimates together with Maddock's bound [Mad16, Corollary 1.2.6] and a careful study of α-torsors to prove Theorem 1.2. In section 5, we apply our results to the BAB conjecture over arbitrary fields, and we prove Theorem 1.3.

2. Preliminaries

2.1. Notations

1. Given a field k, we denote by $\overline{k}$ (resp. k^{sep}) an algebraic (resp. separable) closure. We denote by k^{1/p^∞} the perfect closure of k.
2. Given a field k, a scheme X is a k-variety if it is an integral separated scheme of finite type over k. If X has dimension 1 (resp. 2, 3), we say X is a curve (resp. surface, 3-fold).
3. Given a projective integral k-variety X, we let $d_X := [H^0(X,\mathcal{O}_X) : k]$.
4. Given an $\mathbb{F}_p$-scheme X, we denote by $F\colon X \to X$ the absolute Frobenius morphism of X. We say X is F-finite if F is a finite morphism.
5. For an F-finite field k, its p-degree (or degree of imperfection) is defined as $\mathrm{p}-\deg(k) := \log_p[k : k^p]$.
6. We say (X,Δ) is a *pair* if X is a normal k-variety, Δ is an effective $\mathbb{Q}$-divisor with coefficients in $[0,1]$ and $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor.
7. For the definitions of the singularities of the MMP (as canonical, klt and log canonical), we refer to [Kol13, Definition 2.8].
8. Given an integral scheme X with normalisation $\nu\colon Y \to X$, we denote by $\mathcal{I} \subset \mathcal{O}_X$ the conductor ideal (i.e., the annihilator of the $\mathcal{O}_X$-module $\nu_*(\mathcal{O}_Y)/\mathcal{O}_X$)). The corresponding closed subscheme $D \subset X$ is called the *conductor scheme* of ν. Note that $\mathcal{I}$ is also an ideal of $\mathcal{O}_Y$ and the corresponding subscheme $C \subset Y$ is called *ramification locus* of ν.
9. A projective morphism $f\colon X \to Y$ of normal schemes is a *contraction* if $f_*\mathcal{O}_X = \mathcal{O}_Y$.

2.2. Geometric reducedness and normality

We collect well-known results on the geometry of algebraic varieties, especially surfaces, defined over imperfect fields that we need in this article.

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Definition 2.1. A k-variety X is *geometrically reduced* (resp. *geometrically normal, geometrically regular*) if the base change $X_{\overline{k}}$ is reduced (resp. normal, regular).

We recall Tate's base change formula for purely inseparable field extensions.

Theorem 2.2 [PW22, Theorem 1.1]. *Let X be a normal k-variety such that k is algebraically closed in $K(X)$. Let Y be the normalisation of the reduced scheme $(X \times_k \overline{k})_{\text{red}}$ together with the natural morphism $f\colon Y \to X$. Then there exists an effective divisor $C \geq 0$ such that $K_Y + (p-1)C = f^*K_X$. If X is geometrically integral, then $(p-1)C$ can be chosen to be the ramification divisor of f.*

We start with the behaviour of geometric reducedeness under birational equivalence.

Lemma 2.3 [BT22, Lemma 2.2]. *Let X and Y be two k-birational varieties. Then X is geometrically reduced over k if and only if Y is geometrically reduced over k.*

Next, we note that geometric normality descends under birational contractions. For the definition of the (S_n)-property, we refer to [Sta, Tag 033Q].

Proposition 2.4. *Let $\pi\colon X \to Y$ be a projective birational morphism of normal k-varieties. If X is geometrically normal, so is Y.*

Proof. Recall that a variety X over k has the property (S_n) if and only if $X_{\overline{k}}$ also has, by faithfully flat descent. As Y is (S_2), by Serre's criterion [Sta, Tag 031S], Y is geometrically normal if and only if it is geometrically (R_1). Suppose by contradiction that there exists a codimension 1 point $\eta \in Y$ such that the localisation $\mathcal{O}_{Y,\eta}$ is not geometrically regular. As Y is normal, π is an isomorphism over codimension 1 points of Y, and thus, X is not geometrically (R_1), reaching the contradiction. □

We discuss singularities of the MMP over imperfect fields.

Definition 2.5. Let (X, Δ) be a pair over k such that k is algebraically closed in $K(X)$. We say it is *geometrically canonical* (resp. klt, log canonical) if the base change $(X_{\overline{k}}, \Delta_{\overline{k}})$ is so.

In particular, note that geometrically log canonical implies geometrically normal. If X is geometrically canonical (resp. klt, lc), then X is also canonical (resp. klt, lc) by [BT22, Proposition 2.3]. We now specialise to the case of surfaces. Recall that the existence of resolution of singularities for excellent surfaces has been proven in [Lip78].

Proposition 2.6. *Let X be the spectrum of a local excellent ring $(R, \mathfrak{m})$ with closed point x. If (X, Δ) is a klt surface pair for some $\Delta \geq 0$, then X has rational and $\mathbb{Q}$-factorial singularities. Therefore, if two projective k-surfaces X and Y with klt singularies are k-birational, then $H^i(X, \mathcal{O}_X) \simeq H^i(Y, \mathcal{O}_Y)$ for every $i \geq 0$.*

Proof. Rationality of klt surface singularities follows from [Kol13, Proposition 2.28], and $\mathbb{Q}$-factoriality of rational singularities is proven in [Lip69, Proposition 17.1]. The last statement is obvious by considering a common resolution of X and Y. □

Corollary 2.7. *Let $(x \in X)$ be a Gorenstein normal surface singularity. Then X is canonical if and only if it is rational.*

Proof. If X is canonical, then it is rational by Proposition 2.6. Suppose now that X is rational and let $f\colon Y \to X$ be a resolution of singularities. As X is Gorenstein and X has rational singularities, we have that $f_*\omega_Y = \omega_X$ by [Kol13, Proposition 2.77], which in turn implies that X has canonical singularities by [Kol13, Claim 2.3.1]. □

2.3. Del Pezzo surfaces

In this subsection, we collect some terminology on del Pezzo surfaces and recall previously known results.

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Definition 2.8. We say X is a *Gorenstein* (resp. *canonical*, *regular*) *del Pezzo surface* over k if X is a reduced k-projective Gorenstein (resp. canonical, regular) surface with $H^0(X, \mathcal{O}_X) = k$ and ω_X^{-1} is ample. We say X is a *weak* del Pezzo if ω_X^{-1} is big and nef.

We recall the classification of Gorenstein normal del Pezzo surfaces over algebraically closed fields:

Proposition 2.9 [HW81, Theorem 2.2]. *Let X be a normal Gorenstein del Pezzo surface over an algebraically closed field k. Then one of the following holds:*

1. *X is a canonical del Pezzo surface and the explicit list is described in [Dol12, Section 8], or*
2. *the minimal resolution $Z \to X$ is a ruled surface of the form $\mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{L})$, where E is an elliptic curve and* $\deg \mathcal{L} < 0$. *The surface X is obtained by contracting the negative section of Z.*

In [BT22, Theorem 3.3], it is shown that canonical del Pezzo surfaces which are geometrically normal are geometrically canonical. We present a different proof of this result relying on Proposition 2.9 and the following observation:

Lemma 2.10. *Let $(y \in Y)$ be a geometrically log canonical surface singularity over k. Suppose that Y has rational singularities. Then $Y_{\overline{k}}$ has rational singularities.*

Proof. We can suppose k is separably closed and Y is the spectrum of a local henselian ring $(R, \mathfrak{m})$ by the existence of resolution of singularities [Lip78]. Let $U := \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ be the punctured spectrum. Since Y is rational, the group $\mathrm{Pic}(U)$ is finite by [Lip69, Proposition 17.1]. Therefore, also $X := Y_{\overline{k}}$ is $\mathbb{Q}$-factorial by [Tan18a, Lemma 2.5], and thus, $\mathrm{Pic}(U_{\overline{k}})$ is a torsion group. Let $f\colon W \to X$ be the minimal resolution with exceptional divisor $E = \sum_{i=1}^n E_i$. As defined in [Lip69], $\mathrm{Pic}^0(W)$ is the group of line bundles L on W such that $L \cdot E_i = 0$ for every i and there is an exact sequence of groups $0 \to \mathrm{Pic}^0(W) \to \mathrm{Pic}(W) \to \bigoplus \mathbb{Z}[E_i] \to 0$. By [Lip69, Proposition 14.4], $\mathrm{Pic}^0(W)$ embeds into $\mathrm{Pic}(U_{\overline{k}})$, and thus, we deduce it is a torsion group.

Suppose now by contradiction that X is not rational. By the classification of log canonical singularities [Kol13, Corollary 3.39], the exceptional divisor E is either an elliptic curve, a nodal curve or a circle of smooth rational curves. In the first case, $\mathrm{Pic}^0(E) \simeq E(k)$, while in the latter cases, $\mathrm{Pic}^0(E) \simeq k^*$ by [BLR90, Chapter 9.3, Corollary 11 and 12] and since $h^1(E, \mathcal{O}_E) = 1$. By [Lip69, Lemma 14.3], the restriction map $\mathrm{Pic}(W) \to \mathrm{Pic}(E)$ is surjective. Considering the exact sequence $0 \to \mathrm{Pic}^0(E) \to \mathrm{Pic}(E) \to \mathbb{Z}^n \to 0$, we can deduce that the map $\mathrm{Pic}^0(W) \to \mathrm{Pic}^0(E)$ is surjective. This is a contradiction, as $\mathrm{Pic}^0(W)$ is torsion while k^* and $E(k)$ are not. □

Proposition 2.11. *Let X be a canonical del Pezzo surface. If X is geometrically normal, then it is geometrically canonical.*

Proof. By Proposition 2.9, X is geometrically log canonical. As X has rational singularities, X is geometrically rational by Lemma 2.10. As X is Gorenstein, we conclude that X is geometrically canonical by Corollary 2.7. □

We now recall the results of Reid on the classification of non-normal Gorenstein del Pezzo surfaces [Rei94]. We fix some notations we will use throughout the article (cf. subsection 2.1 for the terminology used).

Definition 2.12. Let X be a non-normal integral Gorenstein del Pezzo surface with normalisation $\nu\colon Y \to X$. We say X is *tame* if $H^1(X, \mathcal{O}_X) = 0$.

One can characterise tame del Pezzo surfaces in terms of the conductor.

Theorem 2.13. *Let X be a non-normal integral Gorenstein del Pezzo surface over an algebraically closed field. Then the conductor $D \subset X$ is integral. Moreover,*

1. *X is tame if and only if $D \simeq \mathbb{P}^1$;*
2. *$(p-1)$ divides $h^1(\mathcal{O}_X)$.*

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Proof. The integrality of the conductor follows from [Rei94, Lemma, page 718] for integral del Pezzo surfaces. Then, (1) follows from the proof of [Rei94, Corollary 4.10], as D is irreducible. (2) is proved in [Rei94, 4.11] □

We will repeatedly use the following:

Lemma 2.14. *Let $\pi\colon X \to Y$ be a proper birational morphism of k-surfaces. If X is a regular (resp. canonical) del Pezzo surface, then so is Y. If X is a regular (or canonical) weak del Pezzo surface, then also Y is a canonical weak del Pezzo surface.*

Proof. We only prove the case where X is a regular weak del Pezzo surface, as the others are similar. As $-K_X$ is π-big and π-nef, we conclude that Y has canonical singularities by the negativity lemma [Tan18b, Lemma 2.11]. As $-K_Y = \pi_*(-K_X)$, we conclude by projection formula that $-K_Y$ is big and nef. □

From the point of view of the MMP, it is natural to consider surfaces of del Pezzo type. For their basic properties, we refer to [BT22, Section 2.3].

Definition 2.15. We say X is a *surface of del Pezzo type* over k if X is a projective k-variety with $H^0(X, \mathcal{O}_X) = k$ and there exists $\Delta \geq 0$ such that (X, Δ) is a log del Pezzo pair (i.e., (X, Δ) klt and $-(K_X + \Delta)$ is big and nef).

The following describes the Picard scheme of del Pezzo surfaces.

Proposition 2.16. *Let X be a surface of del Pezzo type. Then $\mathrm{Pic}^0_{X/k}$ is a unipotent smooth commutative k-group scheme of finite type over k of dimension $h^1(X, \mathcal{O}_X)$.*

Proof. By Serre duality, we have $H^2(X, \mathcal{O}_X) = H^0(X, \omega_X) = 0$, and therefore, by [FGI05, Corollary 9.4.18.3, Corollary 9.5.13 and Remark 9.5.15], the group scheme $\mathrm{Pic}^0_{X/k}$ is smooth of dimension $h^1(\mathcal{O}_X)$. We are left to show that $\mathrm{Pic}^0_{X/k}$ is unipotent. For this, we can suppose k is separably closed. By [BT22, Theorem 1.3], there exists $n > 0$ such that for every $L \in \mathrm{Pic}^0(X)$, we have $L^{\otimes p^n} \simeq \mathcal{O}_X$. In other words, multiplication by p^n on $\mathrm{Pic}^0_{X/k}$ coincides with the zero homomorphism on k-rational points. By density of rational points [Poo17, Proposition 3.5.70] and since $\mathrm{Pic}^0_{X/k}$ is reduced, we conclude that taking p^n-powers on $\mathrm{Pic}^0_{X/k}$ coincides with the zero homomorphism as a morphism of schemes, and thus, $\mathrm{Pic}^0_{X/k}$ is unipotent. □

3. Bounds on the anticanonical volume and effective very ampleness

In this section, we prove bounds on the anticanonical volume and very ampleness statements for canonical del Pezzo surfaces over imperfect fields.

3.1. Bounding volumes

We start by bounding the volume of canonical del Pezzo surfaces in terms of their thickening exponent $\epsilon(X/k)$ (see [Tan24, Definition 7.4] and Definition 3.2 below). First, we need an explicit bound on the Cartier index of a klt surface singularity. For a $\mathbb{Q}$-factorial variety X, we define its *Cartier index* to be the smallest integer $n > 0$ such that for every Weil divisor D in X, the Weil divisor nD is Cartier.

Lemma 3.1. *Let X be the spectrum of a local k-algebra $(R, \mathfrak{m})$, and let x be the closed point corresponding to $\mathfrak{m}$. Suppose (X, Δ) is a klt surface pair for some $\Delta \geq 0$. Let $f\colon Y \to X$ be the minimal resolution of singularities, with exceptional divisor $E = \sum_{i=1}^n E_i$. Let $M = (E_i \cdot_k E_j)_{i,j=1}^n$ be the intersection matrix and let $d = \det(M)$. Then there exists d_x such that $d = d_x[k(x) : k]$ and the Cartier index of X divides d_x.*

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Proof. Recall that X is rational and $\mathbb{Q}$-factorial by Proposition 2.6. Let D be a Weil integral divisor on X, and write $f^*D = f_*^{-1}D + \sum_{i=1}^n a_i E_i$ for some $a_i \in \mathbb{Q}$.

We claim it is sufficient to show $d_x a_i$ is integral. Indeed, then $f^*(d_x D)$ is an integral divisor on a regular surface, and thus, $f^*(d_x D)$ is Cartier. If we write $K_Y + \Delta_Y = f^*(K_X + \Delta)$, then Δ_Y is effective by the negativity lemma and (Y, Δ_Y) is klt. As $f^*(d_x D) - (K_Y + \Delta_Y)$ is f-nef and big, and $f^*(d_x D)$ is f-trivial, there exists $b_0 > 0$ such that for all $b \geq b_0$, we have that $bf^*(d_x D) = f^*A_b$ for a Cartier divisor A_b on X by the base point free theorem for excellent surfaces [Tan18b, Theorem 4.4]. Then $f^*(d_x D) = (b_0+1)f^*d_x D - b_0 f^* d_x D = f^*(A_{b_0+1} - A_{b_0})$, and thus, $d_x D$ is Cartier.

We denote by (a_i) (resp. $f_*^{-1}D \cdot E_j$) the vector $(a_1, \dots, a_n)$ (resp. $(f_*^{-1}D \cdot E_1, \dots, f_*^{-1}D \cdot E_n)$). Given a closed point $x \in X$, we denote by $k(x)$ the residue field of X at x. By the projection formula,

$$(a_i) = M^{-1}(-f_*^{-1}D \cdot E_j) = \frac{1}{d_x[k(x):k]} A(f_*^{-1}D \cdot E_j),$$

where A is a matrix with integer coefficients. We have $(-f_*^{-1}D \cdot E_j) = \sum_j m_j [k(y_j):k]$ for some $m_j \in \mathbb{Z}$, where the y_j are the intersection points of $f_*^{-1}D$ with E_j. As $k(x) \subset k(y_j)$, we conclude that $[k(x):k]$ divides $(f_*^{-1}D \cdot E_j)$, thus showing $d_x a_i$ is an integer. □

We bound the volume of canonical del Pezzo surfaces, generalising the regular case proven in [Tan24, Theorem 4.7].

Definition 3.2 [Tan21, Definition 5.1, Definition 7.4]. Let X be a normal variety over k such that k is algebraically closed in $K(X)$. We define the *Frobenius length of geometric non-normality* $\ell_F(X/k)$ as

$$\ell_F(X/k) := \min\left\{ e \geq 0 \mid (X \times_k k^{1/p^e})_{\mathrm{red}}^{\mathrm{norm}} \text{ is geometrically normal over } k^{1/p^e} \right\}.$$

Set R to be the local ring of $X \times_k k^{1/p^\infty}$ at the generic point. We define the *thickening exponent* $\epsilon(X/k)$ as the non-negative integer such that $\mathrm{length}_R R = p^{\epsilon(X/k)}$

For a discussion of the properties of $\ell_F(X/k)$ and $\epsilon(X/k)$, we refer the reader to [Tan21, Section 5, Section 7].

We fix some notation. For $d \geq 1$, we denote the Hirzebruch surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-d))$ by $\mathbb{F}_d$, a closed rational fibre by F and the negative section by C_d. The contraction of C_d is the morphism $p \colon \mathbb{F}_d \to \mathbb{P}(1,1,d)$, and we denote by $L := p_*F$ the generator of its class group. Recall that $L \in |\mathcal{O}_{\mathbb{P}(1,1,d)}(1)|$ and that $L^2 = \frac{1}{d}$.

Lemma 3.3. *The divisor class $nK_{\mathbb{P}(1,1,d)}$ is Cartier if and only if $d \mid n(d+2)$.*

Proof. As $K_{\mathbb{P}(1,1,d)} \sim (-d-2)L$ and the Cartier index of L is d, the lemma is immediate. □

Proposition 3.4. *Let X be a canonical del Pezzo surface. Then*

1. *if X is geometrically normal, then it is geometrically canonical and $K_X^2 \leq 9$;*
2. *if X is not geometrically normal, then $p \in \{2, 3\}$ and*
   (a) *if $p = 3$, $\ell_F(X/k) = 1$ and $K_X^2 \leq 12 \cdot 3^{\epsilon(X/k)}$.*
   (b) *if $p = 2$, $\ell_F(X/k) \leq 2$ and $K_X^2 \leq 16 \cdot 2^{\epsilon(X/k)}$.*

Proof. We can assume k to be separably closed, and we will repeatedly use the fact that $\epsilon(X/k)$ is a k-birational invariant [Tan21, Proposition 7.10]. If X is geometrically normal, then we conclude by Proposition 2.11. So we suppose that X is not geometrically normal and $p = 2, 3$ by [BT22, Theorem 3.7.(1)]. The bounds on $\ell_F(X/k)$ are proven in [BT22, Theorem 3.7.(2)-(3)].

Let $Z \to X$ be the minimal resolution of X. As Z is a regular weak del Pezzo surface, by Lemma 2.14, a K_Z-MMP will end with a regular weak del Pezzo surface Y admitting a Mori fibre space $f \colon Y \to B$ (i.e., f is a contraction where $-K_Y$ is f-ample and $\dim(B) \leq 1$). Note that $K_Y^2 \geq K_Z^2 = K_X^2$.

If $B = \mathrm{Spec}(k)$, then Y is a regular del Pezzo surface, and we conclude by [Tan24, Theorem 4.7]. If B is a curve, as Y is weak del Pezzo, the cone theorem [Tan18b, Theorem 2.14] implies that the Mori cone of Y is

$$\mathrm{NE}(Y) = \mathbb{R}_+[F] + \mathbb{R}_+[\Gamma],$$

where F the class of a closed fibre of f and Γ is the class of an integral curve with self-intersection $\Gamma^2 \leq 0$. If $K_Y \cdot_k \Gamma < 0$, then Y is a regular del Pezzo surface by Kleiman's criterion, and we conclude again by [Tan24, Theorem 4.7].

If $K_Y \cdot_k \Gamma = 0$, by the Hodge index theorem $\Gamma^2 < 0$ and, if we denote $k_\Gamma = H^0(\Gamma, \mathcal{O}_\Gamma)$, by adjunction, the equality $\Gamma^2 = \deg_k \omega_{\Gamma/k} = -2[k_\Gamma : k]$ holds. Then there exists a birational contraction $Y \to T$ where T is a canonical del Pezzo surface of Picard rank 1 with a unique singular point x and $K_T^2 = K_Y^2$. As $k_\Gamma = k(x)$ by [Kol13, Corollary 10.10], we have $\Gamma^2 = -2[k(x) : k]$, which implies that the Cartier index of T divides 2 by Lemma 3.1. If T is geometrically normal, it is geometrically canonical by [BT22, Theorem 3.7]. Moreover, as $T_{\overline{k}}$ has Picard rank 1 and a singular point, we conclude $K_X^2 \leq 8$. If T is not geometrically normal and $g\colon V = (T \times_k \overline{k})_{\mathrm{red}}^{\mathrm{norm}} \to T$ is the normalised base change where $K_V + (p-1)C = g^*K_T$, we deduce that $2p^{\ell_F(T/k)}K_V$ is Cartier by [Tan21, Theorem 5.12]. By the classification of the normalised base changes of canonical del Pezzo surfaces with Picard rank 1 [PW22, Theorem 4.1], the bounds on the Frobenius length [BT22, Theorem 3.7] and Lemma 3.3, we deduce the following:

- if $p = 3$, then $6K_V$ is Cartier, and thus, $V \simeq \mathbb{P}(1,1,d)$ for $d \in \{1,2,3,4,6,12\}$ and $C = L$;
- if $p = 2$, then $8K_V$ is Cartier, and thus, $V \simeq \mathbb{P}(1,1,d)$ for $d \in \{1,2,4,8,16\}$ and $C = L$ or $2L$ by [Tan24, Proposition 4.1].

Using [Tan24, Lemma 4.5], we have

$$p^{\epsilon(X/k)}(g^*K_T)^2 = p^{\epsilon(X/k)}(K_V + (p-1)C)^2 = K_T^2.$$

If $p = 3$, we have $V = \mathbb{P}(1,1,d)$, $C = L$, and thus, $K_T^2 \leq (dL)^2 \cdot 3^{\epsilon(X/k)} = d \cdot 3^{\epsilon(X/k)} \leq 12 \cdot 3^{\epsilon(X/k)}$. Similarly, in the case where $p = 2$, we obtain that $K_T^2 \leq 16 \cdot 2^{\epsilon(X/k)}$. □

Using the bounds on the anticanonical volume, we can restrict the possibilities for the normalised base changes of non-normal canonical del Pezzo surfaces obtained in [PW22, Theorem 4.1]. For the analogous result in the regular case, see [Tan24, Theorem 4.6].

Theorem 3.5. *Let X be a canonical del Pezzo surface. Let $\nu\colon Y \to (X \times_k \overline{k})_{\mathrm{red}}$ be the normalisation morphism and let $f\colon Y \to X \times_k \overline{k}$ be the composite morphism.*

1. *If X is geometrically normal, then it is geometrically canonical.*
2. *If $p \geq 5$, then X is geometrically normal.*
3. *If $p = 3$ and X is not geometrically normal, then $\ell_F(X/k) = 1$ and (Y, C) is isomorphic to $(\mathbb{P}(1,1,d), L)$ for some $d \leq 12$.*
4. *If $p = 2$ and X is not geometrically normal, then $\ell_F(X/k) \in \{1,2\}$ and (Y, C) is isomorphic to one of the following:*
   (a) $(\mathbb{P}^2, L)$ *and* $\ell_F(X/k) = 1$;
   (b) $(\mathbb{P}^2, C \in |2L|)$;
   (c) $(\mathbb{P}(1,1,d), 2L)$ *for* $2 \leq d \leq 16$.
   (d) $(\mathbb{P}^1 \times \mathbb{P}^1, C \in |F_1 + F_2|)$ *and* $\ell_F(X/k) = 1$;
   (e) $(\mathbb{P}^1 \times \mathbb{P}^1, F_i)$ *and* $\ell_F(X/k) = 1$;
   (f) $(\mathbb{F}_d, D \in |C_d + F|)$, *where* C_d *is the negative section and* $\ell_F(X/k) = 1$ *for* $1 \leq d \leq 14$;
   (g) $(\mathbb{F}_d, C_d)$ *and* $\ell_F(X/k) = 1$ *for* $1 \leq d \leq 12$;

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Proof. By Proposition 3.4, we are only left to prove the classification in (3) and (4). Suppose $p = 3$. The only possible normalised base change is $\mathbb{P}(1, 1, d)$ by [Tan24, Proposition 4.1 and Remark 4.3]. However, by Proposition 3.4, we have $K_X^2 = p^{\epsilon(X/k)}d \le 12 \cdot p^{\epsilon(X/k)}$.

Suppose $p = 2$. The list of possibilities without the bounds on d is proved in [Tan24, Proposition 4.1]. It is now sufficient to note that in Case (4f), $K_X^2 = p^{\epsilon(X/k)}(d+2)$; in Case (4g), $K_X^2 = p^{\epsilon(X/k)}(d+4)$; and in Case (4c), $K_X^2 = p^{\epsilon(X/k)}d$. Using Proposition 3.4, we deduce the desired bounds on d. □

3.2. Effective Kodaira vanishing and very ampleness on del Pezzo surfaces

In this section, we prove an effective version of the Kawamata–Viehweg vanishing theorem on canonical del Pezzo surfaces. From this, we deduce bounds on the effective global generation and very ampleness for the anti-pluricanonical linear systems.

We start by giving an effective version of [PW22, Theorem 1.9] in the 2-dimensional case.

Proposition 3.6. *Let X be a canonical del Pezzo surface and let A be a big and nef Cartier divisor on X. Then*

1. *if $p > 3$, then $H^1(X, \mathcal{O}_X(-A)) = 0$;*
2. *if $p = 3$, then $H^1(X, \mathcal{O}_X(-dA)) = 0$ if $d \ge 2$;*
3. *if $p = 2$, then $H^1(X, \mathcal{O}_X(-dA)) = 0$ if $d \ge 4$.*

If X is a normal Gorenstein del Pezzo surface, the same results hold if A is ample.

Proof. We let $\mathcal{A}_m = \mathcal{O}_X(mA)$ for $m \in \mathbb{Z}$. We fix $d > 0$. We show that $H^1(X, \mathcal{A}_{-dn}) = 0$ for n sufficiently large. If A is ample, we conclude by Serre duality and Serre vanishing. If A is only big and nef and X is a canonical del Pezzo surface, by the base point free theorem [Ber21b, Proposition 2.1], there is a birational contraction $\pi\colon X \to Y$ such that $A = \pi^*H$, where H is an ample Cartier divisor and by Lemma 2.14, Y is a del Pezzo surface with canonical singularities. Thus, the singularities of Y are rational by Proposition 2.6, and the projection formula implies $H^1(X, \mathcal{A}_{-dn}) = H^1(Y, \mathcal{H}_{-dn})$. As Y is a normal, we can apply Serre duality to deduce $H^1(Y, \mathcal{H}_{-dn}) \simeq H^1(Y, \omega_X \otimes \mathcal{H}_{dn})$, which vanishes for n large enough by Serre vanishing.

Suppose $H^1(X, \mathcal{A}_{-d}) \ne 0$. Without loss of generality by the previous paragraph, we can assume $F^*\colon H^1(X, \mathcal{A}_{-d}) \to H^1(X, \mathcal{A}_{-pd})$ has a nontrivial element ζ in the kernel $H^1_{\mathrm{fppf}}(X, \alpha_{\mathcal{A}_{-d}})$. By [PW22, Theorem 2.11], associated to ζ there exists a degree p purely inseparable morphism $\pi\colon Z \to X$ such that Z is an integral Gorenstein surface with $\omega_Z = \pi^*(\omega_X(-(p-1)dA))$. Let $\nu\colon Z^{\mathrm{norm}} \to Z$ be the normalisation and let $\mu\colon Y := (Z^{\mathrm{norm}} \times_k \overline{k})^{\mathrm{norm}}_{\mathrm{red}} \to Z^{\mathrm{norm}}$ be the normalised base change to the algebraic closure. We denote by Γ the divisorial part of the ramification locus. We have $\mathcal{O}_{Z^{\mathrm{norm}}}(K_{Z^{\mathrm{norm}}} + \Gamma) = \nu^*(\omega_Z)$, and there exists an effective Weil divisor $D \ge 0$ such that $K_Y + (p-1)D = \mu^*K_{Z^{\mathrm{norm}}}$ by [PW22, Theorem 1.1], and we conclude

$$K_Y + (p-1)D + \mu^*\Gamma = f^*(K_X - (p-1)dA),$$

where $f = \pi \circ \nu \circ \mu$. Consider a general curve C on Y of genus $g \ge 1$ so that C is contained in the smooth locus of Y and $C \cdot ((p-1)D + \mu^*\Gamma) \ge 0$. Therefore, $K_Y \cdot C < 0$, and the bend and break lemma [Kol96, Chapter II, Theorem 5.8] shows that for every point $x \in C$, there exists a rational curve L_x such that

$$-(K_Y + (p-1)D + \mu^*\Gamma) \cdot L_x \le 4\frac{-(K_Y + (p-1)D + \mu^*\Gamma) \cdot C}{-K_Y \cdot C} \le 4,$$

as $-(K_Y + (p-1)D + \mu^*\Gamma)$ is big and nef. Since $-K_X$ is ample, we infer the inequality

$$f^*((p-1)dA) \cdot L_x < f^*(-K_X + (p-1)dA) \cdot L_x \le 4.$$

As $A = \pi^*H$ where H is an ample Cartier divisor and x is a general point on C, we have $f^*A \cdot L_x \ge 1$, and thus, we have $(p-1)d \le 3$, which concludes the proof. □

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Lemma 3.7. *Let X be a canonical del Pezzo surface such that X is not geometrically normal. Let A be a big and nef Cartier divisor on X. Then*

1. *if* $p = 3$*, then* $\mathcal{O}_X(3A)$ *is globally generated;*
2. *if* $p = 2$ *and* $\ell_F(X/k) = 1$*, then* $\mathcal{O}_X(2A)$ *is globally generated;*
3. *if* $p = 2$ *and* $\ell_F(X/k) = 2$*, then* $\mathcal{O}_X(4A)$ *is globally generated.*

Proof. The proof is the same as [Tan24, Theorem 3.5]. There is a factorisation of the iterated Frobenius morphism by [Tan21, Theorem 5.9]:

$$F^{\ell_F(X/k)}_{X\times_k \overline{k}} : X \times_k \overline{k} \to (X \times_k \overline{k})^{\mathrm{norm}}_{\mathrm{red}} \xrightarrow{\mu} X \times_k \overline{k},$$

where $(X \times_k \overline{k})^{\mathrm{norm}}_{\mathrm{red}}$ is a toric variety by Theorem 3.5. Thus, $\mu^* A$ is globally generated and also $(F^{\ell}_F(X/k)_{X\times_k \overline{k}})^* A = A^{p^{\ell_F(X/k)}}$. □

We recall the following very ampleness criterion for line bundles. For the notion of Castelnuovo–Mumford regularity and its basic properties, we refer to [Laz04, Section 1.8].

Proposition 3.8 [Tan21, Lemma 11.2]. *Let X be a geometrically irreducible k-projective variety of dimension n. Let* $\mathcal{A}$ *be a globally generated ample line bundle and suppose* $\mathcal{L}$ *is an ample line bundle which is* 0*-regular with respect to* $\mathcal{A}$*. Then* $\mathcal{A} \otimes \mathcal{L}$ *is very ample.*

The following is a generalisation of [Tan24, Theorem 3.5] including the case of canonical del Pezzo surfaces.

Proposition 3.9. *Let X be a canonical del Pezzo surface such that X is not geometrically normal. Then*

1. *if* $p = 3$*, then* ω_X^{-9} *is very ample;*
2. *if* $p = 2$ *and* $\ell_F(X/k) = 1$*, then* ω_X^{-7} *is very ample;*
3. *if* $p = 2$ *and* $\ell_F(X/k) = 2$*, then* ω_X^{-12} *is very ample.*

Proof. By Lemma 3.7 and Proposition 3.8, it is sufficient to verify that for $p = 3$ (resp. $p = 2, \ell_F(X/k) = 1$ and $p = 2, \ell_F(X/k) = 2$), the line bundle ω_X^{-6} (resp. ω_X^{-5} and ω_X^{-8}) is 0-regular with respect to ω_X^{-3} (resp. ω_X^{-2} and ω_X^{-4}) to show the statement. We prove only the case $p = 2$ and $\ell_F(X/k) = 2$, as the others are analogous. In this case, $H^1(X, \omega_X^{-8} \otimes \omega_X^{4}) = H^1(X, \omega_X^{-4}) = H^1(X, \omega_X^{5}) = 0$ by Proposition 3.6 and $H^2(X, \mathcal{O}_X) = H^0(X, \omega_X) = 0$. □

We now show the effective statements on very ampleness for the pluri-anticanonical systems.

Theorem 3.10. *Let X be a canonical del Pezzo surface. Then* ω_X^{-12} *is very ample.*

Proof. If X is geometrically normal, then it is geometrically canonical by Proposition 2.11 and $\omega_X^{\otimes -6}$ is very ample by [BT22, Proposition 2.14]. If X is not geometrically normal, we apply Proposition 3.9. □

4. Bounds on the irregularity

In this section, we study geometrically integral geometrically non-normal Gorenstein del Pezzo surfaces X. The additional condition on geometric integrality allows to find additional constraints on the normalised base changes to the algebraic closure and the irregularity of X.

4.1. A bound on $\gamma(X/k)$ *for geometrically integral varieties*

Given a geometrically integral normal variety X over k, we relate the δ-invariant measuring the singularities in codimension 1 of $X_{\overline{k}}$ with the capacity of denormalising extensions $\gamma(X/k)$ introduced by Tanaka [Tan21, Section 4].

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Definition 4.1. For an integral k-variety X with normalization $\nu\colon Y \to X$ with ramification $C \subseteq Y$ and conductor $D \subseteq X$, we define the *δ-invariant* of X over k as

$$\delta(X/k) := \max_{\eta \in D} \operatorname{length}_{\mathcal{O}_{D,\eta}}(\mathcal{O}_{C,\eta}/\mathcal{O}_{D,\eta}),$$

where η runs over all generic points of irreducible components of D.

Proposition 4.2. *Let X be a geometrically integral normal variety over a field k. Then*

$$\ell_F(X/k) \le \gamma(X/k) \le \delta(X_{\bar{k}}/\bar{k}).$$

Proof. The inequality $\ell_F(X/k) \le \gamma(X/k)$ is shown in [Tan21, Proposition 8.7], so we are left to show $\gamma(X/k) \le \delta(X_{\bar{k}}/\bar{k})$. As the statement can be checked on an open covering of X, we can assume that the conductor D of $X_{\bar{k}}$ is irreducible, with generic point η.

By definition of $\gamma(X/k)$ [Tan21, Definition 4.1], we can find a sequence of purely inseparable field extensions $k =: k_0 \subseteq k_1 \subseteq \ldots \subseteq k_{\gamma(X/k)}$ such that, if we inductively define $X_0 := X$ and $X_i := (X_{i-1,k_i})^{\mathrm{norm}}$, then $X_{i,k_{i+1}}$ is not normal and there is no longer sequence of fields with this property. In particular, $X_{\gamma(X/k)}$ is geometrically normal, and the normalization $\nu : Y \to X_{\bar{k}}$ of $X_{\bar{k}}$ factors as

$$\nu = \nu_1 \circ \cdots \circ \nu_{\gamma(X/k)} : Y = X_{\gamma(X/k),\bar{k}} \to \ldots \to X_{0,\bar{k}} = X_{\bar{k}}.$$

Note that each $X_{i,\bar{k}}$ has the property (S_2), being the base change of a normal variety along a field extension.

Now, after localizing at η, the factorization of ν corresponds to an ascending chain of subrings $\mathcal{O}_{X_{\bar{k}},\eta} = \mathcal{O}_{X_{0,\bar{k}},\eta} \subseteq \mathcal{O}_{X_{1,\bar{k}},\eta} \ldots \subseteq \mathcal{O}_{X_{\gamma(X/k),\bar{k}},\eta} = \mathcal{O}_{Y,\eta}$. Each inclusion $\mathcal{O}_{X_{i-1,\bar{k}},\eta} \subseteq \mathcal{O}_{X_{i,\bar{k}},\eta}$ is strict: otherwise, ν_i would be an isomorphism in codimension 1, and hence so would be $X_i \to X_{i-1,k_i}$. Since X_i is normal and X_{i-1,k_i} has property (S_2), this would imply that X_{i-1,k_i} is normal as well, contradicting our choice of k_i.

By definition, we have isomorphisms of $(\mathcal{O}_{X_{\bar{k}},\eta})$-modules

$$\mathcal{O}_{Y,\eta}/\mathcal{O}_{X_{\bar{k}},\eta} \cong (\mathcal{O}_{Y,\eta}/\mathcal{C}_\eta)/(\mathcal{O}_{X_{\bar{k}},\eta}/\mathcal{C}_\eta) \cong \mathcal{O}_{C,\eta}/\mathcal{O}_{D,\eta}.$$

Note that both sides are annihilated by the conductor ideal $\mathcal{C}_\eta$; hence, this is also an isomorphism of $(\mathcal{O}_{D,\eta})$-modules. Therefore, by strictness of $\mathcal{O}_{X_{i-1,\bar{k}},\eta} \subseteq \mathcal{O}_{X_{i,\bar{k}},\eta}$ for every $i \le \gamma(X/k)$, we have

$$\gamma(X/k) \le \operatorname{length}_{\mathcal{O}_{X_{\bar{k}},\eta}}(\mathcal{O}_{Y,\eta}/\mathcal{O}_{X_{\bar{k}},\eta}) = \operatorname{length}_{\mathcal{O}_{D,\eta}}(\mathcal{O}_{C,\eta}/\mathcal{O}_{D,\eta}) = \delta(X_{\bar{k}}/\bar{k}),$$

as claimed. □

Proposition 4.3. *Let X be a geometrically integral normal Gorenstein variety. Then, $\ell_F(X/k) \le \gamma(X/k) \le \delta(X_{\bar{k}}/\bar{k}) = \max_{\eta \in D} \operatorname{length}_{\mathcal{O}_{D,\eta}}(\mathcal{O}_{D,\eta})$. In particular, if every component of D is reduced, then $\ell_F(X/k) \le 1$.*

Proof. By Proposition 4.2, we only have to show the last equality. Let η be the generic point of an irreducible component of the conductor $D \subset X$. The Gorenstein condition implies $\operatorname{length}_{\mathcal{O}_{D,\eta}}\mathcal{O}_{C,\eta} = 2\operatorname{length}_{\mathcal{O}_{D,\eta}}\mathcal{O}_{D,\eta}$ by [FS20, Proposition A.2], which shows that $\delta(X_{\bar{k}}/\bar{k}) = \max_{\eta \in D} \operatorname{length}_{\mathcal{O}_{D,\eta}}\mathcal{O}_{D,\eta}$ by [Sta, Tag 00IV], as claimed.

The last statement is immediate as $\operatorname{length}_{\mathcal{O}_{D,\eta}}\mathcal{O}_{D,\eta} = 1$ if D is reduced. □

We can improve the bounds of [BT22] in the geometrically integral case.

Corollary 4.4. *Let X be a geometrically integral normal Gorenstein del Pezzo surface. Then $\ell_F(X/k) \le 1$. Moreover, if L is a torsion line bundle, then $L^{\otimes p} \cong \mathcal{O}_X$. In particular,* $\mathrm{Pic}^0_{X_{\bar{k}}/\bar{k}} \cong \mathbb{G}_{a,\bar{k}}^{h^1(X,\mathcal{O}_X)}$.

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Proof. By Theorem 2.13, the conductor D is reduced, and thus, we can apply Proposition 4.3 to conclude. The proof of the assertion on torsion line bundles follows as in [BT22, Theorem 4.1]. For the last statement, by Proposition 2.16, the Picard scheme $\mathrm{Pic}^0_{X_{\bar{k}}/\bar{k}}$ is a smooth commutative unipotent algebraic group of dimension $h^1(X, \mathcal{O}_X)$. As it is annihilated by p, we conclude by [Ser88, Proposition VII.11]. □

The previous analysis allows to obtain better estimates for the global generation than Proposition 3.9 in the geometrically integral canonical case.

Corollary 4.5. *Let X be a geometrically integral canonical del Pezzo surface. Let A be a big and nef Cartier divisor on X and suppose X is not geometrically normal. Then $p \in \{2, 3\}$ and the following hold:*

1. *If $p = 3$, then $\mathcal{O}_X(3A)$ is globally generated and ω_X^{-9} is very ample;*
2. *If $p = 2$, then $\mathcal{O}_X(2A)$ is globally generated and ω_X^{-7} is very ample.*

Proof. By Corollary 4.4, $\ell_F(X/k) = 1$, and we conclude by combining Lemma 3.7 and Proposition 3.9. □

4.2. Anti-pluricanonical maps of non-normal del Pezzo surfaces

In this section, we assume k is algebraically closed. Let X be a non-normal integral Gorenstein del Pezzo surface with normalization $\nu\colon Y \to X$ with ramification $C \subseteq Y$ and conductor $D \subseteq X$. As Gorenstein del Pezzo surfaces have the property (S_2), by [Rei94, Theorem, Section 2.6], there is an exact sequence

$$0 \to \omega_X \to \nu_*\nu^*\omega_X \xrightarrow{\mathrm{Tr}\circ\mathrm{Res}} \omega_D \to 0,$$

where Res is the pushforward of the classical residue map $\omega_Y(C) \to \omega_C$ (where we identify $\omega_Y(C) \cong \nu^*\omega_X$ and $\omega_Y(C)|_C \cong \omega_C$ by adjunction). The homomorphism Tr is the trace map which, over the generic point η of D, is given by the $(\mathcal{O}_{D,\eta})$-dual of the inclusion $\mathcal{O}_{D,\eta} \subseteq \nu_*\mathcal{O}_{C,\eta}$ by [Rei94, Remark 2.9]. Tensoring with $\omega_X^{-(n+1)}$ and applying the projection formula, we obtain

$$0 \to \omega_X^{-n} \to \nu_*\nu^*\omega_X^{-n} \to \omega_D \otimes \omega_X^{-(n+1)} \to 0.$$

As $\nu_*\nu^*\omega_X^{-\otimes n}$ is canonically isomorphic to $\nu_*(\omega_Y^{-\otimes n}(-nC))$, taking global sections, we deduce the following:

Lemma 4.6. *We have the following equality of subspaces of $H^0(Y, \omega_Y^{-n}(-nC))$:*

$$\nu^*H^0(X, \omega_X^{-n}) = \mathrm{Ker}(H^0((\mathrm{Tr}\circ\mathrm{Res}) \otimes \omega_X^{\otimes -(n+1)})).$$

We now prove a useful lower bound on the dimension of the space of anti-pluricanonical sections on del Pezzo surfaces. It will be the main tool to bound the irregularity of del Pezzo surfaces.

Corollary 4.7. *There is an inclusion of k-vector spaces:*

$$V := \{s \in H^0(Y, \mathcal{O}_Y(-n(K_Y + C))) \mid s|_C = 0\} \subseteq \nu^*H^0(X, \omega_X^{-n}).$$

Thus, if ω_X^{-n} is globally generated, then

$$h^0(X, \omega_X^{-n}) \geq \dim V + 2.$$

Proof. By the natural identifiation $\omega_Y(C)|_C \cong \omega_C$ given by adjunction, the space V is equal to the kernel of the homomorphism $H^0(\mathrm{Res}\otimes\omega_X^{-n-1})$; hence, it is contained in the kernel of $H^0((\mathrm{Tr}\circ\mathrm{Res})\otimes\omega_X^{-n-1}) = \nu^*H^0(X, \omega_X^{-n})$ by Lemma 4.6.

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If ω_X^{-n} is globally generated, then the linear system $|\nu^*H^0(X,\omega_X^{-n})|$ has no base points on C. Since all sections in V vanish on C and ω_X^{-1} is ample, there are at least two more linearly independent sections of $\nu^*H^0(X,\omega_X^{-n})$ that are nonzero when restricted to C, thus concluding the inequality. □

4.3. Irregularity of geometrically integral l.c.i. del Pezzo surfaces

We prove effective bounds on the values of the irregularity of locally complete intersection (lci) del Pezzo surfaces.

Proposition 4.8. *Let X be a geometrically integral normal locally complete intersection del Pezzo surface over a field k of characteristic $p > 0$. Let $\nu\colon Y \to X_{\bar{k}}$ be the normalization of $X_{\bar{k}}$ and let $C \subseteq Y$ be the ramification of ν. Then, one of the following holds:*

1. $h^1(X,\omega_X^n) = 0$ *for all* $n \in \mathbb{Z}$.
2. $p = 3$, $(Y,C) = (\mathbb{P}^2, 2L)$, $h^1(X,\mathcal{O}_X) = 2$, *and* $h^1(X,\omega_X^n) = 0$ *for all* $n \geq 2$.
3. $p = 2$, $(Y,C) = (\mathbb{P}^2, 2L)$, $h^1(X,\mathcal{O}_X) = 1$, *and* $h^1(X,\omega_X^n) = 0$ *for all* $n \geq 2$.
4. $p = 2$, $(Y,C) = (\mathbb{P}(1,1,2), 2L)$, $h^1(X,\mathcal{O}_X) = 1$, *and* $h^1(X,\omega_X^n) = 0$ *for all* $n \geq 2$.

Proof. If X is geometrically normal, then X is geometrically canonical by Proposition 2.11. By Serre duality, it is sufficient to show that $h^1(X_{\bar{k}},\omega_{X_{\bar{k}}}^{\otimes n}) = 0$ for $n > 0$. This follows from [Ber21a, Theorem 5.6.a]. If X is not geometrically normal and the ramification divisor contains a reduced component, then $X_{\bar{k}}$ is tame and $h^1(X,\omega_X^{\otimes n}) = 0$ for all $n \in \mathbb{Z}$ by [Rei94, Corollary 4.10]. Therefore, by [PW22, Theorem 4.1], we may assume that $p \in \{2,3\}$, $h^1(X,\mathcal{O}_X) > 0$ and $(Y,C) = (\mathbb{P}(1,1,d), 2L)$ for some $d \geq 1$ where L is a line through the vertex of the cone.

Choose weighted coordinates x, y, z of degree $1, 1, d$ on Y such that $L = \{x = 0\}$, hence $2L = \{x^2 = 0\}$ in weighted coordinates. Let $n \geq 1$ and $V_{n,d} \subseteq H^0(\mathbb{P}(1,1,d), \mathcal{O}(nd))$ be the subspace of sections vanishing along $2L$. Then, $V_{n,d}$ consists of weighted homogeneous polynomials of the form $x^2 f_{nd-2}(x,y,z)$; hence, $\dim V_{n,d} = \sum_{j=1}^n (jd-1) = \frac{n^2+n}{2}d - n$. As $\nu^*\omega_{X_{\bar{k}}} \cong \mathcal{O}(-dL)$, we have $\nu^*\omega_{X_{\bar{k}}}^{-n} \cong \mathcal{O}(ndL)$. By Corollary 4.7, we have

$$h^0(X,\omega_X^{-n}) \geq \begin{cases} \frac{n^2+n}{2}d - n. \\ \frac{n^2+n}{2}d - n + 2 \text{ if, additionally, } \omega_X^{-n} \text{ is globally generated.} \end{cases} \tag{4.1}$$

By the Riemann–Roch formula [Tan18b, Theorem 2.10], we have

$$h^0(X,\omega_X^{-n}) - h^1(X,\omega_X^{-n}) = 1 - h^1(X,\mathcal{O}_X) + \frac{n^2+n}{2}K_X^2 = 1 - h^1(X,\mathcal{O}_X) + \frac{n^2+n}{2}d.$$

Thus, if we assume $h^1(X,\omega_X^{-n}) = 0$, we deduce from Equation (4.1) that

$$h^1(X,\mathcal{O}_X) = 1 - h^0(X,\omega_X^{-n}) + \frac{n^2+n}{2}d \leq \begin{cases} n+1 \\ n-1 \text{ if, additionally, } \omega_X^{-n} \text{is g.g.} \end{cases} \tag{4.2}$$

We also recall Maddock's bound [Mad16, Corollary 1.2.6]: if $h^1(X,\omega_X^n) \neq 0$ but $h^1(X,\omega_X^{pn}) = 0$, then

$$h^1(X,\mathcal{O}_X) \geq \frac{nd(p-1)(3+n(2p-1))}{12}. \tag{4.3}$$

Now assume $p = 3$. By Serre vanishing and $h^1(\mathcal{O}_X) \neq 0$, there exists a largest $N \geq 0$ such that $h^1(X,\omega_X^{-N}) = h^1(X,\omega_X^{(N+1)}) \neq 0$. By (4.2) and (4.3), we have the following chain:

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$$N+2 \geq h^1(X,\mathcal{O}_X) \geq \frac{(N+1)d(p-1)(3+(N+1)(2p-1))}{12} = \frac{(N+1)d(8+5N)}{6}.$$

Hence, $N=0$, $d=1$, showing $h^1(X,\mathcal{O}_X) \leq 2$. Finally, $h^1(X,\mathcal{O}_X)=2$ by Theorem 2.13.

Now assume $p=2$. Then, the argument of the previous paragraph yields

$$N+2 \geq h^1(X,\mathcal{O}_X) \geq \frac{(N+1)d(p-1)(3+(N+1)(2p-1))}{12} = \frac{(N+1)d(N+2)}{4}. \tag{4.4}$$

Hence, $N \leq 3$. Therefore, $h^1(X,\omega_X^{-4})=0$, and, by Corollary 4.5, ω_X^{-4} is globally generated, so $h^1(X,\mathcal{O}_X) \leq 3$ by (4.2). If $h^1(X,\mathcal{O}_X)=1$, then $N=0$ and $d \in \{1,2\}$ by (4.4), and we get Cases (3) and (4).

So, it remains to exclude the possibility $h^1(X,\mathcal{O}_X) \geq 2$ in characteristic $p=2$. By Corollary 4.5, ω_X^{-2} is globally generated, so by (4.4), the inequality $h^1(X,\mathcal{O}_X) \geq 2$ implies $N=2$, $d=1$, and $h^1(X,\mathcal{O}_X)=3$.

Seeking a contradiction, assume that there exists an X with these invariants. Since $N=2$, we have $H^1(X,\omega_X^3) \neq 0$ and $H^1(X,\omega_X^6)=0$. Let $Z \to X$ be a nontrivial $\alpha_{\omega_X^3}$-torsor and let $k_Z := H^0(Z,\mathcal{O}_Z)$. Note that Z is an l.c.i. del Pezzo surface by [Mad16, Theorem 1.2.3], and by [Mad16, Equation (1.2.4)], we have $[k_Z : k](1-h^1(Z,\mathcal{O}_Z))=2$ so that we have $[k_Z:k]=2$ and $H^1(Z,\mathcal{O}_Z)=0$. By [Mad16, Equation (1.2.5)], we then conclude that $K_Z^2=16$. Now, we consider the following diagram:

$$\begin{array}{ccccccc}
(Z_{\bar{k}})^{\mathrm{norm}} & \longrightarrow & Z_{\bar{k}} & \longrightarrow & Z & & \\
\downarrow{\scriptstyle (\pi_{\bar{k}})^{\mathrm{norm}}} & & \downarrow{\scriptstyle \pi_{\bar{k}}} & & \downarrow{\scriptstyle \pi} & \searrow^{f} & \\
(X_{\bar{k}})^{\mathrm{norm}} & \longrightarrow & X_{\bar{k}} & \longrightarrow & X_{k_Z} & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \operatorname{Spec} \bar{k} & \longrightarrow & \operatorname{Spec} k_Z & \longrightarrow & \operatorname{Spec} k,
\end{array}$$

where $Z_{\bar{k}} = Z \times_{\operatorname{Spec} k_Z} \operatorname{Spec} \bar{k}$. Since f and $X_{k_Z} \to X$ are finite of degree 2, the morphism π is finite and birational. In particular, Z, considered as a k_Z-scheme, is geometrically integral and the induced map $(\pi_{\bar{k}})^{\mathrm{norm}}$ of the normalisations is an isomorphism. As $(X_{\bar{k}})^{\mathrm{norm}} \cong \mathbb{P}^2$, $16 = K_Z^2 \leq K^2_{(Z_{\bar{k}})^{\mathrm{norm}}} = 9$, reaching a contradiction. □

Corollary 4.9. *Let X be a geometrically integral normal locally complete intersection del Pezzo surface over a field k of characteristic p. Let $\nu\colon Y \to X_{\bar{k}}$ be the normalization of $X_{\bar{k}}$ and let $C \subseteq Y$ be the ramification of ν. Then, one of the following holds:*

1. $h^1(X,\mathcal{O}_X)=0$, $K_X^2 \geq 3$, *and* ω_X^{-1} *is very ample.*
2. $h^1(X,\mathcal{O}_X)=0$, $K_X^2=2$, *and* ω_X^{-2} *is very ample.*
3. $h^1(X,\mathcal{O}_X)=0$, $K_X^2=1$, *and* ω_X^{-3} *is very ample.*
4. $h^1(X,\mathcal{O}_X)=1$, $p=2$, $(Y,C,K_X^2) \in \{(\mathbb{P}^2,2L,1),(\mathbb{P}(1,1,2),2L,2)\}$, *and* ω_X^{-6} *is very ample.*
5. $h^1(X,\mathcal{O}_X)=2$, $p=3$, $(Y,C,K_X^2)=(\mathbb{P}^2,2L,1)$, *and* ω_X^{-7} *is very ample.*

Proof. Claims (1), (2) and (3) are a consequence of [BT22, Proposition 2.14] if X is geometrically normal, and hence geometrically canonical, and [Rei94, Corollary 4.10] if X is not geometrically normal.

Let us prove Claim (4). By Proposition 4.8, we have $p=2$ and the desired classification of (Y,C). By Lemma 3.7, ω_X^{-2} is globally generated. Using Proposition 4.8, it is easy to check that ω_X^{-4} is 0-regular with respect to ω_X^{-2}. Claim (5) is proven similarly. □

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4.4. Refinements in the regular and canonical case

We show various refinements of the bounds of Proposition 4.8 in the case where we assume X to be a regular or canonical del Pezzo surface. We start with the case $p = 3$.

Proposition 4.10. *Let X be a geometrically integral canonical del Pezzo surface over a field k of characteristic $p = 3$. Then, X is tame.*

Proof. Without loss of generality, we may assume that k is separably closed.

First, assume that X is regular. Seeking a contradiction, we assume that $h^1(X, \mathcal{O}_X) \neq 0$. Let $\nu \colon Y \to X_{\bar{k}}$ be the normalisation of $X_{\bar{k}}$ and let $C \subseteq Y$ be the ramification of ν. By Proposition 4.8 and Serre duality, we know that $h^1(X, \mathcal{O}_X) = 2$, $K_X^2 = 1$, $h^1(X, \omega_X^{-n}) = 0$ for $n > 0$, and $(Y, C) = (\mathbb{P}^2, 2L)$.

First, we claim that $h^0(X, \omega_X^{-n} \otimes \mathcal{L}) = 0$ for all nontrivial torsion line bundles $\mathcal{L}$ and for $n \in \{0, 1\}$. Since X is reduced, this holds if $n = 0$. For the case $n = 1$, by the Riemann–Roch theorem, we have

$$\chi(\omega_X^{-1} \otimes \mathcal{L}) = 0,$$

so if $h^1(X, \omega_X^{-1} \otimes \mathcal{L}) \neq 0$, then $h^1(X, \omega_X^2 \otimes \mathcal{L}^{-1}) \neq 0$ by Serre duality. Since $\omega_X^6 \otimes \mathcal{L}^{-3} \cong \omega_X^6$ by Corollary 4.4 and $h^1(X, \omega_X^6) = 0$ by Proposition 4.8, there exists a nontrivial $\alpha_{(\omega_X^2 \otimes \mathcal{L}^{-1})}$-torsor $Z \to X$ such that Z is an l.c.i. del Pezzo surface. Moreover, by [Mad16, Equation (1.2.5)], we have $2^e(1 - q_Z) = 10$ for some integers $0 \leq e \leq 1$ and $q_Z \geq 0$, contradicting our assumption.

By Riemann–Roch, we have $h^0(X, \omega_X^{-2}) = 2$. Write

$$|-2K_X| = F + |M|,$$

where F is the fixed part and M is the movable part of the linear system. Since $M \neq 0$, we have $F \in |-nK_X + E|$ for some $0 \leq n \leq 1$ and a divisor E such that $\mathcal{O}_X(E)$ is torsion. By the previous paragraph, we have $h^0(X, \omega_X^{-n}(E)) = 0$; hence, $F = 0$.

Since the linear system $|-2K_X|$ does not have fixed components, its base locus Z is 0-dimensional, and we denote by A the ring of global section $H^0(Z, \mathcal{O}_Z)$. Since $(-2K_X)^2 = 4$, we have $\mathrm{length}_k(A) = 4$, so A is an Artinian k-algebra of length 4. As k is separably closed, we can write $A = \prod_{i=1}^s A_i$ where each A_i is a local Artinian k-algebra of dimension n_i over its residue field k_i and k_i is a purely inseparable extension of k. Since

$$4 = \mathrm{length}_k(A) = \sum_{i=1}^s \mathrm{length}_k(A_i) = \sum_{i=1}^s n_i[k_i : k]$$

and $p = 3$, we have $k_i = k$ for at least one i. In other words, at least one of the base points of $|-2K_X|$ is a k-rational point P.

If P is in the image of the conductor D under the natural map $X_{\bar{k}} \to X$, then P lies in the non-smooth locus of $X \to \mathrm{Spec}\, k$, so X cannot be regular at P by [FS20, Corollary 2.6]. Hence, in this case, the proof is finished.

So, seeking a contradiction, assume that P is not in the image of D. Let P' be the unique preimage of P under the map $Y \to X_{\bar{k}} \to X$. Since ν is an isomorphism around P', the point P' lies in the base locus of $\nu^*|-2K_{X_{\bar{k}}}|$. By Corollary 4.7, we know that $C = 2L \in \nu^*|-2K_{X_{\bar{k}}}|$ hence, $P' \in C$, and thus, P is in the image of D under $X_{\bar{k}} \to X$. This contradicts our choice of P.

Finally, assume that X is canonical. By Proposition 2.6, we can replace X with its minimal resolution, which is a regular weak del Pezzo surface. By running a K_X-MMP, we can suppose X is a weak del Pezzo surface admitting a Mori fibre space structure $\pi \colon X \to B$. If X is a regular del Pezzo surface, we conclude by the previous case. If B is a curve and X is a weak del Pezzo surface, then the generic fibre F is a regular conic. As $p = 3$, F is smooth by [BT22, Lemma 2.17], and thus, $-K_B$ is ample by [Eji19, Corollary 4.10.c]. Therefore, $H^1(B, \mathcal{O}_B) = 0$ and, as $H^1(X, \mathcal{O}_X) = H^1(B, \mathcal{O}_B) = 0$ by the relative Kawamata–Viehweg vanishing theorem [Tan18b, Theorem 4.2], we conclude. □

In the following proposition, we describe the geometry of α_{ω_X}-torsors over wild regular del Pezzo surfaces in characteristic $p = 2$. The reader should compare this with the construction of the regular wild del Pezzo surfaces of degree 1 in [Mad16].

Proposition 4.11. *Let X be a geometrically integral regular del Pezzo surface over a field k of characteristic $p = 2$. Assume that $h^1(X, \mathcal{O}_X) \neq 0$. Then, $\mathrm{p}-\deg(k) \geq 2$, $K_X^2 \leq 2$, and there exists an α_{ω_X}-torsor $Z \to X$ such that Z satisfies the following properties:*

1. *If $K_X^2 = 2$, then $k_Z := h^0(Z, \mathcal{O}_Z)$ is a purely inseparable extension of k of degree 2 and Z is a twisted form of $\mathbb{P}(1,1,2)$ over k_Z.*
2. *If $K_X^2 = 1$, then Z is a normal tame del Pezzo surface such that $\epsilon(Z/k) = 1$, $K_Z^2 = 8$ and the normalised base change $((Z_{\bar{k}})^{\mathrm{norm}}_{\mathrm{red}}, E)$ is $(\mathbb{P}^2, L)$.*

Proof. If X is not tame, then $\rho(X) = 1$ by Proposition 4.8. If $\mathrm{p}-\deg(k) = 1$, then X is geometrically canonical by [FS20], contradicting $h^1(X, \mathcal{O}_X) \neq 0$.

By Proposition 4.8, we have $h^1(X, \mathcal{O}_X) = h^1(X, \omega_X) = 1$, $h^1(X, \omega_X^n) = 0$ for $n \geq 2$, and $K_X^2 \in \{1, 2\}$. In particular, there exists a nontrivial α_{ω_X}-torsor $f\colon Z \to X$. In the following, we treat the cases $K_X^2 = 2$ and $K_X^2 = 1$ separately. We set $k_Z := H^0(Z, \mathcal{O}_Z)$. Note that, by the same proof as for the second paragraph of the proof of Proposition 4.10, we have $h^1(X, \omega_X^n \otimes \mathcal{L}) = 0$ for all torsion line bundles $\mathcal{L}$ and all $n \geq 2$.

Assume $K_X^2 = 2$. In this case, by [Mad16, Equation (1.2.5)], we have $[k_Z : k] = 2$, $h^1(Z, \mathcal{O}_Z) = 0$, and $K_Z^2 = 8$, where we compute the self-intersection over k_Z. Now, consider the commutative diagram

$$\begin{array}{ccccccc}
(Z_{\bar{k}})^{\mathrm{norm}} & \longrightarrow & Z_{\bar{k}} & \longrightarrow & Z & & \\
\downarrow{\scriptstyle (\pi_{\bar{k}})^{\mathrm{norm}}} & & \downarrow{\scriptstyle \pi_{\bar{k}}} & & \downarrow{\scriptstyle \pi} & \overset{f}{\searrow} & \\
(X_{\bar{k}})^{\mathrm{norm}} & \longrightarrow & X_{\bar{k}} & \longrightarrow & X_{k_Z} & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \operatorname{Spec} \bar{k} & \longrightarrow & \operatorname{Spec} k_Z & \longrightarrow & \operatorname{Spec} k,
\end{array}$$

where $Z_{\bar{k}} = Z \times_{\operatorname{Spec} k_Z} \operatorname{Spec} \bar{k}$. As in the end of the proof of Proposition 4.8, the diagram shows that Z is geometrically integral when considered as a k_Z-scheme and $(\pi_{\bar{k}})^{\mathrm{norm}}$ is an isomorphism; hence, $(Z_{\bar{k}})^{\mathrm{norm}} \cong (X_{\bar{k}})^{\mathrm{norm}} \cong \mathbb{P}(1,1,2)$ by Proposition 4.8. In particular, we have $K_Z^2 = 8 = K^2_{(Z_{\bar{k}})^{\mathrm{norm}}}$, so Z is in fact geometrically normal. Therefore, $Z_{\bar{k}} \cong \mathbb{P}(1,1,2)$, so Z is a twisted form of $\mathbb{P}(1,1,2)$ over k_Z.

Assume $K_X^2 = 1$. In this case, by [Mad16, Equation (1.2.5)], we have $k_Z = k$ and $h^1(Z, \mathcal{O}_Z) = 0$. We first claim that Z is not geometrically reduced. Indeed, let $\psi\colon \mathbb{P}^2 \to X$ be the normalised base change and consider the $\alpha_{\psi^*\omega_X}$-torsor $T := Z \times_X \mathbb{P}^2 \to \mathbb{P}^2$ obtained by base changing along ψ. As $\psi^*\omega_X$ is anti-ample, considering the exact sequence

$$0 = H^0(\mathbb{P}^2, \psi^*\omega_X^p) \to H^1_{\mathrm{fppf}}(\mathbb{P}^2, \alpha_{\psi^*\omega_X}) \to H^1(\mathbb{P}^2, \psi^*\omega_X) = 0,$$

we have that $H^1_{\mathrm{fppf}}(\mathbb{P}^2, \alpha_{\psi^*\omega_X}) = 0$, so that $T \to \mathbb{P}^2$ is a trivial torsor, and thus, T is not reduced. Since $T \to Z_{\bar{k}}$ is generically an isomorphism, Z is not geometrically reduced.

Next, we claim that Z is normal. Suppose by contradiction it is not and let $\nu\colon Z^{\mathrm{norm}} \to Z$ be the normalisation. In this case, as ν is not an isomorphism, we have that $K^2_{Z^{\mathrm{norm}}} > 8$, where we calculate the self-intersection number over k. Let $g\colon Z^{\mathrm{norm}} \to X$ be the composition $f \circ \nu$. As X is regular and Z^{norm} is integral, by [Mad16, Proposition 2.2.1], there exists a line bundle $\mathcal{M}$ numerically equivalent to mK_X for some integer m such that g is a nontrivial $\alpha_{\mathcal{M}}$-torsor. In particular, Z^{norm} is Gorenstein, and thus, as it is the normalization of a del Pezzo surface, Z^{norm} is also a del Pezzo surface. Therefore, $m \geq 0$. We now distinguish two cases:

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1. if $H^0(Z^{\mathrm{norm}}, \mathcal{O}_{Z^{\mathrm{norm}}}) = k$, by [Mad16, Equation (1.2.4)], we have $2(1+m)^2 = K^2_{Z^{\mathrm{norm}}} > 8$, which implies that $m > 1$, contradicting Proposition 4.8;
2. if $[H^0(Z^{\mathrm{norm}}, \mathcal{O}_{Z^{\mathrm{norm}}}) : k] = 2$, we have that $(1+m)^2 = K^2_{Z^{\mathrm{norm}}} > 4$, which also implies that $m > 1$, contradicting Proposition 4.8 as well. Note that, here, we calculate the self-intersection $K^2_{Z^{\mathrm{norm}}}$ over $H^0(Z^{\mathrm{norm}}, \mathcal{O}_{Z^{\mathrm{norm}}})$.

Thus, Z must be normal.

By [Kle66, Example 1], we have that $8 = K_Z^2 = 2^{\epsilon(Z/k)}(K_{(Z_{\bar{k}})^{\mathrm{norm}}_{\mathrm{red}}} + E)^2$. Since Z is geometrically non-reduced, we have $\epsilon(Z/k) \geq 1$, and since f is finite flat of degree 2 and $\epsilon(X/k) = 0$, we have $\epsilon(Z/k) \leq 1$; hence, $\epsilon(Z/k) = 1$. As Z is normal and Gorenstein, we can apply [PW22, Theorem 4.1] to conclude that $Z = \mathbb{P}^2$ and E is a line. □

Proof of Theorem 1.2. Combine Proposition 4.8, Proposition 4.10 and Proposition 4.11. □

5. On the BAB conjecture for surfaces over arbitrary fields

In this section, we prove boundedness results for del Pezzo surfaces over arbitrary fields.

We recall some terminology when discussing boundedness in birational geometry. For the following definition, we say that a scheme X is a *projective variety* if X is integral, $H^0(X, \mathcal{O}_X)$ is a field k and the natural morphism $\pi_X : X \to \mathrm{Spec}(k)$ is projective. We will always consider X as a k-variety via the natural morphism π_X.

Definition 5.1. We say that a class of projective varieties $\mathcal{X}$ is *bounded* (resp. *birationally bounded*) if there exists a projective flat morphism $Y \to T$ of finite type $\mathbb{Z}$-schemes such that for every $X \in \mathcal{X}$ with $k := H^0(X, \mathcal{O}_X)$, there exists a morphism $\mathrm{Spec}(k) \to T$ and a k-isomorphism (resp. a k-birational map) $X \to Y \times_V \mathrm{Spec}(k)$.

The Borisov–Alexeev–Borisov (BAB) conjecture states that mildly singular Fano varieties form a bounded family in every dimension.

Conjecture 5.2 (BAB). *For any rational number $\varepsilon > 0$, the class*

$$\mathcal{X}_{d,\varepsilon} = \{X \mid X \text{ is a } \varepsilon\text{-klt Fano variety of dimension } d\}$$

is bounded.

Remark 5.3. The presence of the ε–klt hypothesis is necessary in the BAB conjecture, already in dimension 2. Indeed, Gorenstein del Pezzo surfaces with general log canonical singularities are not bounded as cones over elliptic curves of Proposition 2.9 show. Moreover, boundedness already fails for klt del Pezzo surfaces as the set of weighted projective planes $\{\mathbb{P}(1,1,d)\}_{d\geq 1}$ shows.

We discuss the BAB conjecture for surfaces defined over arbitrary fields. The result of [Ale94, CTW17] shows that the class of geometrically ε-klt del Pezzo surfaces form a bounded family. However, the conjecture is still open for ε-klt del Pezzo surfaces defined over an imperfect field.

In subsection 5.1, we settle the BAB conjecture for geometrically integral canonical del Pezzo surfaces. In the remaining two subsections, we discuss the general ε-klt case. In subsection 5.2, we prove boundedness of the anticanonical volumes for ε-klt del Pezzo surfaces over imperfect fields. This, together with a bound on the $\mathbb{Q}$-Gorenstein index proven in subsection 5.3, implies the BAB Conjecture 5.2 for such surfaces in characteristic $p > 5$ (cf. Theorem 5.12).

5.1. Boundedness of geometrically integral canonical del Pezzo surfaces

In [Tan24], Tanaka proves boundedness for geometrically integral regular del Pezzo surfaces. As a consequence of our results in section 3 and section 4, we can extend Tanaka's result to the canonical case.

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Theorem 5.4. *The class*

$$\mathcal{X}_{\mathrm{dP,can}} = \{X \mid X \text{ is a geometrically integral canonical del Pezzo surface}\}$$

is bounded.

Proof. As X is geometrically integral, $\epsilon(X/k) = 0$, and thus, $K_X^2 \leq 16$ by Proposition 3.4. Since X is canonical, K_X is Cartier and K_X^2 is an integer. By Proposition 4.8, Theorem 3.10 and Riemann–Roch, there exists an $N > 0$ such that ω_X^{-12} embeds X into $\mathbb{P}_k^n$ for some $n \leq N$. Again by Proposition 4.8, and since $K_X^2 \leq 16$, the possibilities for the Hilbert polynomial $\chi(X, \omega_X^{-12})$ are finite. Therefore, all X arise via pullback from a universal family over a suitable finite union of Hilbert scheme of finite type over $\mathrm{Spec}\mathbb{Z}$ [Kol96, Theorem 1.4]. □

5.2. Bounds on the volume of ε-klt del Pezzo surfaces

We prove an explicit bound for the volumes of ε-klt del Pezzo surfaces, generalising the results of [Ale94, AM04] to imperfect fields. To do so, we start with some elementary computations on surfaces of del Pezzo type admitting a Mori fibration onto a curve. Recall the definition of surfaces of del Pezzo type from Definition 2.15.

Lemma 5.5. *Let X be a regular surface of del Pezzo type. Let $\pi: X \to B$ be a Mori fibre space onto a regular curve and let $F_b = \pi^* b$, where b is a closed point of B. Then, there exists an integral curve Γ such that*

$$\mathrm{NE}(X) = \mathbb{R}_+[F_b] + \mathbb{R}_+[\Gamma].$$

Moreover, setting $d_\Gamma := [H^0(\Gamma, \mathcal{O}_\Gamma) : k]$ and $m_\Gamma = [k(\Gamma) : k(B)]$, there exists $n \geq 0$ such that

$$\Gamma^2 = -d_\Gamma \cdot n, \quad K_X \cdot \Gamma = d_\Gamma (n-2), \tag{5.1}$$

and

$$K_X^2 = \frac{d_\Gamma}{m_\Gamma^2}(8m_\Gamma + 4n(1-m_\Gamma)) \leq 8. \tag{5.2}$$

Proof. The existence of Γ is a consequence of the cone theorem [Tan18b, Theorem 2.14], while Equation (5.1) is proved in [BT22, Lemmas 4.3, 4.6]. To prove Equation (5.2), we write $K_X \equiv xF_b + y\Gamma$ for some $x, y \in \mathbb{Q}$. Set $d_b = [k(b) : k]$. As $K_X \cdot F_b = -2d_b$, we conclude that $y = -\frac{2}{m_\Gamma}$. Therefore,

$$d_\Gamma(n-2) = K_X \cdot \Gamma = x m_\Gamma d_b + \frac{2nd_\Gamma}{m_\Gamma},$$

which implies

$$K_X \equiv \frac{d_\Gamma(m_\Gamma(n-2) - 2n)}{m_\Gamma^2 d_b} F_b - \frac{2}{m_\Gamma}\Gamma.$$

A straightforward computation with intersection numbers then shows Equation (5.2). Finally, as $(1 - m_\Gamma) \leq 0$, we have $K_X^2 \leq \frac{d_\Gamma}{m_\Gamma} 8$ and, as $d_\Gamma \leq m_\Gamma$, we conclude. □

We now prove bounds on the anticanonical volume of ε-klt del Pezzo.

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Theorem 5.6. *Fix a rational number $\varepsilon > 0$. Then for every geometrically integral ε-klt del Pezzo surface X, we have*

$$K_X^2 \leq \max\left\{9, 8 + 20\frac{(1-\varepsilon)^2}{\varepsilon}\right\}.$$

Proof. Let $f\colon Y \longrightarrow X$ be the minimal resolution, and write $K_Y + \sum_i b_i E_i = f^*K_X$. By the ε-klt hypothesis and minimality of f, we have $0 < b_i < 1-\varepsilon$. We run a K_Y-MMP which ends with $\psi\colon Y \longrightarrow Z$, where Z is a regular projective surface admitting a Mori fibre space structure $\pi\colon Z \to B$. Since $-(K_Y + \sum_i b_i E_i)$ is big and nef, so is $-(K_Z + \Delta_Z)$, where $\Delta_Z = \psi_*(\sum b_i E_i)$. Moreover, $K_X^2 = (K_Y + \sum_i b_i B_i)^2 \leq (K_Z + \Delta_Z)^2$.

Suppose $\dim(B) = 0$. Then Z is a regular del Pezzo surface of Picard rank 1. As $-(K_Z+\Delta_Z)$ is ample, there exists $0 \leq \lambda < 1$ such that $\Delta_Z \equiv -\lambda K_Z$. Therefore, we deduce $(K_Z+\Delta_Z)^2 = (1-\lambda)^2 K_Z^2 \leq K_Z^2 \leq 9$, where the last inequality follows by [Tan24, Theorem 1.2].

Suppose $\dim(B) = 1$. Let Γ be the extremal curve described in Lemma 5.5. We write $\Delta_Z = \alpha\Gamma + G$, where $\mathrm{Supp}(G)$ does not contain Γ. Since the Picard rank of Z is 2, G is a $\mathbb{Q}$-Cartier nef divisor. As $-(K_Z+\Delta_Z)$ and $-(K_Z+\alpha\Gamma)$ are big and nef classes, their intersection with G is non-positive, and thus, we have

$$(K_Z + \Delta_Z)^2 = (K_Z + \alpha\Gamma)^2 + (K_Z + \alpha\Gamma)\cdot G + (K_Z + \Delta_Z)\cdot G \leq (K_Z + \alpha\Gamma)^2.$$

Therefore, it is sufficient to bound the volume of del Pezzo surface pairs $(Z, \alpha\Gamma)$, where $Z \longrightarrow B$ is a Mori fibre space onto a curve, Γ is the extremal curve of Lemma 5.5 and $0 \leq \alpha < 1-\varepsilon$. Note that

$$(K_Z + \alpha\Gamma)^2 = K_Z^2 + d_\Gamma\alpha(2(n-2) - n\alpha) \text{ and } 0 \leq \alpha < 1-\varepsilon. \tag{5.3}$$

Claim 5.7. The self-intersection of Γ is bounded:

$$n \leq \frac{2}{\varepsilon}.$$

Proof of Claim. By adjunction,

$$-2d_\Gamma = (K_Z + \Gamma)\cdot\Gamma = (K_Z + \alpha\Gamma)\cdot\Gamma + (1-\varepsilon-\alpha)\Gamma^2 + \varepsilon\Gamma^2.$$

As $-(K_Z+\alpha\Gamma)$ is big and nef and $1-\varepsilon > \alpha$, we have $(K_Z+\alpha\Gamma)\cdot\Gamma + (1-\varepsilon-\alpha)\Gamma^2 \leq 0$, and therefore, we deduce $2d_\Gamma \geq \varepsilon d_\Gamma \cdot n$. □

If $K_Z\cdot\Gamma \leq 0$, then $(K_Z+\alpha\Gamma)^2 \leq K_Z^2 \leq 8$ by Lemma 5.5, and we are done. So, assume $K_Z\cdot\Gamma > 0$, or, equivalently, $n > 2$. In this case, we have $d_\Gamma \leq m_\Gamma \leq 5$ by [BT22, Proposition 4.7]. Therefore, by Equation (5.3) and Claim 5.7, we deduce the following series of inequalities:

$$\begin{aligned}(K_Z + \alpha\Gamma)^2 &= K_Z^2 + d_\Gamma\alpha(2(n-2) - n\alpha) \\ &\leq K_Z^2 + 5\alpha(2n-4) \leq K_Z^2 + 5(1-\varepsilon)\left(\frac{4}{\varepsilon} - 4\right) \\ &\leq 8 + 20\frac{(1-\varepsilon)^2}{\varepsilon}.\end{aligned}$$

□

As a consequence, we can show a boundedness result for klt del Pezzo surfaces of bounded Gorenstein index in characteristic $p > 5$ (cf. [HMX14, Corollary 1.8] for the analogue in characteristic 0). In the following, we say that a klt del Pezzo surface is *tame* if $h^1(X, \mathcal{O}_X) = 0$.

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Corollary 5.8. *Let $n > 0$ be an integer. Then, the classes*

$$\mathcal{X}^{\text{tame}}_{\text{dP},n} = \{X \mid X \text{ is a geometrically integral tame klt del Pezzo surface s.t. } nK_X \text{ is Cartier}\}, \text{ and}$$
$$\mathcal{X}^{>5}_{\text{dP},n} = \left\{X \mid X \text{ is a klt del Pezzosurface s.t. } nK_X \text{ is Cartier and } \operatorname{char}(H^0(X,\mathcal{O}_X)) \neq 2,3,5\right\}$$

are bounded.

Proof. By [BT22, Corollary 5.5] and [BT22, Theorem 5.7], klt del Pezzo surfaces in characteristic bigger than 5 are geometrically integral and tame, so it suffices to show that $\mathcal{X}^{\text{tame}}_{\text{dP},n}$ is bounded.

So, let $X \in \mathcal{X}^{\text{tame}}_{\text{dP},n}$. As X is $\left(\frac{1}{n}\right)$-klt, Theorem 5.6 implies that K_X^2 is bounded. As the Cartier index of K_X is fixed, the set of volumes $\left\{K_X^2 \mid X \in \mathcal{X}^{\text{tame}}_{\text{dP},n}\right\}$ is a finite set. As X has rational singularities by Proposition 2.6, we can apply Riemann–Roch to compute for all $t \geq 1$:

$$\chi(X, \mathcal{O}_X(-ntK_X)) = \chi(X, \mathcal{O}_X) + \frac{nt(nt+1)K_X^2}{2}.$$

As X is tame, $\chi(X, \mathcal{O}_X) = 1$, and therefore, there are only a finite number of possibilities for the Hilbert polynomials $P_n(t) := \chi(X, \mathcal{O}_X(-ntK_X))$. Finally, we apply [Kol85, Theorem 2.1.2] to conclude that $X_{\overline{K}}$ form a bounded family over $\operatorname{Spec}(\mathbb{Z}[1/30])$. In particular, there exists $m := m(n)$ such that $-mnK_{X_{\overline{K}}}$ is very ample, and thus, by faithfully flat descent, $-mnK_X$ is very ample. Therefore, there exists $N = N(n) > 0$ such that $-mnK_X$ embeds X into some $\mathbb{P}^N$ with a finite number of possibilities for the Hilbert polynomial. This concludes that X belongs to a bounded family by a classical Hilbert scheme argument. □

5.3. *Bounds on the $\mathbb{Q}$-Gorenstein index of ε-klt del Pezzo surfaces*

After Corollary 5.8, to conclude the proof of boundedness of ε-klt del Pezzo surfaces, we are only left to prove a bound on the Cartier index of K_X depending only on ε. We start with the following result, which is well known over perfect fields.

Lemma 5.9. *Fix $\varepsilon \in \mathbb{Q}_{>0}$ and $n \in \mathbb{Z}_{>0}$. Then, there exists $N = N(\varepsilon, n)$ such that for every ε-klt surface X admitting a minimal resolution $f: Y \to X$ with $\rho(Y) \leq n$, the divisor NK_X is Cartier.*

Proof. Without loss of generality, we assume that X is the spectrum of a local ring and that $\rho(Y) = n$. Let $E = \sum_{i=1}^n E_i$ be the sum of the exceptional divisors of f, and write $K_Y + \sum_{i=1}^n b_i E_i = f^*K_X$ for some $0 \leq b_i < 1 - \varepsilon$. By the base point free theorem [Tan18b, Theorem 4.2], it suffices to find an effective $N = N(\varepsilon, n)$ such that Nb_i is integral. For each i, we write $E_i^2 = -d_{E_i} n_i$ for some integer $n_i > 0$, where $d_{E_i} = [H^0(E_i, \mathcal{O}_{E_i}) : k]$. Moreover, for each $i \neq j$, we write $E_i \cdot E_j = d_{E_j} n_{ij}$ for some $n_{ij} > 0$. The b_i are determined by the following system of equations:

$$(2 - n_j + b_j n_j) = \sum_{i \neq j} b_i n_{ij} \text{ for } j = 1, \ldots, n. \tag{5.4}$$

Since X is ε-klt, we have

$$n_j \leq \frac{2}{\varepsilon}. \tag{5.5}$$

Indeed,

$$\begin{aligned} 0 = (K_Y + \sum b_i E_i) \cdot E_j &\geq (K_Y + b_j E_j) \cdot E_j \\ &\geq -2d_{E_j} + (b_j - 1) \cdot (-d_{E_j} n_j) \geq -2d_{E_j} + \varepsilon n_j d_{E_j}. \end{aligned}$$

Moreover, n_{ij} is bounded from above by [Kol13, Corollary 3.31 and Section 3.41] and [Sat23, Appendix A]. As the n_i and n_{ij} are integers, we only have a finite number of possibilities for the coefficients in Equation 5.4, so we conclude that there are only finitely many possibilities for the solutions b_i, thus showing the existence of a $N(\varepsilon, n)$ for which Nb_i is integral. □

Lemma 5.10. *Let X be a geometrically integral regular projective surface of del Pezzo type over k, and let $\pi\colon X \to B$ be a Mori fibre space. Let $\Delta = \sum b_i E_i$ be an effective $\mathbb{Q}$-divisor such that (X, Δ) is klt and $-(K_X + \Delta)$ is nef. Then*

1. *if* $\dim(B) = 0$, *then* $\sum b_i \leq 3$;
2. *if* $\dim(B) = 1$, *then* $\sum b_i \leq 4$.

Proof. Suppose $\dim(B) = 0$. Let H be an ample Cartier divisor generating $\mathrm{Num}(X)$, and let $d \geq 0$ such that $-K_X \equiv dH$. As X is a regular del Pezzo surface, we have $K_X^2 \leq 9$ by [Tan24, Theorem 1.2], and therefore, $d \leq 3$. Since X is regular, the B_i are Cartier divisors, and thus, $\sum b_i \leq 3$.

Suppose $\dim(B) = 1$. Let F_b be a closed fibre of π and Γ the curve given by Lemma 5.5. Set $d_b := [k(b) : k]$. We write $\Delta = b_0\Gamma + \sum b_i E_i$ as a sum of pairwise distinct prime divisors. As $0 \geq (K_X + \Delta) \cdot F_b$, adjunction implies

$$2d_b \geq \Delta \cdot F_b \geq b_0 d_b + \sum_{(E_i \cdot F_b) \neq 0} b_i d_b. \tag{5.6}$$

As $K_X \cdot \Gamma = d_\Gamma(n-2)$, $0 \geq (K_X + \Delta) \cdot \Gamma$, and $b_0 \leq 1$, adjunction implies

$$2d_\Gamma \geq nd_\Gamma - b_0 n d_\Gamma + \sum_i b_i (E_i \cdot \Gamma) \geq \sum_{(E_i \cdot \Gamma) \neq 0} b_i d_\Gamma. \tag{5.7}$$

Summing the two equations and using that every curve on X intersects either Γ or F_b, we obtain $4 \geq \sum b_i$, as desired. □

Proposition 5.11. *Let $\varepsilon > 0$. Then, there exists a constant $D(\varepsilon)$ such that for all geometrically integral ε-klt del Pezzo surfaces X, the minimal resolution $f\colon Y \to X$ satisfies $\rho(Y) \leq D(\varepsilon)$.*

Proof. We can suppose k is separably closed. We follow the computations of [AM04, Theorem 1.8], verifying that the explicit classification of rational Mori fibre spaces over algebraically closed fields is not needed. Without loss of generality, we can suppose $\varepsilon < \frac{2}{3}$. Since $-K_X$ is ample, by Bertini's theorem, we can choose $H \sim_{\mathbb{Q}} -K_X$ to be an effective $\mathbb{Q}$-divisor whose support is regular and contained in the regular locus of X such that (X, H) is ε-klt. Let $f\colon Y \to X$ be the minimal resolution and write $K_Y + \Gamma_Y = f^*(K_X + H) \sim_{\mathbb{Q}} 0$, where $\Gamma_Y := \sum_{i \in I} b_i E_i + f_*^{-1} H$ and $b_i < 1 - \varepsilon$ by hypothesis. We run a K_Y-MMP which ends with $g\colon Y \to Z$, where Z is a regular projective surface admitting a Mori fibre space structure $\pi\colon Z \to B$. We summarise the situation in the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \pi} \\ X & & B. \end{array}$$

We fix the following notation:

- $E_{i,Z} := g_* E_i$.
- $I_Z := \{i \in I \mid E_{i,Z} \neq 0\}$.
- $\Delta_Y := \sum_{i \in I} b_i E_i$.
- $\Delta_Z := \sum_{i \in I_Z} b_i E_{i,Z}$.
- $\Gamma_Z := g_* \Gamma_Y$

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Note that, by construction, g is a $(K_Y+\Delta_Y)$-non-positive (resp. $(K_Y+\Gamma_Y)$-trivial) birational contraction and (Z,Δ_Z) (resp. (Z,Γ_Z)) is a log del Pezzo pair (resp. Calabi–Yau pair) with ε-klt singularities.

The morphism g is a composition of blow-ups of closed points on regular surfaces by [Sta, Tag 0C5R]. We can decompose g as

$$g\colon Y \xrightarrow{\psi} W \xrightarrow{\varphi} Z,$$

where ψ and φ are proper birational morphisms between regular surfaces such that

1. φ is a composition of blow-ups at closed points P such that $\mathrm{mult}_P(\widetilde{\Gamma_Z})$ has multiplicity at least $\nu := \frac{\varepsilon}{2}$, where $\widetilde{\Gamma_Z}$ denotes the strict transform of Γ_Z;
2. ψ is a composition of blow-ups at closed points P, where $\mathrm{mult}_P(\widetilde{\Gamma_Z})$ has multiplicity $< \nu$.

We first bound $\rho(W/Z)$ in terms of ε. On Y, as $f_*^{-1}H$ is big and nef, we can apply Equation (5.5) to obtain

$$(g_*^{-1}\Gamma_Z)^2 \geq \sum_{i\in I} b_i^2 (g_*^{-1}E_{i,Z})^2 \geq \sum_{i\in I_Z} b_i(1-\varepsilon)\left(\frac{-2}{\varepsilon}\right). \tag{5.8}$$

If $\dim(B)=0$ (resp. 1), we have $\sum_{i\in I_Z} b_i \leq 4$ (resp. ≤ 3) by Lemma 5.10, and thus,

$$(g_*^{-1}\Gamma_Z)^2 \geq \begin{cases} 6-\frac{6}{\varepsilon} \text{ if } \dim(B)=0 \\ 8-\frac{8}{\varepsilon} \text{ if } \dim(B)=1. \end{cases} \tag{5.9}$$

After each of the blow-ups in φ, the self-intersection of $\widetilde{\Gamma_Z}$ decreases by at least ν^2, and we deduce that $(g_*^{-1}\Gamma_Z)^2 \leq \Gamma_Z^2 - \nu^2\rho(W/Z)$. If $\dim(B)=0$ (resp. 1), then $\Gamma_Z^2 = K_Z^2 \leq 9$ (resp. 8) by [Tan24, Theorem 1.2] (resp. Lemma 5.5). Therefore, $(g_*^{-1}\Gamma_Z)^2 \leq 9-\nu^2\rho(W/Z)$ (resp. $8-\nu^2\rho(W/Z)$). Together with (5.9), we conclude

$$\rho(W/Z) \leq \begin{cases} \frac{(3\varepsilon+6)}{\varepsilon\nu^2} \text{if } \dim(B)=0; \\ \frac{8}{\varepsilon\nu^2} \text{ if } \dim(B)=1. \end{cases} \tag{5.10}$$

As we chose $\varepsilon < \frac{2}{3}$, we have $\frac{(3\varepsilon+6)}{\varepsilon\nu^2} < \frac{8}{\varepsilon\nu^2}$.

We now prove a bound on $\rho(Y/W)$ depending only on ε. Let $F=\sum_{i\in J} F_i$ be the sum of the exceptional divisors of φ and let $f_i := \mathrm{coeff}_{F_i}(\Gamma_W)$, where $\Gamma_W := \psi_*\Gamma_Y$. As W and Z are regular surfaces, $\mathrm{Supp}(F)$ is an snc divisor. Let $\psi\colon Y \xrightarrow{s} T \xrightarrow{t} W$ be a factorisation of ψ, where t is a blow-up at a point P of W with exceptional divisor C. Write

$$K_T+\Gamma_T = K_T+\widetilde{\Gamma_Z}+\sum f_i\widetilde{F_i}+cC \sim_{\mathbb{Q}} t^*(K_W+\widetilde{\Gamma}_Z+\sum f_iF_i).$$

We claim that P must lie on the intersection of two components F_1 and F_2 of F. For this, let $J_F := \{i\in J \mid P\in F_i\}$. Then, since $\mathrm{mult}_P(\widetilde{\Gamma}_Z) \leq \nu < \varepsilon$ and $f_i < 1-\varepsilon$, we have that

$$0 < c = (\mathrm{mult}_P(\widetilde{\Gamma}_Z)+\sum_{i\in J_F} f_i - 1) \leq \nu + |J_F|(1-\varepsilon)-1.$$

Hence, $|J_F|\geq 2$. Since $\mathrm{Supp}(F)$ is an snc divisor, this implies $|J_F|=2$, as desired. This argument can be repeated for each blow-up $T\to W$ factorising $Y\to W$.

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As the number of nodes of F is bounded by $\rho(W/Z) - 1$, it remains to bound the number of times we are allowed to blow-up along a node to obtain a bound on $\rho(Y/W)$. This follows from a straightforward induction as in [AM04, Lemma 1.9]. An explicit computation shows that

$$\rho(Y) = \rho(Y/W) + \rho(W/Z) + \rho(Z) \leq \left(\frac{1}{(\varepsilon - \nu)^2} - 1\right) \cdot \left(\frac{8}{\varepsilon\nu^2} - 1\right) + \left(\frac{8}{\varepsilon\nu^2}\right) + 2.$$

As $\nu = \frac{\varepsilon}{2}$, we deduce that

$$\rho(Y) \leq \frac{128}{\varepsilon^5} + \left(3 - \frac{4}{\varepsilon^2}\right) \leq \frac{128}{\varepsilon^5}. \qquad \square$$

We now have all the ingredients to prove the BAB conjecture in dimension 2 and characteristic $p \neq 2, 3$ and 5.

Theorem 5.12. *Let $\varepsilon > 0$ be a rational number. Then, the classes*

$$\mathcal{X}^{\text{tame}}_{\text{dP},\varepsilon} = \{X \mid X \text{ is a geometrically integral tame } \varepsilon\text{-klt del Pezzo surface}\}, \text{ and}$$
$$\mathcal{X}^{>5}_{\text{dP},\varepsilon} = \left\{X \mid X \text{ is an } \varepsilon\text{-klt del Pezzo surface s.t. } \operatorname{char}(H^0(X, \mathcal{O}_X)) \neq 2, 3, 5\right\}$$

are bounded.

Proof. By Lemma 5.9 and Proposition 5.11, there exists $n = n(\varepsilon) > 0$ such that $-nK_X$ is Cartier for all geometrically integral ε-klt del Pezzo surfaces X. Hence, we can apply Corollary 5.8 to conclude that $\mathcal{X}^{\text{tame}}_{\text{dP},\varepsilon}$ and $\mathcal{X}^{>5}_{\text{dP},\varepsilon}$ are bounded. $\square$

Remark 5.13. To prove the geometrically integral case of the BAB conjecture in characteristic $p \leq 5$, the missing ingredient is a bound on the irregularity for ε-klt del Pezzo surface. While the canonical case (the characteristic $p > 5$ klt case) has been treated in Theorem 1.2 (resp. [BT22, Theorem 5.7]), we are not able to prove a similar bound in the general case. Note that klt del Pezzo surfaces with $h^1(X, \mathcal{O}_X) = 1$ are constructed in [Tan20] over fields k of characteristic $p = 2, 3$ and $\text{p} - \deg(k) = 1$.

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