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MULTI-POLY-BERNOULLI NUMBERS AND RELATED ZETA FUNCTIONS

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Abstract. We construct and study a certain zeta function which interpolates multi-poly-Bernoulli numbers at nonpositive integers and whose values at positive integers are linear combinations of multiple zeta values. This function can be regarded as the one to be paired up with the ξ-function defined by Arakawa and Kaneko. We show that both are closely related to the multiple zeta functions. Further we define multi-indexed poly-Bernoulli numbers, and generalize the duality formulas for poly-Bernoulli numbers by introducing more general zeta functions.

§1. Introduction

In this paper, we investigate the function defined by

$$\eta(k_1,\ldots,k_r;s)=\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^t)}{1-e^t}\,dt \tag{1}$$

and its generalizations, in connection with multi-poly-Bernoulli numbers and multiple zeta values (we shall give the precise definitions later in Section 2). This function can be viewed as a twin sibling of the function $\xi(k_1,\ldots,k_r;s)$,

$$\xi(k_1,\ldots,k_r;s)=\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{e^t-1}\,dt, \tag{2}$$

which was introduced and studied in [4]. The present paper may constitute a natural continuation of the work [4].

To explain our results in some detail, we first give an overview of the necessary background. For an integer $k\in\mathbb{Z}$, two types of poly-Bernoulli numbers $\{B_n^{(k)}\}$ and $\{C_n^{(k)}\}$ are defined as follows (see Kaneko [20] and

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Arakawa–Kaneko [4], also Arakawa–Ibukiyama–Kaneko [3]):

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \tag{3}$$

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!}, \tag{4}$$

where $\mathrm{Li}_k(z)$ is the polylogarithm function defined by

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad (|z|<1). \tag{5}$$

Since $\mathrm{Li}_1(z) = -\log(1-z)$, we see that $B_n^{(1)}$ (resp. $C_n^{(1)}$) coincides with the ordinary Bernoulli number B_n defined by

$$\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \left(\text{resp. } \frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\right).$$

A number of formulas, including closed formulas of $B_n^{(k)}$ and $C_n^{(k)}$ in terms of the Stirling numbers of the second kind as well as the duality formulas

$$B_n^{(-k)} = B_k^{(-n)}, \tag{6}$$

$$C_n^{(-k-1)} = C_k^{(-n-1)} \tag{7}$$

that hold for $k, n \in \mathbb{Z}_{\geqslant 0}$, have been established (see [20, Theorems 1 and 2] and [21, Section 2]). We also mention that Brewbaker [9] gave a purely combinatorial interpretation of the number $B_n^{(-k)}$ of negative upper index as the number of 'Lonesum-matrices' with n rows and k columns.

A multiple version of $B_n^{(k)}$ is defined in [4, p. 202, Remarks (ii)] by

$$\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!} \quad (k_1,\ldots,k_r \in \mathbb{Z}), \tag{8}$$

where

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{1 \leqslant m_1 < \cdots < m_r} \frac{z^{m_r}}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} \tag{9}$$

is the multiple polylogarithm. Hamahata and Masubuchi [14, 15] investigated some properties of $\mathbb{B}_n^{(k_1,\ldots,k_r)}$, and gave several generalizations of the known results in the single-index case. Based on this research, Bayad and Hamahata [8] further studied these numbers. Furusho [12, p. 269] also refers to (8).

More recently, Imatomi, Takeda and Kaneko [18] defined and studied another type of multi-poly-Bernoulli numbers given by

$$\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!}, \tag{10}$$

$$\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_n^{(k_1,\ldots,k_r)} \frac{t^n}{n!} \tag{11}$$

for $k_1, \ldots, k_r \in \mathbb{Z}$. They proved several formulas for $B_n^{(k_1,\ldots,k_r)}$ and $C_n^{(k_1,\ldots,k_r)}$, and further gave an important relation between $C_{p-2}^{(k_1,\ldots,k_r)}$ and the "finite multiple zeta value", that is,

$$\sum_{1 \leqslant m_1 < \cdots < m_r < p} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \equiv -C_{p-2}^{(k_1,\ldots,k_{r-1},k_r-1)} \mod p \tag{12}$$

for any prime number p.

The function (2) for $k_1, \ldots, k_r \in \mathbb{Z}_{\geqslant 1}$ can be analytically continued to an entire function of the complex variable $s \in \mathbb{C}$ [4, Sections 3 and 4]. The particular case $r = k = 1$ gives $\xi(1; s) = s\zeta(s+1)$. Hence $\xi(k_1, \ldots, k_r; s)$ can be regarded as a multi-indexed zeta function. It is shown in [4] that the values at nonpositive integers of $\xi(k; s)$ interpolate poly-Bernoulli numbers $C_m^{(k)}$,

$$\xi(k; -m) = (-1)^m C_m^{(k)} \tag{13}$$

for $k \in \mathbb{Z}_{\geqslant 1}$ and $m \in \mathbb{Z}_{\geqslant 0}$. And also by investigating $\xi(k_1, \ldots, k_r; s)$ and its values at positive integer arguments, one produces many relations among multiple zeta values defined by

$$\zeta(l_1, \ldots, l_r) = \sum_{1 \leqslant m_1 < \cdots < m_r} \frac{1}{m_1^{l_1} \cdots m_r^{l_r}} \quad (= \mathrm{Li}_{l_1,\ldots,l_r}(1)) \tag{14}$$

for $l_1, \ldots, l_r \in \mathbb{Z}_{\geqslant 1}$ with $l_r \geqslant 2$ [4, Corollary 11].

Recently, further properties of $\xi(k_1, \ldots, k_r; s)$ and related results have been given by several authors (see, for example, Bayad–Hamahata [6, 7], Coppo–Candelpergher [10], Sasaki [28], and Young [31]).

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In this paper, we conduct a basic study of the function (1) and relate it to the multi-poly-Bernoulli numbers $B_n^{(k_1,\ldots,k_r)}$ as well as multiple zeta (or "zeta-star") values. Note that the only difference in both definitions (1) and (2) is, up to sign, the arguments $1-e^t$ and $1-e^{-t}$ of $\mathrm{Li}_{k_1,\ldots,k_r}(z)$ in the integrands. One sees in the main body of the paper a remarkable contrast between "B-type" poly-Bernoulli numbers and those of "C-type", and between multiple zeta and zeta-star values. We further investigate the case of nonpositive indices k_i in connection with a yet more generalized "multi-indexed" poly-Bernoulli number.

The paper is organized as follows. In Section 2, we give the analytic continuation of $\eta(k_1,\ldots,k_r;s)$ in the case of positive indices, and formulas for values at integer arguments (Theorems 2.3 and 2.5). In Section 3, we study relations between two functions $\eta(k_1,\ldots,k_r;s)$ and $\xi(k_1,\ldots,k_r;s)$ (Proposition 3.2), as well as relations with the single-variable multiple zeta function (Definition (26) and Theorem 3.6). We turn in Section 4 to the study of $\eta(k_1,\ldots,k_r;s)$ in the negative index case and give a certain duality formula for $B_m^{(-k_1,\ldots,-k_r)}$ (Definition 4.3 and Theorems 4.4 and 4.7). We carry forward the study of negative index case in Section 5 and define the "multi-indexed" poly-Bernoulli numbers $\{B_{m_1,\ldots,m_r}^{(k_1,\ldots,k_r),(d)}\}$ for $(k_1,\ldots,k_r)\in\mathbb{Z}^r$, $(m_1,\ldots,m_r)\in\mathbb{Z}_{\geqslant 0}^r$ and $d\in\{1,\ldots,r\}$ (Definition 5.1), which include (8) and (10) as special cases. We prove the "multi-indexed" duality formula for them in the case $d=r$ (Theorem 5.4).

§2. The function $\eta(k_1,\ldots,k_r;s)$ for positive indices and its values at integers

2.1 Analytic continuation and the values at nonpositive integers

We start with the definition in the case of positive indices.

Definition 2.1. For positive integers $k_1,\ldots,k_r\in\mathbb{Z}_{\geqslant 1}$, let

$$\eta(k_1,\ldots,k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^t)}{1-e^t}\,dt$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1-r$, where $\Gamma(s)$ is the gamma function. When $r=1$, we often denote $\eta(k;s)$ by $\eta_k(s)$.

The integral on the right-hand side converges absolutely in the domain $\mathrm{Re}(s)>1-r$, as is seen from the following lemma.

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LEMMA 2.2.

(i) *For* $k_1, \ldots, k_r \in \mathbb{Z}_{\geqslant 1}$, *the function* $\mathrm{Li}_{k_1,\ldots,k_r}(1-e^t)$ *is holomorphic for* $t \in \mathbb{C}$ *with* $|\mathrm{Im}(t)| < \pi$.

(ii) *For* $k_1, \ldots, k_r \in \mathbb{Z}_{\geqslant 1}$ *and* $t \in \mathbb{R}_{>0}$, *we have the estimates*

$$\mathrm{Li}_{k_1,\ldots,k_r}(1-e^t) = O\left(t^r\right) \quad (t \to 0) \tag{15}$$

and

$$\mathrm{Li}_{k_1,\ldots,k_r}(1-e^t) = O\left(t^{k_1+\cdots+k_r}\right) \quad (t \to \infty). \tag{16}$$

Proof. As is well-known, we can regard the function $\mathrm{Li}_{k_1,\ldots,k_r}(z)$ as a single-valued holomorphic function in the simply connected domain $\mathbb{C} \smallsetminus [1,\infty)$, via the process of iterated integration starting with

$$\mathrm{Li}_1(z) = \int_0^z dz/(1-z).$$

Noting that $1-e^t \in [1,\infty)$ is equivalent to $\mathrm{Im}(t) = (2j+1)\pi$ for some $j \in \mathbb{Z}$, we have the assertion (i).

The estimate (15) is clear from the definition of $\mathrm{Li}_{k_1,\ldots,k_r}(z)$, because its Taylor series at $z = 0$ starts with the term $z^r/1^{k_1} \cdots r^{k_r}$. As for (16), we proceed by induction on the "weight" $k_1 + \cdots + k_r$ as follows by using the formula

$$\frac{d}{dz}\mathrm{Li}_{k_1,\ldots,k_r}(z) = \begin{cases} \dfrac{1}{z}\,\mathrm{Li}_{k_1,\ldots,k_{r-1},k_r-1}(z) & (k_r > 1) \\ \dfrac{1}{1-z}\,\mathrm{Li}_{k_1,\ldots,k_{r-1}}(z) & (k_r = 1), \end{cases} \tag{17}$$

which is easy to derive and is the basis of the analytic continuation of $\mathrm{Li}_{k_1,\ldots,k_r}(z)$ mentioned above. If $r = k_1 = 1$, then we have $\mathrm{Li}_1(1-e^t) = -t$ and the desired estimate holds. Suppose the weight k is larger than 1 and the assertion holds for any weight less than k. If $k_r > 1$, then by (17) we have

$$\begin{aligned} |\mathrm{Li}_{k_1,\ldots,k_r}(1-e^t)| &= \left| \int_0^{1-e^t} \frac{\mathrm{Li}_{k_1,\ldots,k_r-1}(u)}{u}\, du \right| \\ &= \left| \int_0^t \frac{1}{1-e^v}\,\mathrm{Li}_{k_1,\ldots,k_r-1}(1-e^v)(-e^v)\, dv \right| \quad (u := 1-e^v) \end{aligned}$$

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$$\leqslant \int_0^{\varepsilon} \left| e^v \frac{\mathrm{Li}_{k_1,\ldots,k_r-1}(1-e^v)}{e^v-1} \right| dv + \int_{\varepsilon}^{t} \left| \frac{e^v}{e^v-1} \mathrm{Li}_{k_1,\ldots,k_r-1}(1-e^v) \right| dv$$

for small $\varepsilon > 0$. The former integral is $O(1)$ because the integrand is continuous on $[0,\varepsilon]$. On the other hand, by induction hypothesis, the integrand of the latter integral is $O\left(v^{k_1+\cdots+k_r-1}\right)$ as $v \to \infty$. Therefore, the latter integral is $O\left(t^{k_1+\cdots+k_r}\right)$ as $t \to \infty$. The case of $k_r = 1$ is similarly proved also by using (17), and is omitted here. ∎

We now show that the function $\eta(k_1,\ldots,k_r;s)$ can be analytically continued to an entire function, and interpolates multi-poly-Bernoulli numbers $B_m^{(k_1,\ldots,k_r)}$ at nonpositive integer arguments.

THEOREM 2.3. *For positive integers $k_1,\ldots,k_r \in \mathbb{Z}_{\geqslant 1}$, the function $\eta(k_1,\ldots,k_r;s)$ can be analytically continued to an entire function on the whole complex plane. And the values of $\eta(k_1,\ldots,k_r;s)$ at nonpositive integers are given by*

$$\eta(k_1,\ldots,k_r;-m) = B_m^{(k_1,\ldots,k_r)} \quad (m \in \mathbb{Z}_{\geqslant 0}). \tag{18}$$

In particular, $\eta_k(-m) = B_m^{(k)}$ for $k \in \mathbb{Z}_{\geqslant 1}$ and $m \in \mathbb{Z}_{\geqslant 0}$.

Proof. In order to prove this theorem, we adopt here the method of contour integral representation (see, for example, [30, Theorem 4.2]). Let $\mathcal{C}$ be the standard contour, namely the path consisting of the positive real axis from the infinity to (sufficiently small) ε ("top side"), a counterclockwise circle C_ε around the origin of radius ε, and the positive real axis from ε to the infinity ("bottom side"). Let

$$\begin{aligned} H(k_1,\ldots,k_r;s) &= \int_{\mathcal{C}} t^{s-1} \frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^t)}{1-e^t}\, dt \\ &= (e^{2\pi i s}-1) \int_{\varepsilon}^{\infty} t^{s-1} \frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^t)}{1-e^t}\, dt \\ &\quad + \int_{C_\varepsilon} t^{s-1} \frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^t)}{1-e^t}\, dt. \end{aligned}$$

It follows from Lemma 2.2 that $H(k_1,\ldots,k_r;s)$ is entire, because the integrand has no singularity on $\mathcal{C}$ and the contour integral is absolutely

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convergent for all $s \in \mathbb{C}$. Suppose $\mathrm{Re}(s) > 1 - r$. The last integral tends to 0 as $\varepsilon \to 0$. Hence

$$\eta(k_1, \ldots, k_r; s) = \frac{1}{(e^{2\pi i s} - 1)\Gamma(s)} H(k_1, \ldots, k_r; s),$$

which can be analytically continued to $\mathbb{C}$, and is entire. In fact $\eta(k_1, \ldots, k_r; s)$ is holomorphic for $\mathrm{Re}(s) > 0$, hence has no singularity at any positive integer. Set $s = -m \in \mathbb{Z}_{\leqslant 0}$. Then, by (10),

$$\begin{aligned} \eta(k_1, \ldots, k_r; -m) &= \frac{(-1)^m m!}{2\pi i} H(k_1, \ldots, k_r; -m) \\ &= \frac{(-1)^m m!}{2\pi i} \int_{C_\varepsilon} t^{-m-1} \sum_{n=0}^{\infty} B_n^{(k_1, \ldots, k_r)} \frac{(-t)^n}{n!} \, dt = B_m^{(k_1, \ldots, k_r)}. \end{aligned}$$

This completes the proof. □

REMARK 2.4. Using the same method as above or the method used in [4], we can establish the analytic continuation of $\xi(k_1, \ldots, k_r; s)$ to an entire function, and see that

(19) $$\xi(k_1, \ldots, k_r; -m) = (-1)^m C_m^{(k_1, \ldots, k_r)} \quad (m \in \mathbb{Z}_{\geqslant 0})$$

for $k_1, \ldots, k_r \in \mathbb{Z}_{\geqslant 1}$, which is a multiple version of (13).

2.2 Values at positive integers

About the values at positive integer arguments, we prove formulas for both $\xi(k_1, \ldots, k_r; s)$ and $\eta(k_1, \ldots, k_r; s)$, for general index $(k_1, \ldots, k_r)$. These formulas generalize [4, Theorem 9(i)], and have remarkable similarity in that one obtains the formula for $\eta(k_1, \ldots, k_r; s)$ just by replacing multiple zeta values in the one for $\xi(k_1, \ldots, k_r; s)$ with multiple "zeta-star" values. Recall the multiple zeta-star value is a real number defined by

(20) $$\zeta^\star(l_1, \ldots, l_r) = \sum_{1 \leqslant m_1 \leqslant \cdots \leqslant m_r} \frac{1}{m_1^{l_1} \cdots m_r^{l_r}}$$

for $l_1, \ldots, l_r \in \mathbb{Z}_{\geqslant 1}$ with $l_r \geqslant 2$. This was first studied (for general r) by Hoffman in [16].

To state our theorem, we further introduce some notation. For an index set $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geqslant 1}^r$, put $\mathbf{k}_+ = (k_1, \ldots, k_{r-1}, k_r + 1)$. The usual dual index of an *admissible* index (i.e. the one that the last entry is greater than one) $\mathbf{k}$ is denoted by $\mathbf{k}^*$. For $\mathbf{j}=(j_1,\ldots,j_r)\in\mathbb{Z}^r_{\geqslant 0}$, we write $|\mathbf{j}|=j_1+\cdots+j_r$ and call it the weight of $\mathbf{j}$, and $d(\mathbf{j})=r$, the depth of $\mathbf{j}$. For two such indices $\mathbf{k}$ and $\mathbf{j}$ of the same depth, we denote by $\mathbf{k}+\mathbf{j}$ the index obtained by the component-wise addition, $\mathbf{k}+\mathbf{j}=(k_1+j_1,\ldots,k_r+j_r)$, and by $b(\mathbf{k};\mathbf{j})$ the quantity given by

$$b(\mathbf{k};\mathbf{j}):=\prod_{i=1}^{r}\binom{k_i+j_i-1}{j_i}.$$

THEOREM 2.5. *For any index set* $\mathbf{k}=(k_1,\ldots,k_r)\in\mathbb{Z}^r_{\geqslant 1}$ *and any* $m\in\mathbb{Z}_{\geqslant 1}$, *we have*

$$\xi(k_1,\ldots,k_r;m)=\sum_{|\mathbf{j}|=m-1,\,d(\mathbf{j})=n} b((\mathbf{k}_+)^*;\mathbf{j})\,\zeta((\mathbf{k}_+)^*+\mathbf{j}) \tag{21}$$

and

$$\eta(k_1,\ldots,k_r;m)=(-1)^{r-1}\sum_{|\mathbf{j}|=m-1,\,d(\mathbf{j})=n} b((\mathbf{k}_+)^*;\mathbf{j})\,\zeta^\star((\mathbf{k}_+)^*+\mathbf{j}), \tag{22}$$

where both sums are over all $\mathbf{j}\in\mathbb{Z}^r_{\geqslant 0}$ *of weight* $m-1$ *and depth* $n:=d(\mathbf{k}^*_+)$ $(=|\mathbf{k}|+1-d(\mathbf{k}))$.

In particular, we have

$$\xi(k_1,\ldots,k_r;1)=\zeta((\mathbf{k}_+)^*)$$
$$(=\zeta(\mathbf{k}_+),\ \text{by the duality of multiple zeta values})$$

and

$$\eta(k_1,\ldots,k_r;1)=(-1)^{r-1}\zeta^\star((\mathbf{k}_+)^*).$$

In order to prove the theorem, we give certain multiple integral expressions of the functions $\xi(k_1,\ldots,k_r;s)$ and $\eta(k_1,\ldots,k_r;s)$.

PROPOSITION 2.6. *Notations being as above, write* $(\mathbf{k}_+)^*=(l_1,\ldots,l_n)$. *Then we have, for* $\mathrm{Re}(s)>1-r$,

(i) $$\xi(k_1,\ldots,k_r;s)$$
$$=\frac{1}{\prod_{i=1}^n\Gamma(l_i)\cdot\Gamma(s)}\int_0^\infty\cdots\int_0^\infty(x_1+\cdots+x_n)^{s-1}x_1^{l_1-1}\cdots x_n^{l_n-1}$$
$$\times\frac{1}{e^{x_1+\cdots+x_n}-1}\cdot\frac{1}{e^{x_2+\cdots+x_n}-1}\cdot\cdots\cdot\frac{1}{e^{x_n}-1}dx_1\cdots dx_n.$$

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(ii) $$\eta(k_1,\ldots,k_r;s) = \frac{(-1)^{r-1}}{\prod_{i=1}^{n}\Gamma(l_i)\cdot\Gamma(s)}\int_0^\infty\cdots\int_0^\infty (x_1+\cdots+x_n)^{s-1}x_1^{l_1-1}\cdots x_n^{l_n-1} \times \frac{1}{e^{x_1+\cdots+x_n}-1}\cdot\frac{e^{x_2+\cdots+x_n}}{e^{x_2+\cdots+x_n}-1}\cdot\cdots\cdot\frac{e^{x_n}}{e^{x_n}-1}\,dx_1\cdots dx_n.$$

Proof. First write the index $(k_1,\ldots,k_r)$ as

$$(k_1,\ldots,k_r)=(\underbrace{1,\ldots,1}_{a_1-1},b_1+1,\ldots,\underbrace{1,\ldots,1}_{a_h-1},b_h+1),$$

with (uniquely determined) integers $h\geqslant 1$, $a_i\geqslant 1$ $(1\leqslant i\leqslant h)$, $b_i\geqslant 1$ $(1\leqslant i\leqslant h-1)$, and $b_h\geqslant 0$. Then, by performing the intermediate integrals of repeated $dz/(1-z)$ in the standard iterated integral coming from (17), we obtain the following iterated integral expression of the multiple polylogarithm $\mathrm{Li}_{k_1,\ldots,k_r}(z)$:

$$\begin{aligned}
&\mathrm{Li}_{k_1,\ldots,k_r}(z)\\
&=\underbrace{\int_0^z\frac{dx_h}{x_h}\int_0^{x_h}\cdots\cdots\int_0^{x_h}\frac{dx_h}{x_h}}_{b_h}\int_0^{x_h}\frac{1}{a_h!}\log\left(\frac{1-x_{h-1}}{1-x_h}\right)^{a_h}\frac{dx_{h-1}}{x_{h-1}}\\
&\cdot\underbrace{\int_0^{x_{h-1}}\frac{dx_{h-1}}{x_{h-1}}\cdots\cdots\int_0^{x_{h-1}}\frac{dx_{h-1}}{x_{h-1}}}_{b_{h-1}-1}\int_0^{x_{h-1}}\frac{1}{a_{h-1}!}\log\left(\frac{1-x_{h-2}}{1-x_{h-1}}\right)^{a_{h-1}}\\
&\times\frac{dx_{h-2}}{x_{h-2}}\cdots\cdots\cdots\cdots\underbrace{\int_0^{x_3}\frac{dx_3}{x_3}\cdots\cdots\int_0^{x_3}\frac{dx_3}{x_3}}_{b_3-1}\int_0^{x_3}\frac{1}{a_3!}\log\left(\frac{1-x_2}{1-x_3}\right)^{a_3}\\
&\times\frac{dx_2}{x_2}\underbrace{\int_0^{x_2}\frac{dx_2}{x_2}\cdots\int_0^{x_2}\frac{dx_2}{x_2}}_{b_2-1}\cdot\int_0^{x_2}\frac{1}{a_2!}\log\left(\frac{1-x_1}{1-x_2}\right)^{a_2}\\
&\times\underbrace{\frac{dx_1}{x_1}\int_0^{x_1}\cdots\cdots\int_0^{x_1}\frac{dx_1}{x_1}\int_0^{x_1}}_{b_1-1}\frac{\left(-\log(1-x)\right)^{a_1}}{a_1!}\frac{dx}{x}.
\end{aligned}$$

Here, to ease notation, we used the same variable in the repetitions of integrals $\int_0^x dx/x$, and we understand $x_h=z$ if $b_h=0$. The paths of integrations are in the domain $\mathbb{C}\setminus[1,\infty)$, and the formula is valid for $z\in\mathbb{C}\setminus[1,\infty)$.

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We may check this formula by differentiating both sides repeatedly and using (17). Putting $z = 1 - e^{-t}$ and $1 - e^{t}$, changing variables accordingly, and suitably labeling the variables, we obtain

$$
\begin{aligned}
&\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})\\
&= \int_0^t \int_0^{t_{b_1+\cdots+b_h}} \cdots \int_0^{t_2} \underbrace{\frac{1}{e^{t_{b_1+\cdots+b_h}}-1} \cdots\cdots \frac{1}{e^{t_{b_1+\cdots+b_{h-1}+2}}-1}}_{b_h-1}\\
&\quad \times \frac{1}{a_h!} \frac{(t_{b_1+\cdots+b_{h-1}+1}-t_{b_1+\cdots+b_{h-1}})^{a_h}}{e^{t_{b_1+\cdots+b_{h-1}+1}}-1}\\
&\quad \cdot \underbrace{\frac{1}{e^{t_{b_1+\cdots+b_{h-1}}}-1} \cdots\cdots \frac{1}{e^{t_{b_1+\cdots+b_{h-2}+2}}-1}}_{b_{h-1}-1}\\
&\quad \times \cdots\cdots\\
&\quad \times \frac{1}{a_3!} \frac{(t_{b_1+b_2+1}-t_{b_1+b_2})^{a_3}}{e^{t_{b_1+b_2+1}}-1} \cdot \underbrace{\frac{1}{e^{t_{b_1+b_2}}-1} \cdots\cdots \frac{1}{e^{t_{b_1+2}}-1}}_{b_2-1}\\
&\quad \times \frac{1}{a_2!} \frac{(t_{b_1+1}-t_{b_1})^{a_2}}{e^{t_{b_1+1}}-1} \cdot \underbrace{\frac{1}{e^{t_{b_1}}-1} \cdots\cdots \frac{1}{e^{t_2}-1}}_{b_1-1}\\
&\quad \cdot \frac{1}{a_1!} \frac{t_1^{a_1}}{e^{t_1}-1}\, dt_1\, dt_2 \cdots dt_{b_1+\cdots+b_h},
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
&\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{t})\\
&= (-1)^r \int_0^t \int_0^{t_{b_1+\cdots+b_h}} \cdots \int_0^{t_2} \underbrace{\frac{e^{t_{b_1+\cdots+b_h}}}{e^{t_{b_1+\cdots+b_h}}-1} \cdots\cdots \frac{e^{t_{b_1+\cdots+b_{h-1}+2}}}{e^{t_{b_1+\cdots+b_{h-1}+2}}-1}}_{b_h-1}\\
&\quad \times \frac{1}{a_h!} \frac{(t_{b_1+\cdots+b_{h-1}+1}-t_{b_1+\cdots+b_{h-1}})^{a_h} e^{t_{b_1+\cdots+b_{h-1}+1}}}{e^{t_{b_1+\cdots+b_{h-1}+1}}-1}\\
&\quad \cdot \underbrace{\frac{e^{t_{b_1+\cdots+b_{h-1}}}}{e^{t_{b_1+\cdots+b_{h-1}}}-1} \cdots\cdots \frac{e^{t_{b_1+\cdots+b_{h-2}+2}}}{e^{t_{b_1+\cdots+b_{h-2}+2}}-1}}_{b_{h-1}-1}\\
&\quad \times \cdots\cdots
\end{aligned}
$$

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$$\times \frac{1}{a_3!}\frac{(t_{b_1+b_2+1}-t_{b_1+b_2})^{a_3}e^{t_{b_1+b_2+1}}}{e^{t_{b_1+b_2+1}}-1}\cdot\underbrace{\frac{e^{t_{b_1+b_2}}}{e^{t_{b_1+b_2}}-1}\cdots\cdots\frac{e^{t_{b_1+2}}}{e^{t_{b_1+2}}-1}}_{b_2-1}$$

$$\times \frac{1}{a_2!}\frac{(t_{b_1+1}-t_{b_1})^{a_2}e^{t_{b_1+1}}}{e^{t_{b_1+1}}-1}\cdot\underbrace{\frac{e^{t_{b_1}}}{e^{t_{b_1}}-1}\cdots\cdots\frac{e^{t_2}}{e^{t_2}-1}}_{b_1-1}$$

$$\cdot\frac{1}{a_1!}\frac{t_1^{a_1}e^{t_1}}{e^{t_1}-1}\,dt_1\,dt_2\,\cdots\,dt_{b_1+\cdots+b_h}. \tag{24}$$

The factor $(-1)^r$ on the right of (24) comes from $(-1)^{a_1+\cdots+a_h}=(-1)^r$. Plugging (23) and (24) into the definitions (2) and (1) respectively and making the change of variables

$$t=x_1+\cdots+x_n,\ t_{b_1+\cdots+b_h}=x_2+\cdots+x_n,\ t_{b_1+\cdots+b_h-1}=x_3+\cdots+x_n,$$
$$\ldots,t_2=x_{n-1}+x_n,\ t_1=x_n,$$

we obtain the proposition. One should note that the dual index $(\mathbf{k}_+)^*=(l_1,\ldots,l_n)$ is given by

$$(\mathbf{k}_+)^*=(\underbrace{1,\ldots,1}_{b_h},a_h+1,\underbrace{1,\ldots,1}_{b_{h-1}-1},a_{h-1}+1,\ldots,\underbrace{1,\ldots,1}_{b_1-1},a_1+1)$$

and the depth n is equal to $b_1+\cdots+b_h+1$, and that (the trivial) $x_i^{l_i-1}=1$ when $l_i=1$. ∎

Proof of Theorem 2.5. Set $s=m$ in the integral expressions in the proposition, and expand $(x_1+\cdots+x_k)^{m-1}$ by the multinomial theorem. Then the formula in the theorem follows from the lemma below.

LEMMA 2.7. *For $l_1,\ldots,l_r\in\mathbb{Z}_{\geqslant 1}$ with $l_r\geqslant 2$, we have*

$$\zeta(l_1,\ldots,l_r)=\frac{1}{\prod_{j=1}^r\Gamma(l_j)}\int_0^\infty\cdots\int_0^\infty\frac{x_1^{l_1-1}\cdots x_r^{l_r-1}}{e^{x_1+\cdots+x_r}-1}\cdot\frac{1}{e^{x_2+\cdots+x_r}-1}\cdots\frac{1}{e^{x_r}-1}dx_1\cdots dx_r$$

and

$$\zeta^\star(l_1,\ldots,l_r)=\frac{1}{\prod_{j=1}^r\Gamma(l_j)}\int_0^\infty\cdots\int_0^\infty\frac{x_1^{l_1-1}\cdots x_r^{l_r-1}}{e^{x_1+\cdots+x_r}-1}\cdot\frac{e^{x_2+\cdots+x_r}}{e^{x_2+\cdots+x_r}-1}\cdots\frac{e^{x_r}}{e^{x_r}-1}dx_1\cdots dx_r.$$

Proof. The first formula is given in [4, Theorem 3(i)]. As for the second, we may proceed similarly by using $n^{-s} = \Gamma(s)^{-1}\int_0^\infty t^{s-1}e^{-nt}\,dt$ to have

$$\begin{aligned}
\zeta^\star(l_1,\ldots,l_r) &= \sum_{m_1=1}^\infty \sum_{m_2,\ldots,m_r=0}^\infty \frac{1}{m_1^{l_1}(m_1+m_2)^{l_2}\cdot\cdots\cdot(m_1+\cdots+m_r)^{l_r}}\\
&= \frac{1}{\prod_{j=1}^r \Gamma(l_j)} \sum_{m_1=1}^\infty \sum_{m_2,\ldots,m_r=0}^\infty \int_0^\infty \cdots \int_0^\infty x_1^{l_1-1}e^{-m_1x_1}\cdot x_2^{l_2-1}e^{-(m_1+m_2)x_2}\cdots\\
&\qquad \cdots x_r^{l_r-1}e^{-(m_1+\cdots+m_r)x_r}\,dx_1\cdots dx_r\\
&= \frac{1}{\prod_{j=1}^r \Gamma(l_j)} \sum_{m_1=1}^\infty \sum_{m_2,\ldots,m_r=0}^\infty \int_0^\infty \cdots \int_0^\infty x_1^{l_1-1}\cdots x_r^{l_r-1}\\
&\quad \times e^{-m_1(x_1+\cdots+x_r)}\cdot e^{-m_2(x_2+\cdots+x_r)}\cdots e^{-m_rx_r}\,dx_1\cdots dx_r\\
&= \frac{1}{\prod_{j=1}^r \Gamma(l_j)} \int_0^\infty \cdots \int_0^\infty \frac{x_1^{l_1-1}\cdots x_r^{l_r-1}}{e^{x_1+\cdots+x_r}-1}\\
&\quad \cdot \frac{e^{x_2+\cdots+x_r}}{e^{x_2+\cdots+x_r}-1}\cdots\frac{e^{x_r}}{e^{x_r}-1}dx_1\cdots dx_r.
\end{aligned}$$

□

We record here one corollary to the theorem in the case of $\eta_k(m)$ (compare with the similar formula in [4, Theorem 9(i)]). Noting $(k+1)^* = (\underbrace{1,\ldots,1}_{k-1},2)$, we have

COROLLARY 2.8. *For $k, m \geqslant 1$, we have*

$$\eta_k(m) = \sum_{\substack{j_1,\ldots,j_{k-1}\geqslant 1, j_k\geqslant 2\\ j_1+\cdots+j_k=k+m}} (j_k-1)\zeta^\star(j_1,\ldots,j_{k-1},j_k). \tag{25}$$

§3. Relations among the functions ξ, η and ζ, and their consequences to multiple zeta values and multi-poly-Bernoulli numbers

In this section, we first deduce that each of the functions η and ξ can be written as a linear combination of the other by the *same* formula. This is a consequence of the so-called Landen-type connection formula for the multiple polylogarithm $\mathrm{Li}_{k_1,\ldots,k_r}(z)$. We then establish a formula for $\xi(k_1,\ldots,k_r;s)$ in terms of the single-variable multiple zeta function

$$\zeta(l_1,\ldots,l_r;s) = \sum_{1\leqslant m_1<\cdots<m_r<m} \frac{1}{m_1^{l_1}\cdots m_r^{l_r}m^s} \tag{26}$$

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defined for positive integers $l_1, \ldots, l_r$, the analytic continuation of which has been given in [4] (the analytic continuation of a more general multi-variable multiple zeta function is established in [1]). This answers the question posed in §5 of [4]. As a result, the function $\eta(k_1, \ldots, k_r; s)$ can also be written by the multiple zeta functions of the type above. We then give a formula for values at positive integers of $\xi(k_1, \ldots, k_r; s)$, and hence of $\eta(k_1, \ldots, k_r; s)$, in terms of the "shuffle regularized values" of multiple zeta values, and thereby derive some consequences on the values of $\eta_k(s)$.

Let $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r_{\geqslant 1}$ be an index set. Recall that $\mathbf{k}$ is said to be admissible if the last entry k_r is greater than 1, the weight of $\mathbf{k}$ is the sum $k_1 + \cdots + k_r$, and the depth is the length r of the index. For two indices $\mathbf{k}$ and $\mathbf{k}'$ of the same weight, we say $\mathbf{k}'$ refines $\mathbf{k}$, denoted $\mathbf{k} \preceq \mathbf{k}'$, if $\mathbf{k}$ is obtained from $\mathbf{k}'$ by replacing some commas by +'s. For example, $(5) = (2+3) \preceq (2,3)$, $(2,3) = (1+1, 2+1) \preceq (1,1,2,1)$, etc. The standard expression of a multiple zeta-star value as a sum of multiple zeta values is written as

$$\zeta^\star(\mathbf{k}) = \sum_{\substack{\mathbf{k}' \preceq \mathbf{k} \\ \text{admissible}}} \zeta(\mathbf{k}'),$$

where the sum on the right runs over the admissible indices $\mathbf{k}'$ such that $\mathbf{k}$ refines $\mathbf{k}'$.

The following formula is known as the Landen connection formula for the multiple polylogarithm [26, Proposition 9].

Lemma 3.1. *For any index* $\mathbf{k}$ *of depth* r*, we have*

$$\mathrm{Li}_{\mathbf{k}}\left(\frac{z}{z-1}\right) = (-1)^r \sum_{\mathbf{k} \preceq \mathbf{k}'} \mathrm{Li}'_{\mathbf{k}}(z). \tag{27}$$

We can prove this by induction on weight and by using (17), see [26].

By using this and noting $z/(z-1) = 1 - e^t$ (resp. $1 - e^{-t}$) if $z = 1 - e^{-t}$ (resp. $1 - e^t$), we immediately obtain the following proposition.

Proposition 3.2. *Let* $\mathbf{k}$ *be any index set and* r *its depth. We have the relations*

$$\eta(\mathbf{k}; s) = (-1)^{r-1} \sum_{\mathbf{k} \preceq \mathbf{k}'} \xi(\mathbf{k}'; s) \tag{28}$$

and

$$\xi(\mathbf{k}; s) = (-1)^{r-1} \sum_{\mathbf{k} \preceq \mathbf{k}'} \eta(\mathbf{k}'; s). \tag{29}$$

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COROLLARY 3.3. *Let k be a positive integer. Then we have*

$$\eta_k(s) = \sum_{\mathbf{k}:\text{weight } k} \xi(\mathbf{k}; s) \tag{30}$$

and

$$\xi_k(s) = \sum_{\mathbf{k}:\text{weight } k} \eta(\mathbf{k}; s), \tag{31}$$

where the sums run over all indices of weight k. Here we have written $\xi_k(s)$ for $\xi(k; s)$.

Proof. The index (k) is of depth 1 and all indices of weight k (admissible or nonadmissible) refine (k). ∎

We mention here that, also by taking $\mathbf{k} = (k)$ in Lemma 3.1 and setting $z = 1 - e^t$ or $1 - e^{-t}$, one immediately obtains a kind of sum formulas for multi-poly-Bernoulli numbers as follows (compare with similar formulas in [17, Theorem 3.1]).

COROLLARY 3.4. *For $k \geqslant 1$ and $m \geqslant 0$, we have*

$$B_m^{(k)} = (-1)^m \sum_{\substack{k_1+\cdots+k_r=k \\ k_i, r \geqslant 1}} C_m^{(k_1,\ldots,k_r)} \tag{32}$$

and

$$C_m^{(k)} = (-1)^m \sum_{\substack{k_1+\cdots+k_r=k \\ k_i, r \geqslant 1}} B_m^{(k_1,\ldots,k_r)}. \tag{33}$$

Next, we prove an Euler-type connection formula for the multiple polylogarithm. If an index $\mathbf{k}$ is of weight $|\mathbf{k}|$, we also say the multiple zeta value $\zeta(\mathbf{k})$ is of weight $|\mathbf{k}|$.

LEMMA 3.5. *Let $\mathbf{k}$ be any index. Then we have*

$$\mathrm{Li}_{\mathbf{k}}(1-z) = \sum_{\mathbf{k}',\, j \geqslant 0} c_{\mathbf{k}}(\mathbf{k}'; j) \mathrm{Li}_{\underbrace{1, \ldots, 1}_{j}}(1-z)\, \mathrm{Li}_{\mathbf{k}'}(z), \tag{34}$$

where the sum on the right runs over indices $\mathbf{k}'$ and integers $j \geqslant 0$ that satisfy $|\mathbf{k}'| + j \leqslant |\mathbf{k}|$, and $c_{\mathbf{k}}(\mathbf{k}'; j)$ is a $\mathbb{Q}$-linear combination of multiple zeta values of weight $|\mathbf{k}| - |\mathbf{k}'| - j$. We understand $\mathrm{Li}_{\emptyset}(z) = 1$ and $|\emptyset| = 0$ for the empty index $\emptyset$, and the constant 1 is regarded as a multiple zeta value of weight 0.

Proof. We proceed by induction on the weight of $\mathbf{k}$. When $\mathbf{k}=(1)$, the trivial identity $\mathrm{Li}_1(1-z)=\mathrm{Li}_1(1-z)$ is the one asserted. Suppose the weight $|\mathbf{k}|$ of $\mathbf{k}$ is greater than 1 and assume the statement holds for any index of weight less than $|\mathbf{k}|$. For $\mathbf{k}=(k_1,\ldots,k_r)$, set $\mathbf{k}_-=(k_1,\ldots,k_{r-1},k_r-1)$ and $\mathbf{k}_+=(k_1,\ldots,k_{r-1},k_r+1)$.

First assume that $\mathbf{k}$ is admissible. Then, by (17) and induction hypothesis, we have

$$\begin{aligned}\frac{d}{dz}\,\mathrm{Li}_{\mathbf{k}}(1-z) &= -\frac{\mathrm{Li}_{\mathbf{k}_-}(1-z)}{1-z}\\ &= -\frac{1}{1-z}\sum_{\mathbf{l},\,j\geqslant 0} c_{\mathbf{k}_-}(\mathbf{l};j)\,\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(1-z)\,\mathrm{Li}_{\mathbf{l}}(z),\end{aligned}$$

the right-hand side being of a desired form. Here, again by (17), we see that

$$\frac{1}{1-z}\,\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(1-z)\,\mathrm{Li}_{\mathbf{l}}(z)=\frac{d}{dz}\left(\sum_{i=0}^{j}\mathrm{Li}_{\underbrace{1,\ldots,1}_{j-i}}(1-z)\,\mathrm{Li}_{\mathbf{l},1+i}(z)\right).$$

We therefore conclude

$$\mathrm{Li}_{\mathbf{k}}(1-z)=-\sum_{\mathbf{l},\,j\geqslant 0} c_{\mathbf{k}_-}(\mathbf{l};j)\sum_{i=0}^{j}\mathrm{Li}_{\underbrace{1,\ldots,1}_{j-i}}(1-z)\,\mathrm{Li}_{\mathbf{l},1+i}(z)+C$$

with some constant C. Since $\lim_{z\to 0}\mathrm{Li}_{\underbrace{1,\ldots,1}_{j-i}}(1-z)\mathrm{Li}_{\mathbf{l},1+i}(z)=0$, we find $C=\zeta(\mathbf{k})$ by setting $z\to 0$, and obtain the desired expression for $\mathrm{Li}_{\mathbf{k}}(1-z)$.

When $\mathbf{k}$ is not necessarily admissible, write $\mathbf{k}=(\mathbf{k}_0,\underbrace{1,\ldots,1}_{q})$ with an admissible $\mathbf{k}_0$ and $q\geqslant 0$. We prove the identity by induction on q. The case $q=0$ ($\mathbf{k}$ is admissible) is already done. Suppose $q\geqslant 1$ and assume the claim is true for smaller q. Then by assumption we have the expression

$$\mathrm{Li}_{\mathbf{k}_0,\underbrace{1,\ldots,1}_{q-1}}(1-z)=\sum_{\mathbf{m},\,j\geqslant 0} c_{\mathbf{k}'}(\mathbf{m};j)\,\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(1-z)\,\mathrm{Li}_{\mathbf{m}}(z),$$

where we have put $\mathbf{k}'=(\mathbf{k}_0,\underbrace{1,\ldots,1}_{q-1})$. We multiply $\mathrm{Li}_1(1-z)$ on both sides. Then, by the shuffle product, the left-hand side becomes the sum of the form

$$q\,\mathrm{Li}_{\mathbf{k}}(1-z)+\sum_{\mathbf{k}_0':\text{admissible}}\mathrm{Li}_{\mathbf{k}_0',\underbrace{1,\ldots,1}_{q-1}}(1-z),$$

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and each term in the sum is written in the claimed form by induction hypothesis. On the other hand, the right-hand side becomes also of the form desired because

$$\mathrm{Li}_1(1-z)\,\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(1-z) = (j+1)\,\mathrm{Li}_{\underbrace{1,\ldots,1}_{j+1}}(1-z).$$

Hence $\mathrm{Li}_{\mathbf{k}}(1-z)$ is of the form as claimed. ∎

With the lemma, we are now able to establish the following (see [4, Section 5, Problem (i)]).

THEOREM 3.6. *Let* $\mathbf{k}$ *be any index set. The function* $\xi(\mathbf{k};s)$ *can be written in terms of multiple zeta functions as*

$$\xi(\mathbf{k};s) = \sum_{\mathbf{k}',\, j\geqslant 0} c_{\mathbf{k}}(\mathbf{k}';j)\binom{s+j-1}{j}\zeta(\mathbf{k}';s+j).$$

Here, the sum is over indices $\mathbf{k}'$ *and integers* $j \geqslant 0$ *satisfying* $|\mathbf{k}'|+j \leqslant |\mathbf{k}|$, *and* $c_{\mathbf{k}}(\mathbf{k}';j)$ *is a* $\mathbb{Q}$*-linear combination of multiple zeta values of weight* $|\mathbf{k}|-|\mathbf{k}'|-j$. *The index* $\mathbf{k}'$ *may be* $\emptyset$ *and for this we set* $\zeta(\emptyset;s+j) = \zeta(s+j)$.

Proof. By setting $z = e^{-t}$ in the lemma and using

$$\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(z) = \frac{(-\log(1-z))^j}{j!}, \tag{35}$$

we have

$$\mathrm{Li}_{\mathbf{k}}(1-e^{-t}) = \sum_{\mathbf{k}',\, j\geqslant 0} c_{\mathbf{k}}(\mathbf{k}';j)\frac{t^j}{j!}\,\mathrm{Li}_{\mathbf{k}}'(e^{-t}).$$

Substituting this into the definition (2) of $\xi(\mathbf{k};s)$ and using the formula [4, Proposition 2, (i)]

$$\zeta(\mathbf{k};s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^t-1}\,\mathrm{Li}_{\mathbf{k}}(e^{-t})\,dt,$$

we immediately obtain the theorem. ∎

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REMARK 3.7. This theorem generalizes [4, Theorem 8], where the corresponding formula for $\mathrm{Li}_{\underbrace{1,\ldots,1}_{r-1},k}(1-z)$ is

$$\mathrm{Li}_{\underbrace{1,\ldots,1}_{r-1},k}(1-z) = (-1)^{k-1} \sum_{\substack{j_1+\cdots+j_k=r+k \\ \forall j_i \geqslant 1}} \mathrm{Li}_{\underbrace{1,\ldots,1}_{j_k-1}}(1-z)\,\mathrm{Li}_{j_1,\ldots,j_{k-1}}(z) + \sum_{j=0}^{k-2}(-1)^j \zeta(\underbrace{1,\ldots,1}_{r-1},k-j)\,\mathrm{Li}_{\underbrace{1,\ldots,1}_{j}}(z).$$

As pointed out by Shu Oi, one can deduce Lemma 3.5 by induction using [27, Proposition 5]. However, to describe the right-hand side of the lemma explicitly is a different problem and neither proof gives such a formula in general. See also [25] for a related topic.

EXAMPLE 3.8. Apart from the trivial case $(1,\ldots,1)$, examples of the identity in Lemma 3.5 up to weight 4 are:

$$\begin{aligned}
\mathrm{Li}_2(1-z) &= -\mathrm{Li}_2(z) - \mathrm{Li}_1(1-z)\mathrm{Li}_1(z) + \zeta(2),\\
\mathrm{Li}_3(1-z) &= \mathrm{Li}_{1,2}(z) + \mathrm{Li}_{2,1}(z) + \mathrm{Li}_1(1-z)\,\mathrm{Li}_{1,1}(z) - \zeta(2)\,\mathrm{Li}_1(z) + \zeta(3),\\
\mathrm{Li}_{1,2}(1-z) &= -\mathrm{Li}_3(z) - \mathrm{Li}_1(1-z)\,\mathrm{Li}_2(z) - \mathrm{Li}_{1,1}(1-z)\,\mathrm{Li}_1(z) + \zeta(3),\\
\mathrm{Li}_{2,1}(1-z) &= 2\mathrm{Li}_3(z) + \mathrm{Li}_1(1-z)\,\mathrm{Li}_2(z) + \zeta(2)\,\mathrm{Li}_1(1-z) - 2\zeta(3),\\
\mathrm{Li}_4(1-z) &= -\mathrm{Li}_{1,1,2}(z) - \mathrm{Li}_{1,2,1}(z) - \mathrm{Li}_{2,1,1}(z) - \mathrm{Li}_1(1-z)\,\mathrm{Li}_{1,1,1}(z)\\
&\quad + \zeta(2)\mathrm{Li}_{1,1}(z) - \zeta(3)\mathrm{Li}_1(z) + \zeta(4),\\
\mathrm{Li}_{1,3}(1-z) &= \mathrm{Li}_{1,3}(z) + \mathrm{Li}_{2,2}(z) + \mathrm{Li}_{3,1}(z) + \mathrm{Li}_1(1-z)\,\mathrm{Li}_{1,2}(z)\\
&\quad + \mathrm{Li}_1(1-z)\,\mathrm{Li}_{2,1}(z) + \mathrm{Li}_{1,1}(1-z)\,\mathrm{Li}_{1,1}(z)\\
&\quad - \zeta(3)\mathrm{Li}_1(z) + \tfrac{1}{4}\zeta(4),\\
\mathrm{Li}_{2,2}(1-z) &= -\mathrm{Li}_{2,2}(z) - 2\mathrm{Li}_{3,1}(z) - \mathrm{Li}_1(1-z)\,\mathrm{Li}_{2,1}(z)\\
&\quad - \zeta(2)\mathrm{Li}_1(1-z)\,\mathrm{Li}_1(z) - \zeta(2)\mathrm{Li}_2(z) + 2\zeta(3)\mathrm{Li}_1(z) + \tfrac{3}{4}\zeta(4),\\
\mathrm{Li}_{3,1}(1-z) &= -2\mathrm{Li}_{1,3}(z) - \mathrm{Li}_{2,2}(z) - \mathrm{Li}_1(1-z)\mathrm{Li}_{1,2}(z) + \zeta(2)\mathrm{Li}_2(z)\\
&\quad + \zeta(3)\mathrm{Li}_1(1-z) - \tfrac{5}{4}\zeta(4),\\
\mathrm{Li}_{1,1,2}(1-z) &= -\mathrm{Li}_4(z) - \mathrm{Li}_1(1-z)\,\mathrm{Li}_3(z) - \mathrm{Li}_{1,1}(1-z)\,\mathrm{Li}_2(z)\\
&\quad - \mathrm{Li}_{1,1,1}(1-z)\,\mathrm{Li}_1(z) + \zeta(4),
\end{aligned}$$

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$$
\begin{aligned}
\mathrm{Li}_{1,2,1}(1-z) &= 3\mathrm{Li}_4(z) + 2\mathrm{Li}_1(1-z)\,\mathrm{Li}_3(z) + \mathrm{Li}_{1,1}(1-z)\,\mathrm{Li}_2(z) \\
&\quad + \zeta(3)\mathrm{Li}_1(1-z) - 3\zeta(4), \\
\mathrm{Li}_{2,1,1}(1-z) &= -3\mathrm{Li}_4(z) - \mathrm{Li}_1(1-z)\,\mathrm{Li}_3(z) + \zeta(2)\mathrm{Li}_{1,1}(1-z) \\
&\quad - 2\zeta(3)\mathrm{Li}_1(1-z) + 3\zeta(4).
\end{aligned}
$$

Accordingly, we have

$$
\begin{aligned}
\xi(2;s) &= -\zeta(2;s) - s\zeta(1;s+1) + \zeta(2)\zeta(s), \\
\xi(3;s) &= \zeta(1,2;s) + \zeta(2,1;s) + s\zeta(1,1;s+1) - \zeta(2)\zeta(1;s) + \zeta(3)\zeta(s), \\
\xi(1,2;s) &= -\zeta(3;s) - s\zeta(2;s+1) - \frac{s(s+1)}{2}\zeta(1;s+2) + \zeta(3)\zeta(s), \\
\xi(2,1;s) &= 2\zeta(3;s) + s\zeta(2;s+1) + \zeta(2)s\zeta(s+1) - 2\zeta(3)\zeta(s), \\
\xi(4;s) &= -\zeta(1,1,2;s) - \zeta(1,2,1;s) - \zeta(2,1,1;s) - s\zeta(1,1,1;s+1) \\
&\quad + \zeta(2)\zeta(1,1;s) - \zeta(3)\zeta(1;s) + \zeta(4)\zeta(s), \\
\xi(1,3;s) &= \zeta(1,3;s) + \zeta(2,2;s) + \zeta(3,1;s) \\
&\quad + s\zeta(1,2;s+1) + s\zeta(2,1;s+1) + \frac{s(s+1)}{2}\zeta(1,1;s+2) \\
&\quad - \zeta(3)\zeta(1;s) + \frac{1}{4}\zeta(4)\zeta(s), \\
\xi(2,2;s) &= -\zeta(2,2;s) - 2\zeta(3,1;s) - s\zeta(2,1;s+1) - \zeta(2)s\zeta(1;s+1) \\
&\quad - \zeta(2)\zeta(2;s) + 2\zeta(3)\zeta(1;s) + \tfrac{3}{4}\zeta(4)\zeta(s), \\
\xi(3,1;s) &= -2\zeta(1,3;s) - \zeta(2,2;s) - s\zeta(1,2;s+1) + \zeta(2)\zeta(2;s) \\
&\quad + \zeta(3)s\zeta(s+1) - \tfrac{5}{4}\zeta(4)\zeta(s), \\
\xi(1,1,2;s) &= -\zeta(4;s) - s\zeta(3;s+1) - \frac{s(s+1)}{2}\zeta(2;s+2) \\
&\quad - \frac{s(s+1)(s+2)}{6}\zeta(1;s+3) + \zeta(4)\zeta(s), \\
\xi(1,2,1;s) &= 3\zeta(4;s) + 2s\zeta(3;s+1) + \frac{s(s+1)}{2}\zeta(2;s+2) + \zeta(3)s\zeta(s+1) \\
&\quad - 3\zeta(4)\zeta(s), \\
\xi(2,1,1;s) &= -3\zeta(4;s) - s\zeta(3;s+1) + \zeta(2)\frac{s(s+1)}{2}\zeta(s+2) \\
&\quad - 2\zeta(3)s\zeta(s+1) + 3\zeta(4)\zeta(s).
\end{aligned}
$$

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From these and (30) of Corollary 3.3, we have for instance

$$\begin{aligned}
\eta_2(s) &= \xi(2;s)+\xi(1,1;s)\\
&= -\zeta(2;s)-s\zeta(1;s+1)+\zeta(2)\zeta(s)+\frac{s(s+1)}{2}\zeta(s+2),\\
\eta_3(s) &= \xi(3;s)+\xi(1,2;s)+\xi(2,1;s)+\xi(1,1,1;s)\\
&= \zeta(3;s)+\zeta(1,2;s)+\zeta(2,1;s)+s\zeta(1,1;s+1)-\frac{s(s+1)}{2}\zeta(1;s+2)\\
&\quad -\zeta(2)\zeta(1;s)+\zeta(2)s\zeta(s+1)+\frac{s(s+1)(s+2)}{6}\zeta(s+3),\\
\eta_4(s) &= \xi(4;s)+\xi(1,3;s)+\xi(2,2;s)+\xi(3,1;s)+\xi(1,1,2;s)\\
&\quad +\xi(1,2,1;s)+\xi(2,1,1;s)+\xi(1,1,1,1;s)\\
&= -\zeta(4;s)-\zeta(1,3;s)-\zeta(2,2;s)-\zeta(3,1;s)-\zeta(1,1,2;s)\\
&\quad -\zeta(1,2,1;s)-\zeta(2,1,1;s)-s\zeta(1,1,1;s+1)+\zeta(2)\zeta(1,1;s)\\
&\quad +\frac{s(s+1)}{2}\zeta(1,1;s+2)-\zeta(2)s\zeta(1;s+1)\\
&\quad +\zeta(2)\frac{s(s+1)}{2}\zeta(s+2)-\frac{s(s+1)(s+2)}{6}\zeta(1;s+3)\\
&\quad +\frac{7}{4}\zeta(4)\zeta(s)+\frac{s(s+1)(s+2)(s+3)}{24}\zeta(s+4).
\end{aligned}$$

Before closing this section, we present a curious observation. Recall the formula

$$\xi_k(m)=\zeta^\star(\underbrace{1,\ldots,1}_{m-1},k+1)$$

discovered by Ohno [24]. Comparing this with the two formulas (25) and [4, Corollary 10], one may expect

$$\eta_k(m)\stackrel{?}{=}\zeta(\underbrace{1,\ldots,1}_{m-1},k+1).$$

This is not true in fact. However, we found experimentally the identities

$$\eta_k(m)=\eta_m(k) \tag{36}$$

and

$$\sum_{j=1}^{k-1}(-1)^{j-1}\eta_{k-j}(j)=\begin{cases}2(1-2^{1-k})\zeta(k) & (k\text{: even}),\\ 0 & (k\text{: odd}).\end{cases} \tag{37}$$

These are respectively analogous to the duality relation

$$\zeta(\underbrace{1,\ldots,1}_{m-1},k+1)=\zeta(\underbrace{1,\ldots,1}_{k-1},m+1)$$

and the relation

$$\sum_{j=1}^{k-1}(-1)^{j-1}\zeta(\underbrace{1,\ldots,1}_{j-1},k-j+1)=\begin{cases}2(1-2^{1-k})\zeta(k) & (k\text{: even}),\\ 0 & (\text{k: odd}),\end{cases}$$

which is a special case of the Le–Murakami relation [23] (or one can derive this from the well-known generating series identity [2, 11]

$$1-\sum_{k>j\geqslant 1}\zeta(\underbrace{1,\ldots,1}_{j-1},k-j+1)X^{k-j}\,Y^{j}=\frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}$$

by setting $Y=-X$ and using the reflection formula for the gamma function.)

We are still not able to prove (36)[1], but could prove (37) by using the following general formula for the value $\xi(\mathbf{k};m)$ and the relation (28) in Proposition 3.2. For other aspects of "height one" multiple zeta values, see [22].

PROPOSITION 3.9. *Let* $\mathbf{k}$ *be any index and* $m\geqslant 1$ *an integer. Then we have*

$$\xi(\mathbf{k};m)=(-1)^{m-1}\zeta^{ш}(\mathbf{k}_+,\underbrace{1,\ldots,1}_{m-1}),\tag{38}$$

where $\zeta^{ш}$ *stands for the "shuffle regularized" value, which is the constant term of the shuffle regularized polynomial defined in [19].*

Proof. By making the change of variable $x=1-e^{-t}$ in the definition (2), we have

$$\xi(\mathbf{k};s)=\frac{1}{\Gamma(s)}\int_0^1\bigl(-\log(1-x)\bigr)^{s-1}\,\mathrm{Li}_{\mathbf{k}}(x)\,\frac{dx}{x}.$$

Put $s=m$ and use (35) to obtain

$$\xi(\mathbf{k};m)=\int_0^1\mathrm{Li}_{\underbrace{1,\ldots,1}_{m-1}}(x)\,\mathrm{Li}_{\mathbf{k}}(x)\,\frac{dx}{x}.$$

[1] Quite recently, Shuji Yamamoto communicated to the authors that he found a proof.

The regularization formula [19, Equation (5.2)], together with the shuffle product of $\mathrm{Li}_{\underbrace{1,\ldots,1}_{m-1}}(x)\,\mathrm{Li}_{\mathbf{k}}(x)$, immediately gives (38). ∎

By using (38) and (30), we can write $\eta_k(m)$ in terms of shuffle regularized values. The following expression seems to follow from that formula by taking the dual, but we have not yet worked it out in detail.

$$\eta_k(m) \overset{?}{=} \binom{m+k}{k}\zeta(m+k) - \sum_{\substack{2\leqslant r\leqslant k+1 \\ j_1+\cdots+j_r=m+k-r-1}} \binom{j_1+\cdots+j_{r-1}}{k-r+1}\zeta(j_1+1,\ldots,j_{r-1}+1,j_r+2).$$

§4. The function $\eta(k_1,\ldots,k_r;s)$ for nonpositive indices

In this section, as in the case of positive indices, we construct η-functions with nonpositive indices. It is known that $\mathrm{Li}_{-k}(z)$ can be expressed as

$$\mathrm{Li}_{-k}(z) = \frac{P(z;k)}{(1-z)^{k+1}}$$

for $k\in\mathbb{Z}_{\geqslant 0}$, where $P(x;k)\in\mathbb{Z}[x]$ is a monic polynomial satisfying

$$\deg P(x;k) = \begin{cases} 1 & (k=0) \\ k & (k\geqslant 1), \end{cases}$$

$$x \mid P(x;k)$$

(see, e.g., Shimura [29, Equations (2.17), (4.2) and (4.6)]; Note that the above $P(x;k)$ coincides with $xP_{k+1}(x)$ in [29]). We first extend this fact to multiple polylogarithms with nonpositive indices as follows.

Lemma 4.1. *For $k_1,\ldots,k_r\in\mathbb{Z}_{\geqslant 0}$, there exists a polynomial $P(x;k_1,\ldots,k_r)\in\mathbb{Z}[x]$ such that*

(39) $$\mathrm{Li}_{-k_1,\ldots,-k_r}(z) = \frac{P(z;k_1,\ldots,k_r)}{(1-z)^{k_1+\cdots+k_r+r}},$$

(40) $$\deg P(x;k_1,\ldots,k_r) = \begin{cases} r & (k_1=\cdots=k_r=0) \\ k_1+\cdots+k_r+r-1 & (\textit{otherwise}), \end{cases}$$

(41) $$x^r \mid P(x;k_1,\ldots,k_r).$$

More explicitly, $P(x;\underbrace{0,0,\ldots,0}_{r}) = x^r$.

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Proof. We prove this lemma by the double induction on $r \geqslant 1$ and $K = k_1 + \cdots + k_r \geqslant 0$. The case $r = 1$ is as mentioned above. For $r \geqslant 2$, we assume the case of $r-1$ holds and consider the case of r. When $K = k_1 + \cdots + k_r = 0$, namely $k_1 = \cdots = k_r = 0$, we have

$$\begin{aligned}\mathrm{Li}_{0,\ldots,0}(z) &= \sum_{m_1<\cdots<m_r} x^{m_r} = \sum_{m_1<\cdots<m_{r-1}} \sum_{m_r=m_{r-1}+1}^{\infty} z^{m_r} \\ &= \frac{z}{1-z} \sum_{m_1<\cdots<m_{r-1}} z^{m_{r-1}} = \cdots = \frac{z^r}{(1-z)^r},\end{aligned}$$

which implies (39)–(41) hold, and also $P(x; 0, \ldots, 0) = x^r$. Hence we assume the case $K = k_1 + \cdots + k_r - 1$ holds and consider the case $K = k_1 + \cdots + k_r (\geqslant 1)$. We consider the two cases $k_r = 0$ and $k_r \geqslant 1$ separately. First we assume $k_r = 0$. Then, by induction hypothesis, we have

$$\begin{aligned}\mathrm{Li}_{-k_1,\ldots,-k_{r-1},0}(z) &= \sum_{m_1<\cdots<m_{r-1}} m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} \sum_{m_r=m_{r-1}+1}^{\infty} z^{m_r} \\ &= \frac{z}{1-z} \sum_{m_1<\cdots<m_{r-1}} m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} z^{m_{r-1}} \\ &= \frac{z}{1-z} \frac{P(z : k_1, \ldots, k_{r-1})}{(1-z)^{k_1+\cdots+k_{r-1}+r-1}}.\end{aligned}$$

Let $P(z; k_1, \ldots, k_{r-1}, 0) = zP(z; k_1, \ldots, k_{r-1})$. Then (39)–(41) hold.

Next we assume $k_r \geqslant 1$. Then, using the same formula as in (17) and the induction hypothesis, we have

$$\begin{aligned}&\mathrm{Li}_{-k_1,\ldots,-k_{r-1},-k_r}(z) = z\frac{d}{dz}\,\mathrm{Li}_{-k_1,\ldots,-k_r+1}(z) \\ &= z\frac{d}{dz}\left(\frac{P(z : k_1, \ldots, k_r - 1)}{(1-z)^{k_1+\cdots+k_r-1+r}}\right) \\ &= \frac{z\,\{P'(z : k_1, \ldots, k_r - 1)(1-z) + (k_1 + \cdots + k_r - 1 + r)P(z; k_1, \ldots, k_r - 1)\}}{(1-z)^{k_1+\cdots+k_r+r}}.\end{aligned}$$

If $k_1 = \cdots = k_{r-1} = 0$ and $k_r = 1$, then the numerator, that is, $P(0, \ldots, 0, -1)$ equals rz^r, using the above results. If not, the degree of the numerator equals $k_1 + \cdots + k_r + r - 1$ by induction hypothesis. Both the cases satisfy (39)–(41). This completes the proof of the lemma. ∎

REMARK 4.2. In the case $r \geqslant 2$, $P(x; k_1, \ldots, k_r)$ is not necessarily a monic polynomial. For example, we have $\mathrm{Li}_{0,-1}(z) = 2z^2/(1-z)^3$, so $P(x; 0, 1) = 2x^2$.

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We obtain from (39) and (40) that

(42)
$$\mathrm{Li}_{-k_1,\ldots,-k_r}(1-e^t)=\frac{P(1-e^t;k_1,\ldots,k_r)}{e^{(k_1+\cdots+k_r+r)t}}=\begin{cases}O(1) & (k_1=\cdots=k_r=0)\\ O(e^{-t}) & (\text{otherwise})\end{cases}$$

as $t\to\infty$, and from (41) that

(43) $$\mathrm{Li}_{-k_1,\ldots,-k_r}(1-e^t)=O(t^r)\quad (t\to 0).$$

Therefore, we can define the following.

DEFINITION 4.3. For $k_1,\ldots,k_r\in\mathbb{Z}_{\geqslant 0}$, define

(44) $$\eta(-k_1,\ldots,-k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_{-k_1,\ldots,-k_r}(1-e^t)}{1-e^t}\,dt$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1-r$. In the case $r=1$, denote $\eta(-k;s)$ by $\eta_{-k}(s)$.

We see that the integral on the right-hand side of (44) is absolutely convergent for $\mathrm{Re}(s)>1-r$. Hence $\eta(-k_1,\ldots,-k_r;s)$ is holomorphic for $\mathrm{Re}(s)>1-r$. By the same method as in the proof of Theorem 2.3 for $\eta(k_1,\ldots,k_r;s)$, we can similarly obtain the following.

THEOREM 4.4. *For $k_1,\ldots,k_r\in\mathbb{Z}_{\geqslant 0}$, $\eta(-k_1,\ldots,-k_r;s)$ can be analytically continued to an entire function on the whole complex plane, and satisfies*

(45) $$\eta(-k_1,\ldots,-k_r;-m)=B_m^{(-k_1,\ldots,-k_r)}\quad (m\in\mathbb{Z}_{\geqslant 0}).$$

In particular, $\eta_{-k}(-m)=B_m^{(-k)}$ $(k\in\mathbb{Z}_{\geqslant 0},\ m\in\mathbb{Z}_{\geqslant 0})$.

It should be noted that $\xi(-k_1,\ldots,-k_r;s)$ cannot be defined by replacing $\{k_j\}$ by $\{-k_j\}$ in (2). In fact, even if $r=1$ and $k=0$ in (2), we see that

$$\xi_0(s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_0(1-e^{-t})}{e^t-1}\,dt=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,dt,$$

which is not convergent for any $s\in\mathbb{C}$. Therefore, we modify the definition (2) as follows.

DEFINITION 4.5. For $k_1,\ldots,k_r\in\mathbb{Z}_{\geqslant 0}$ with $(k_1,\ldots,k_r)\neq(0,\ldots,0)$, define

(46) $$\widetilde{\xi}(-k_1,\ldots,-k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{Li}_{-k_1,\ldots,-k_r}(1-e^t)}{e^{-t}-1}\,dt$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1-r$. In the case $r=1$, denote $\widetilde{\xi}(-k;s)$ by $\widetilde{\xi}_{-k}(s)$ for $k\geqslant 1$.

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We see from (42) and (43) that (46) is well-defined. Also it is noted that $\widetilde{\xi}(k_1,\ldots,k_r;s)$ cannot be defined by replacing $\{-k_j\}$ by $\{k_j\}$ in (46) for $(k_j)\in\mathbb{Z}_{\geqslant 1}^r$.

In a way parallel to deriving Theorem 4.4, we can obtain the following.

THEOREM 4.6. *For $k_1,\ldots,k_r\in\mathbb{Z}_{\geqslant 0}$ with $(k_1,\ldots,k_r)\neq(0,\ldots,0)$, $\widetilde{\xi}(-k_1,\ldots,-k_r;s)$ can be analytically continued to an entire function on the whole complex plane, and satisfies*

$$\widetilde{\xi}(-k_1,\ldots,-k_r;-m)=C_m^{(-k_1,\ldots,-k_r)}\quad(m\in\mathbb{Z}_{\geqslant 0}).\tag{47}$$

In particular, $\widetilde{\xi}_{-k}(-m)=C_m^{(-k)}$ $(k\in\mathbb{Z}_{\geqslant 1},\ m\in\mathbb{Z}_{\geqslant 0})$.

Next we give certain duality formulas for $B_n^{(k_1,\ldots,k_r)}$ which is a generalization of (6). To state this, we define another type of multi-poly-Bernoulli numbers by

$$\begin{aligned}&\sum_{a=0}^{r-1}(-1)^a\binom{r-1}{a}\sum_{l_1,\ldots,l_r\geqslant 1}\frac{\prod_{j=1}^r\left(1-e^{-\sum_{\nu=j}^r x_\nu}\right)^{l_j-1}}{(l_1+\cdots+l_r-a)^s}\\ &\qquad=\sum_{m_1,\ldots,m_r\geqslant 0}\mathfrak{B}_{m_1,\ldots,m_r}^{(s)}\frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!}\end{aligned}\tag{48}$$

for $s\in\mathbb{C}$. In the case $r=1$, we see that $\mathfrak{B}_m^{(k)}=B_m^{(k)}$ for $k\in\mathbb{Z}$. Then we obtain the following result which is a kind of the duality formula. In fact, this coincides with (6) in the case $r=1$.

THEOREM 4.7. *For $k_1,\ldots,k_r\in\mathbb{Z}_{\geqslant 0}$,*

$$\eta(-k_1,\ldots,-k_r;s)=\mathfrak{B}_{k_1,\ldots,k_r}^{(s)}.\tag{49}$$

Therefore, for $m\in\mathbb{Z}_{\geqslant 0}$,

$$B_m^{(-k_1,\ldots,-k_r)}=\mathfrak{B}_{k_1,\ldots,k_r}^{(-m)}.\tag{50}$$

Proof. We first prepare the following relation which will be proved in the next section (see Lemma 5.9):

$$\prod_{j=1}^r\frac{e^{\sum_{\nu=j}^r x_\nu}(1-e^t)}{1-e^{\sum_{\nu=j}^r x_\nu}(1-e^t)}=\sum_{k_1,\ldots,k_r\geqslant 0}\mathrm{Li}_{-k_1,\ldots,-k_r}(1-e^t)\frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!}\tag{51}$$

holds around the origin. Let

$$\mathcal{F}(x_1,\ldots,x_r;s)=\sum_{k_1,\ldots,k_r\geqslant 0}\eta(-k_1,\ldots,-k_r;s)\frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!}.$$

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As a generalization of [18, Proposition 5], we have from (51) that

$$
\begin{aligned}
&\mathcal{F}(x_1,\dots,x_r;s)\\
&=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{1-e^t}\prod_{j=1}^{r}\frac{e^{\sum_{\nu=j}^{r}x_\nu}(1-e^t)}{1-e^{\sum_{\nu=j}^{r}x_\nu}(1-e^t)}\,dt\\
&=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(1-e^t)^{r-1}e^{-rt}\prod_{j=1}^{r}\frac{1}{1-e^{-t}\bigl(1-e^{-\sum_{\nu=j}^{r}x_\nu}\bigr)}\,dt\\
&=\frac{1}{\Gamma(s)}\sum_{a=0}^{r-1}(-1)^a\binom{r-1}{a}\sum_{m_1,\dots,m_r\geqslant 0}\prod_{j=1}^{r}\Bigl(1-e^{-\sum_{\nu=j}^{r}x_\nu}\Bigr)^{m_j}\\
&\quad\times\int_0^\infty t^{s-1}e^{(a-r)t}\prod_{j=1}^{r}e^{-m_jt}\,dt\\
&=\sum_{a=0}^{r-1}(-1)^a\binom{r-1}{a}\sum_{m_1,\dots,m_r\geqslant 0}\frac{\prod_{j=1}^{r}\bigl(1-e^{-\sum_{\nu=j}^{r}x_\nu}\bigr)^{m_j}}{(m_1+\cdots+m_r+r-a)^s}.
\end{aligned}
$$

Therefore, by (48), we obtain (49). Further, setting $s=-m$ in (49) and using (45), we obtain (50). □

Remark 4.8. In the case $r=1$, (49) implies $\eta_{-k}(s)=B_k^{(s)}$. Thus, using Theorem 4.4, we obtain the duality formula (6), which is also written as

$$
\eta_{-k}(-m)=\eta_{-m}(-k) \tag{52}
$$

for $k,m\in\mathbb{Z}_{\geqslant 0}$. This is exactly contrasted with the positive index case (36). Furthermore, by the same method, we can show that $\widetilde{\xi}_{-k-1}(-m)=C_k^{(-m-1)}$ for $k,m\in\mathbb{Z}_{\geqslant 0}$. Hence, using Theorem 4.6 in the case $r=1$, we obtain the duality formula (7).

Example 4.9. When $r=2$, we can calculate directly from (48) that $\mathfrak{B}_{1,0}^{(s)}=3^{-s}-2^{-s}$. On the other hand, as mentioned in Lemma 4.1, we have $\mathrm{Li}_{-1,0}(z)=z^2/(1-z)^3$. Hence the left-hand side of (10) equals

$$
\frac{\mathrm{Li}_{-1,0}(1-e^{-t})}{1-e^{-t}}=\frac{1-e^{-t}}{e^{-3t}}=e^{3t}-e^{2t},
$$

hence $B_m^{(-1,0)}=3^m-2^m$. Thus we can verify $B_m^{(-1,0)}=\mathfrak{B}_{1,0}^{(-m)}$.

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§5. Multi-indexed poly-Bernoulli numbers and duality formulas

In this section, we define multi-indexed poly-Bernoulli numbers (see Definition 5.1) and prove the duality formula for them, namely a multi-indexed version of (6) (see Theorem 5.4).

For this aim, we first recall multiple polylogarithms of $*$-type and of ш-type in several variables defined by

$$\mathrm{Li}^{*}_{s_1,\ldots,s_r}(z_1,\ldots,z_r) = \sum_{1\leqslant m_1<\cdots<m_r} \frac{z_1^{m_1}\cdots z_r^{m_r}}{m_1^{s_1}m_2^{s_2}\cdots m_r^{s_r}},$$

$$\mathrm{Li}^{ш}_{s_1,\ldots,s_r}(z_1,\ldots,z_r) = \sum_{1\leqslant m_1<\cdots<m_r} \frac{z_1^{m_1}z_2^{m_2-m_1}\cdots z_r^{m_r-m_{r-1}}}{m_1^{s_1}m_2^{s_2}\cdots m_r^{s_r}} \tag{53}$$

$$= \sum_{l_1,\ldots,l_r=1}^{\infty} \frac{z_1^{l_1}z_2^{l_2}\cdots z_r^{l_r}}{l_1^{s_1}(l_1+l_2)^{s_2}\cdots(l_1+\cdots+l_r)^{s_r}} \tag{54}$$

for $s_1,\ldots,s_r\in\mathbb{C}$ and $z_1,\ldots,z_r\in\mathbb{C}$ with $|z_j|\leqslant 1$ $(1\leqslant j\leqslant r)$ (see, e.g., [13]). The symbols $*$ and ш are derived from the harmonic product and the shuffle product in the theory of multiple zeta values. In fact, Arakawa and Kaneko defined the two types of multiple L-values $L^*(k_1,\ldots,k_r;f_1,\ldots,f_r)$ of $*$-type and $L^{ш}(k_1,\ldots,k_r;f_1,\ldots,f_r)$ of ш-type associated to periodic functions $\{f_j\}$ (see [5]), defined by replacing $\{z_j^m\}$ by $\{f_j(m)\}$ and setting $(s_j)=(k_j)\in\mathbb{Z}^r_{\geqslant 1}$ on the right-hand sides of (54) and (53) for $(k_1,\ldots,k_r)\in\mathbb{Z}^r_{\geqslant 1}$. Note that

$$\mathrm{Li}^{*}_{s_1,\ldots,s_r}(z_1,\ldots,z_r) = \mathrm{Li}^{ш}_{s_1,\ldots,s_r}\left(\prod_{j=1}^{r} z_j, \prod_{j=2}^{r} z_j,\ldots,z_{r-1}z_r, z_r\right). \tag{55}$$

Definition 5.1. (Multi-indexed poly-Bernoulli numbers) For $s_1,\ldots,s_r\in\mathbb{C}$ and $d\in\{1,2,\ldots,r\}$, the multi-indexed poly-Bernoulli numbers $\{B^{(s_1,s_2,\ldots,s_r),(d)}_{m_1,\ldots,m_r}\}$ are defined by

$$\begin{aligned} &F(x_1,\ldots,x_r;s_1,\ldots,s_r;d)\\ &\quad=\frac{\mathrm{Li}^{ш}_{s_1,\ldots,s_r}\left(1-e^{-\sum_{\nu=1}^r x_\nu},\ldots,1-e^{-x_{r-1}-x_r},1-e^{-x_r}\right)}{\prod_{j=1}^{d}\left(1-e^{-\sum_{\nu=j}^r x_\nu}\right)}\\ &\quad\times\left(=\sum_{l_1,\ldots,l_r=1}^{\infty}\frac{\prod_{j=1}^{r}\left(1-e^{-\sum_{\nu=j}^r x_\nu}\right)^{l_j-\delta_j(d)}}{\prod_{j=1}^{r}\left(\sum_{\nu=1}^{j} l_\nu\right)^{s_j}}\right)\end{aligned}$$

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$$(56) \qquad = \sum_{m_1,\ldots,m_r=0}^{\infty} B_{m_1,\ldots,m_r}^{(s_1,\ldots,s_r),(d)} \frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!},$$

where $\delta_j(d)=1$ $(j \leqslant d)$, $=0$ $(j>d)$.

REMARK 5.2. Note that $\mathrm{Li}^{ш}_{k_1,\ldots,k_r}(z,\ldots,z)=\mathrm{Li}_{k_1,\ldots,k_r}(z)$ defined by (9). Suppose $x_1=\cdots=x_{r-1}=0$ and $(s_j)=(k_j)\in\mathbb{Z}^r$ in (56). We immediately see that if $d=1$ then

$$B_{0,\ldots,0,m}^{(k_1,\ldots,k_r),(1)} = B_m^{(k_1,\ldots,k_r)} \quad (m\in\mathbb{Z}_{\geqslant 0})$$

(see (6)), and if $d=r$ then

$$B_{0,\ldots,0,m}^{(k_1,\ldots,k_r),(r)} = \mathbb{B}_m^{(k_1,\ldots,k_r)} \quad (m\in\mathbb{Z}_{\geqslant 0})$$

(see (8)).

REMARK 5.3. Let

$$(57) \qquad \Lambda_r = \{(x_1,\ldots,x_r)\in\mathbb{C}^r \mid |1-e^{-\sum_{\nu=j}^r x_\nu}|<1 \ (1\leqslant j\leqslant r)\}.$$

Then we can see that

$$\mathrm{Li}^{ш}_{s_1,\ldots,s_r}\left(1-e^{-\sum_{\nu=1}^r x_\nu},\ldots,1-e^{-x_{r-1}-x_r},1-e^{-x_r}\right) \quad (s_1,\ldots,s_r\in\mathbb{C})$$

is absolutely convergent for $(x_j)\in\Lambda_r$. Also $F(x_1,\ldots,x_r;s_1,\ldots,s_r;d)$ is absolutely convergent in the region $\Lambda_r\times\mathbb{C}^r$, so is holomorphic. Hence $B_{m_1,\ldots,m_r}^{(s_1,\ldots,s_r),(d)}$ is an entire function, because

$$B_{m_1,\ldots,m_r}^{(s_1,\ldots,s_r),(d)} = \left(\frac{\partial}{\partial x_1}\right)^{m_1}\cdots\left(\frac{\partial}{\partial x_r}\right)^{m_r} F(x_1,\ldots,x_r;s_1,\ldots,s_r;d)\Big|_{(x_1,\ldots,x_r)=(0,\ldots,0)}$$

is holomorphic for all $(s_1,\ldots,s_r)\in\mathbb{C}^r$.

In the preceding section, we gave a certain duality formula for $B_m^{(k_1,\ldots,k_r)}$ (see Theorem 4.7). By the similar method, we can prove certain duality formulas for $B_{m_1,\ldots,m_r}^{(k_1,\ldots,k_r),(d)}$, though they may be complicated. Hence, in the rest of this section, we will consider the case $d=r$. For emphasis, we denote $B_{m_1,\ldots,m_r}^{(s_1,\ldots,s_r),(r)}$ by $\mathbb{B}_{m_1,\ldots,m_r}^{(s_1,\ldots,s_r)}$. Note that $\delta_j(r)=1$ for any j. With this notation, we prove the following duality formulas.

THEOREM 5.4. *For $m_1,\ldots,m_r,\ k_1,\ldots,k_r\in\mathbb{Z}_{\geqslant 0}$,*

$$(58) \qquad \mathbb{B}_{m_1,\ldots,m_r}^{(-k_1,\ldots,-k_r)} = \mathbb{B}_{k_1,\ldots,k_r}^{(-m_1,\ldots,-m_r)}.$$

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Now we aim to prove this theorem. First we generalize Lemma 4.1 as follows.

LEMMA 5.5. *For $k_1,\ldots,k_r \in \mathbb{Z}_{\geqslant 0}$, there exists a polynomial $\widetilde{P}(x_1,\ldots,x_r;k_1,\ldots,k_r)\in\mathbb{Z}[x_1,\ldots,x_r]$ such that*

$$\mathrm{Li}^{*}_{-k_1,\ldots,-k_r}(z_1,\ldots,z_r)=\frac{\widetilde{P}(\prod_{j=1}^{r}z_j,\prod_{j=2}^{r}z_j,\ldots,z_{r-1}z_r,z_r;k_1,\ldots,k_r)}{\prod_{j=1}^{r}(1-\prod_{\nu=j}^{r}z_\nu)^{\sum_{\nu=j}^{r}k_\nu+1}}, \tag{59}$$

$$\deg_{x_j}\widetilde{P}(x_1,\ldots,x_r;k_1,\ldots,k_r)\leqslant\sum_{\nu=j}^{r}k_\nu+1, \tag{60}$$

$$(x_1\cdots x_r)\mid\widetilde{P}(x_1,\ldots,x_r;k_1,\ldots,k_r). \tag{61}$$

Set $y_j=\prod_{\nu=j}^{r}z_\nu$ $(1\leqslant j\leqslant r)$. Then (59) implies

$$\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(y_1,\ldots,y_r)=\frac{\widetilde{P}(y_1,\ldots,y_r;k_1,\ldots,k_r)}{\prod_{j=1}^{r}(1-y_j)^{\sum_{\nu=j}^{r}k_\nu+1}}. \tag{62}$$

Proof. In order to prove this lemma, we have only to use the same method as in Lemma 4.1 by induction on r. Since the case of $r=1$ is proven, we consider the case of $r\geqslant 2$. Further, when $K=k_1+\cdots+k_r=0$, it is easy to have the assertion. Hence we think about a general case $K=k_1+\cdots+k_r(\geqslant 1)$. When $k_r=0$, we have

$$\begin{aligned}\mathrm{Li}^{*}_{-k_1,\ldots,-k_r}(z_1,\ldots,z_r)&=\frac{z_r}{1-z_r}\,\mathrm{Li}^{*}_{-k_1,\ldots,-k_{r-1}}(z_1,\ldots,z_{r-2},z_{r-1}z_r)\\&=\frac{z_r}{1-z_r}\frac{\widetilde{P}(\prod_{j=1}^{r}z_j,\ldots,z_{r-1}z_r;k_1,\ldots,k_{r-1})}{\prod_{j=1}^{r-1}(1-\prod_{\nu=j}^{r}z_j)^{\sum_{\nu=j}^{r}k_\nu+1}}.\end{aligned}$$

Therefore, setting $\widetilde{P}(x_1,\ldots,x_r;k_1,\ldots,k_{r-1},0)=x_r\widetilde{P}(x_1,\ldots,x_{r-1};k_1,\ldots,k_{r-1})$, we can verify (59)–(61).

Next we consider the case $k_r\geqslant 1$. For $k\in\mathbb{Z}_{\geqslant 0}$, we inductively define a subset $\{c^{(k)}_{j,\nu}\}_{0\leqslant j,\nu\leqslant k+1}$ of $\mathbb{Z}$ by

$$\frac{d}{dz}\left(\sum_{m>l}m^k z^m\right)=\frac{1}{(1-z)^{k+2}}\sum_{j=0}^{k+1}\sum_{\nu=0}^{k+1}c^{(k)}_{j,\nu}l^\nu z^{l+j}. \tag{63}$$

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In fact, by

$$\frac{d}{dz}\left(\sum_{m>l} z^m\right) = \frac{1}{(1-z)^2}(z^l + lz^l - lz^{l+1}),$$

and

$$\sum_{m>l} m^k z^m = z\frac{d}{dz}\left(\sum_{m>l} m^{k-1} z^m\right) \quad (k \geqslant 1),$$

we can determine $\{c_{j,\nu}^{(k)}\}$ by (63). Using this notation, we have

$$\begin{aligned}
\mathrm{Li}^*_{-k_1,\ldots,-k_r}(z_1,\ldots,z_r) &= z_r \frac{d}{dz_r}\mathrm{Li}^*_{-k_1,\ldots,-k_r+1}(z_1,\ldots,z_r) \\
&= z_r \sum_{m_1<\cdots<m_{r-1}} m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} z_1^{m_1}\cdots z_{r-1}^{m_{r-1}} \\
&\quad \times \frac{\sum_{j=0}^{k_r}\sum_{\nu=0}^{k_r} c_{j,\nu}^{(k_r-1)} m_{r-1}^{\nu} z_r^{m_{r-1}+j}}{(1-z_r)^{k_r+1}} \\
&= \frac{1}{(1-z_r)^{k_r+1}} \sum_{j=0}^{k_r}\sum_{\nu=0}^{k_r} c_{j,\nu}^{(k_r-1)} z_r^{j+1} \\
&\quad \times \sum_{m_1<\cdots<m_{r-1}} m_1^{k_1}\cdots m_{r-1}^{k_{r-1}+\nu} z_1^{m_1}\cdots (z_{r-1}z_r)^{m_{r-1}}.
\end{aligned}$$

By the induction hypothesis in the case $r-1$, this is equal to

$$\begin{aligned}
&\frac{1}{(1-z_r)^{k_r+1}} \sum_{j=0}^{k_r}\sum_{\nu=0}^{k_r} c_{j,\nu}^{(k_r-1)} z_r^{j+1} \\
&\times \frac{\widetilde{P}(\prod_{j=1}^{r-1} z_j,\ldots,z_{r-1}z_r;k_1,\ldots,k_{r-2},k_{r-1}+\nu)}{\prod_{j=1}^{r-2}\left(1-\prod_{\nu=j}^{r} z_j\right)^{\sum_{\nu=j}^{r} k_\nu+1}(1-z_{r-1}z_r)^{k_{r-1}+\nu+1}}.
\end{aligned}$$

Therefore, we set

$$\begin{aligned}
\widetilde{P}(x_1,\ldots,x_r;k_1,\ldots,k_r) = \sum_{j=0}^{k_r}\sum_{\nu=0}^{k_r} c_{j,\nu}^{(k_r-1)} x_r^{j+1}(1-x_{r-1})^{k_r-\nu} \\
\times \widetilde{P}(x_1,\ldots,x_{r-1};k_1,\ldots,k_{r-2},k_{r-1}+\nu).
\end{aligned}$$

Then this satisfies (59)–(61). This completes the proof. ∎

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From this result, we can reach the following definition.

DEFINITION 5.6. For $k_1,\ldots,k_r\in\mathbb{Z}_{\geqslant 0}$, define

$$\begin{aligned}
&\eta(-k_1,\ldots,-k_r;s_1,\ldots,s_r)\\
&\quad=\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^r t_j^{s_j-1}\\
&\qquad\times\frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1-e^{\sum_{\nu=1}^r t_\nu},\ldots,1-e^{t_{r-1}+t_r},1-e^{t_r})}{\prod_{j=1}^r(1-e^{\sum_{\nu=j}^r t_\nu})}\prod_{j=1}^r dt_j
\end{aligned}\tag{64}$$

for $s_1,\ldots,s_r\in\mathbb{C}$ with $\mathrm{Re}(s_j)>0$ $(1\leqslant j\leqslant r)$.

Lemma 5.5 ensures that the integral on the right-hand side of (64) is absolutely convergent for $\mathrm{Re}(s_j)>0$. By the same method as in the proof of Theorem 2.3 for $\eta(k_1,\ldots,k_r;s)$, we can similarly obtain the following.

THEOREM 5.7. *For $k_1,\ldots,k_r\in\mathbb{Z}_{\geqslant 0}$, $\eta(-k_1,\ldots,-k_r;s_1,\ldots,s_r)$ can be analytically continued to an entire function on the whole complex space, and satisfies*

$$\eta(-k_1,\ldots,-k_r;-m_1,\ldots,-m_r)=\mathbb{B}^{(-k_1,\ldots,-k_r)}_{m_1,\ldots,m_r}\quad(m_1,\ldots,m_r\in\mathbb{Z}_{\geqslant 0}).\tag{65}$$

Proof. As in the proof of Theorem 2.3, let

$$\begin{aligned}
&H(-k_1,\ldots,-k_r;s_1,\ldots,s_r)\\
&\quad=\int_{\mathcal{C}^r}\prod_{j=1}^r t_j^{s_j-1}\frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1-e^{\sum_{\nu=1}^r t_\nu},\ldots,1-e^{t_r})}{\prod_{j=1}^r(1-e^{\sum_{\nu=j}^r t_\nu})}\prod_{j=1}^r dt_j\\
&\quad=\prod_{j=1}^r(e^{2\pi i s_j}-1)\int_\varepsilon^\infty\cdots\int_\varepsilon^\infty\prod_{j=1}^r t_j^{s_j-1}\\
&\qquad\times\frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1-e^{\sum_{\nu=1}^r t_\nu},\ldots,1-e^{t_r})}{\prod_{j=1}^r(1-e^{\sum_{\nu=j}^r t_\nu})}\prod_{j=1}^r dt_j\\
&\qquad\cdots+\int_{C_\varepsilon^r}\prod_{j=1}^r t_j^{s_j-1}\frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1-e^{\sum_{\nu=1}^r t_\nu},\ldots,1-e^{t_r})}{\prod_{j=1}^r(1-e^{\sum_{\nu=j}^r t_\nu})}\prod_{j=1}^r dt_j,
\end{aligned}\tag{66}$$

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where $\mathcal{C}^r$ is the direct product of the contour $\mathcal{C}$ defined before. Note that the integrand on the second member has no singularity on $\mathcal{C}^r$. It follows from Lemma 5.5 that $H(-k_1,\ldots,-k_r;s_1,\ldots,s_r)$ is absolutely convergent for any $(s_j)\in\mathbb{C}^r$, namely is entire. Suppose $\mathrm{Re}(s_j)>0$ for each j, all terms except for the first term on the third member of (66) tend to 0 as $\varepsilon\to 0$. Hence

$$\eta(-k_1,\ldots,-k_r;s_1,\ldots,s_r)$$
$$=\frac{1}{\prod_{j=1}^r(e^{2\pi i s_j}-1)\Gamma(s_j)}H(-k_1,\ldots,-k_r;s_1,\ldots,s_r),$$

which can be analytically continued to $\mathbb{C}^r$. Also, setting $(s_1,\ldots,s_r)=(-m_1,\ldots,-m_r)\in\mathbb{Z}^r_{\leqslant 0}$ in (66), we obtain (65) from (56). This completes the proof. □

Next we directly construct the generating function of $\eta(-k_1,\ldots,-k_r;s_1,\ldots,s_r)$. We prepare the following two lemmas which we consider when (x_j) is in Λ_r defined by (57).

LEMMA 5.8. *For* $(s_j)\in\mathbb{C}^r$ *with* $\mathrm{Re}(s_j)>0$ $(1\leqslant j\leqslant r)$,

$$F(x_1,\ldots,x_r;s_1,\ldots,s_r;r)$$
$$=\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^r\left\{t_j^{s_j-1}\frac{e^{\sum_{\nu=j}^r x_\nu}}{1-e^{\sum_{\nu=j}^r x_\nu}(1-e^{\sum_{\nu=j}^r t_\nu})}\right\}\prod_{j=1}^r dt_j. \tag{67}$$

Proof. Substituting $n^{-s}=(1/\Gamma(s))\int_0^\infty t^{s-1}e^{-nt}\,dt$ into the second member of (56), we have

$$F(\{x_j\};\{s_j\};r)=\sum_{l_1,\ldots,l_r=1}^\infty\prod_{j=1}^r\left(1-e^{-\sum_{\nu=j}^r x_\nu}\right)^{l_j-1}\frac{1}{\prod_{j=1}^r\Gamma(s_j)}$$
$$\times\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^r\left\{t_j^{s_j-1}\exp\left(-\left(\sum_{\nu=1}^j l_\nu\right)t_j\right)\right\}\prod_{j=1}^r dt_j.$$

We see that the integrand on the right-hand side can be rewritten as

$$\prod_{j=1}^r t_j^{s_j-1}\prod_{j=1}^r\exp\left(-l_j\left(\sum_{\nu=j}^r t_\nu\right)\right).$$

Hence we have

$$\begin{aligned}
&F(\{x_j\};\{s_j\};r)\\
&=\frac{1}{\prod_{j=1}^r\Gamma(s_j)(1-e^{-\sum_{\nu=j}^r x_\nu})}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^r t_j^{s_j-1}\\
&\quad\times\sum_{l_1,\ldots,l_r=1}^\infty\prod_{j=1}^r(1-e^{-\sum_{\nu=j}^r x_\nu})^{l_j}e^{-l_j(\sum_{\nu=j}^r t_\nu)}\prod_{j=1}^r dt_j\\
&=\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^r t_j^{s_j-1}\frac{e^{-\sum_{\nu=j}^r t_\nu}}{1-(1-e^{-\sum_{\nu=j}^r x_\nu})e^{-\sum_{\nu=j}^r t_\nu}}\prod_{j=1}^r dt_j\\
&=\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^r t_j^{s_j-1}\frac{e^{\sum_{\nu=j}^r x_\nu}}{1-e^{\sum_{\nu=j}^r x_\nu}(1-e^{\sum_{\nu=j}^r t_\nu})}\prod_{j=1}^r dt_j.
\end{aligned}$$

This completes the proof. ∎

LEMMA 5.9. *Let $z_1,\ldots,z_r\in\mathbb{C}$ and assume that $|z_j|$ $(1\leqslant j\leqslant r)$ are sufficiently small. Then*

$$\prod_{j=1}^r\frac{z_je^{\sum_{\nu=j}^r x_\nu}}{1-z_je^{\sum_{\nu=j}^r x_\nu}}=\sum_{k_1,\ldots,k_r=0}^\infty \mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(z_1,\ldots,z_r)\frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!}. \tag{68}$$

Set $z_j=1-e^{\sum_{\nu=j}^r t_\nu}$ $(1\leqslant j\leqslant r)$ for $(t_j)\in\Lambda_r$. Then

$$\begin{aligned}
&\prod_{j=1}^r\frac{e^{\sum_{\nu=j}^r x_\nu}(1-e^{\sum_{\nu=j}^r t_\nu})}{1-e^{\sum_{\nu=j}^r x_\nu}(1-e^{\sum_{\nu=j}^r t_\nu})}\\
&\quad=\sum_{k_1,\ldots,k_r=0}^\infty \mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}\left(1-e^{\sum_{\nu=1}^r t_\nu},\ldots,1-e^{t_r}\right)\frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!}.
\end{aligned}\tag{69}$$

In particular, the case $t_1=\cdots=t_{r-1}=0$ and $t_r=t$ implies (51).

Proof. We have only to prove (68). Actually we have

$$\begin{aligned}
&\sum_{k_1,\ldots,k_r=0}^\infty \mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(z_1,\ldots,z_r)\frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!}\\
&\quad=\sum_{k_1,\ldots,k_r=0}^\infty\sum_{m_1,\ldots,m_r=1}^\infty\prod_{j=1}^r\frac{\left((\sum_{\mu=1}^j m_\mu)x_j\right)^{k_j}}{k_j!}z^{m_j}
\end{aligned}$$

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$$= \sum_{m_1,\ldots,m_r=1}^{\infty} \prod_{j=1}^{r} z_j^{m_j} \prod_{\mu=1}^{j} e^{m_\mu x_j}$$
$$= \sum_{m_1,\ldots,m_r=1}^{\infty} \prod_{j=1}^{r} \left(z_j e^{\sum_{\nu=j}^{r} x_\nu}\right)^{m_j} = \prod_{j=1}^{r} \frac{z_j e^{\sum_{\nu=j}^{r} x_\nu}}{1 - z_j e^{\sum_{\nu=j}^{r} x_\nu}}.$$

Thus we have the assertion. ∎

Using these lemmas, we obtain the following.

THEOREM 5.10. *For* $k_1, \ldots, k_r \in \mathbb{Z}_{\geqslant 0}$,

$$\eta(-k_1, \ldots, -k_r; s_1, \ldots, s_r) = \mathbb{B}_{k_1,\ldots,k_r}^{(s_1,\ldots,s_r)}. \tag{70}$$

Proof. By Lemmas 5.8 and 5.9, we have

$$F(x_1, \ldots, x_r; s_1, \ldots, s_r; r)$$
$$= \frac{1}{\prod_{j=1}^{r} \Gamma(s_j)} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{r} \left\{ t_j^{s_j - 1} \frac{e^{\sum_{\nu=j}^{r} x_\nu}}{1 - e^{\sum_{\nu=j}^{r} x_\nu}(1 - e^{\sum_{\nu=j}^{r} t_\nu})} \right\} \prod_{j=1}^{r} dt_j$$
$$= \frac{1}{\prod_{j=1}^{r} \Gamma(s_j)} \sum_{k_1,\ldots,k_r=0}^{\infty} \left\{ \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{r} t_j^{s_j-1} \right.$$
$$\left. \times \frac{\mathrm{Li}^{ш}_{-k_1,\ldots,-k_r}(1 - e^{\sum_{\nu=1}^{r} t_\nu}, \ldots, 1 - e^{t_r})}{\prod_{j=1}^{r}(1 - e^{\sum_{\nu=j}^{r} t_\nu})} \prod_{j=1}^{r} dt_j \right\} \frac{x_1^{k_1} \cdots x_r^{k_r}}{k_1! \cdots k_r!} \tag{71}$$

for $\mathrm{Re}(s_j) > 0$ $(1 \leqslant j \leqslant r)$. Combining (56), (64) and (71), we obtain (70) for $\mathrm{Re}(s_j) > 0$ $(1 \leqslant j \leqslant r)$, hence for all $(s_j) \in \mathbb{C}$, because both sides of (70) are entire functions (see Remark 5.3). ∎

Proof of Theorem 5.4. Setting $(s_1, \ldots, s_r) = (-m_1, \ldots, -m_r)$ in (70), we obtain (58) from (65). This completes the proof of Theorem 5.4.

EXAMPLE 5.11. We can easily see that

$$\mathrm{Li}^{ш}_{-1,0}(z_1, z_2) = \frac{z_1 z_2}{(1 - z_1)^2 (1 - z_2)}, \qquad \mathrm{Li}^{ш}_{0,-1}(z_1, z_2) = \frac{z_1 z_2 (2 - z_1 - z_2)}{(1 - z_1)^2 (1 - z_2)^2}.$$

Hence we have

$$\mathbb{B}_{m,n}^{(-1,0)} = 2^m 3^n, \qquad \mathbb{B}_{m,n}^{(0,-1)} = (2^m + 1) 3^n \quad (m, n \in \mathbb{Z}_{\geqslant 0}).$$

Therefore, $\mathbb{B}_{0,1}^{(-1,0)} = \mathbb{B}_{1,0}^{(0,-1)} = 3$. Similarly we obtain, for example,

$$\mathbb{B}_{1,0}^{(-1,-2)} = \mathbb{B}_{1,2}^{(-1,0)} = 18, \ \mathbb{B}_{1,2}^{(-3,-1)} = \mathbb{B}_{3,1}^{(-1,-2)} = 1820,$$
$$\mathbb{B}_{2,2}^{(-2,-1)} = \mathbb{B}_{2,1}^{(-2,-2)} = 1958.$$

REMARK 5.12. Hamahata and Masubuchi [14, Corollary 10] showed the special case of (58), namely

$$\mathbb{B}_{0,\dots,0,m}^{(0,\dots,0,-k)} = \mathbb{B}_{0,\dots,0,k}^{(0,\dots,0,-m)} \quad (m, k \in \mathbb{Z}_{\geqslant 0})$$

(see Remark 5.2). On the other hand, Theorem 4.7 corresponds to the case $d = 1 \neq r$ except for $r = 1$ (see Remark 5.2), hence is located in the outside of Theorem 5.4. Therefore, in (50), another type of multi-poly-Bernoulli numbers appear.

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