Part II is entitled Modules. The opening gambit moves from vector spaces ('an ideal of a ring *R* [is] some kind of "vector space over *R*"') to $\mathbb{Z}$-modules: 'A more common name for a $\mathbb{Z}$-module is an abelian group.' Modules are shown to have the same canonical decomposition as rings. There is a chapter on modules over integral domains, and then come abelian groups proper. By this time of course one doesn't need much new theory; most of the key results have already been proved in the ring/module context. Part III, 'Groups', moves quickly through the main families of groups, quotient groups and isomorphism theorems, before going on to subgroups and Lagrange's Theorem, after which Sylow's theorems are reached almost at once. By now it is clear where the book is heading, and the last three chapters (Part IV, 'Fields') duly culminate in the fundamental theorem of Galois theory.

For its chosen route, the book does a largely impressive job, though the author cannot resist putting in a good many nuggets of more advanced mathematics. There are roughly twenty or thirty exercises at the end of each chapter (not each section of each chapter), and about a third of these have solutions given in an appendix.

So why teach rings first? Certainly the sequence presented here provides an intellectually coherent body of fully-developed mathematics, presented as a highly polished monolith, but I must admit to considerable reservations as to whether it represents best pedagogical practice. My own experience of teaching and learning suggests that many students like to begin with the tangibility of geometrical examples of groups, uncomplicated by more than one operation–I have never found any learner to be fazed by this. They also find it easier to appreciate the concept of isomorphism when there are several familiar groups with identical structures, a more common scenario than in rings. Nor do I see why Aluffi's criterion for greater abstraction is merely 'fewer axioms'. To me the whole approach seems teacher-centred rather than learner-centred, and I was reminded of how in my undergraduate days the intellectual purity of a course seemed to count for more than the interests of students (with the tacit implication that 'if you don't like it, you don't belong here'). I was also reminded of being told by a Cambridge professor, himself a superb teacher, that he thought first-year undergraduate abstract algebra should not be taught by specialist algebraists. No doubt students of ability high enough to cope with an initial flood of definitions (rings, PIDs, UFDs, fields, modules, etc) and a purely algebraic style offering little visualisation will succeed, but then such students would succeed with other approaches too. However, I am very ready to accept that my opinion is based on limited evidence, and no doubt there will be readers of this review who will disagree with me. If you have good reason to believe that it is in the best interests of your students to learn abstract algebra like this, then good luck–this book does its self-appointed job well. Meanwhile it also provides a clearer introduction to rings than one generally finds in 'groups-first' texts, and it would be valuable merely for that.

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Essential mathematics for undergraduates by Simon G. Chiossi, pp 490, £54.99 (hard), ISBN 978-3-030-87173-4, Springer Verlag (2021).

Students of mathematics know a fair amount of the subject, but then in every subject worthy of serious study there is always much more to be learned. Many topics, however desirable for inclusion in the curriculum, will be omitted from any particular course. The book being reviewed is a collection of what the author considers to be essential material for

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undergraduates, and he writes in the Preface that the structure of the text "is made of morsels of theory—some bite-size, while other mouthfuls—selected from a broad spectrum". Much of the material involves the language of mathematics, and the abstract nature of the subject, but there are also concrete materials firming up the foundation for a graduate mathematician. The many definitions are followed by examples and exercises, while proofs of theorems and propositions are given only when they are short and succinct.

The nineteen chapters are divided into four parts. Part I: Basic objects and formalisation; Part II: Numbers and structures; Part III: Elementary real functions; Part IV: Geometry through algebra. There is a list of what the author considers to be 'hot topics': soundness and completeness of first-order logic, the Boolean nature of logic, the Galois correspondence, equivalence of the axiom of choice, primitive recursion, cardinal numbers, real numbers as Dedekind cuts, compactifications, stereographic projections, projective spaces, action of matrix groups, metric completions, kinematics and electromagnetism, Birkhoff's axiomatisation of plane geometry, and the Cantor set. There are also historical comments and remarks on undecidable theorems or theories, non-standard models of arithmetic, non-solubility of polynomial equations by radicals, the RSA cryptosystem, the P vs. NP problem, the crisis of naive set theory and Russell's antinomy, division algebras, the Erlangen Programme, the sign of curvature and types of geometries, the 'arithmetisation' of analysis, the four-colour theorem, and classical groups.

Topics in the first two parts, mainly on formalisation and meta-mathematical methods, are rarely in a curriculum, but they are well worth studying. Naive set theory is introduced first and, although the axioms of Zermalo-Fraenkel theory (ZF) are not set out, there is a useful account of why the axiom of choice (C) and its equivalent formulations are desirable, before stating the theorems of Gödel and Cohen: that C is relatively consistent with, and independent of, ZF, respectively. More familiar materials are given in the latter half of the book, and these should reinforce the foundation and understanding of the subject for the undergraduates.

Inevitably, many items being taken on board can only be given superficial descriptions, and some of the comments and remarks on them lack clarity or transparency. Take the simple notions of primality and factorisation. Just before the introduction of the notorious P vs. NP problem the author writes "More interesting is the prime-factorisation question, that is, the computational problem of determining the explicit prime factorisation of a specific integer N. Formulated as a decision problem, it asks to decide whether N has or not a factor less than a given p." Such remarks should be accompanied by an earlier statement that a number can be shown to be composite without having to produce a prime divisor using, for example, Fermat's little theorem. Indeed it should also have been pointed out that, although it is not known whether the factorisation problem is in P or not, the famous AKS primality test [1], given by three computer scientists in 2004, shows that the problem of whether a given number is prime or composite is in P. Also, Euclid's argument that there are infinitely many primes requires only the fact that each number exceeding 1 has a prime divisor, and not the fundamental theorem of arithmetic which states that prime factorisation is essentially unique. The sentence "For example, the irreducible polynomial $x^5 - 16x + 2 = 0$ has simple Galois group S_5, and so it cannot be solved 'by radicals' " is unlikely to help readers not already familiar with the topic. Indeed, readers should be well versed in permutations, automorphisms, field extensions, and Galois groups, in order to profit from reading the section on Galois connections. Again, after the definition of metric completion, there follows the example: "If p is a prime, consider the p-adic metric $d_p(x, y) = p^{-n}$ on $\boldsymbol{Q}$, where n is the largest integer such that $x - y \in p^n\mathbf{Z}$. The corresponding completion $\mathbf{Q}_P$ defines p-adic numbers". Since neither non-Archimedean

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metrics nor solutions of congruences modulo prime powers has previously been aired properly in the text, the reader not already familiar with $\mathbf{Q}_p$ will be left clueless to what exactly p-adic numbers are.

Such a wide-ranging text requires a good index but, unfortunately, the given one is quite inadequate. Many, or even most, items sprinkled all over the text are absent from the index. Thus there is one single entry for W, namely 'well-ordering', none for Y, and only two items for Z, namely 'zero' and 'zeta-function', so it is hopeless trying to find out from the index if an item is covered in the text or not, and much frustration trying to locate it even when it is there. There is an accompanying Author Index, with entries of over two hundred mathematicians, from Abel and Ackermann to Zemalo and Zorn, who had contributed to topics in the book; however, only the years of their birth and death, together with the prestigious honours they had received in their lifetimes, are given in their entries, and not the page numbers for their contributions. I thought 'continuous fraction' was a misprint for 'continued fraction', but it seems to be the author's choice, because the proof-reading for this lengthy text with wide-ranging topics is meticulous.

The author writes with much enthusiasm trying to explain what has been achieved by mathematicians. *Gazette* readers will be pleased to find that he has chosen to present T. Estermann's proof [2] of the irrationality of $N^{1/k}$, which is based on the well ordering of the natural numbers. Notwithstanding the criticisms, it has to be said that many students will find that there is plenty to learn from this well-written book, which would also be a useful reference text had there been a properly compiled index.

References

1. Manindra Agrawal, Neeraj Kayal and Nitin Saxena, "PRIMES is in P", *Annals of Mathematics* **160** (2004) pp. 781-793.
2. T. Estermann, 'The irrationality of $\sqrt{2}$, *Math. Gaz*. **59** (1975) p. 110.

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A first course in group theory by Bijan Davvaz, pp. 291, £44.99 (hard), also available as e-book, ISBN 978-9-81166-364-2, Springer Verlag (2021)

There have been several recent introductory books on group theory. This one, by an Iranian professor, takes a relatively traditional view, using the definition-theorem-proof structure and with not a great deal of informal explanation, though there are helpful, if brief, introductions to most sections as well as nuggets of historical background. It makes a point of using geometrical diagrams, in full colour, to help cement concepts, and in fact chapter 2, 'Symmetries of shapes', illustrates the standard affine transformations, without going any further than GCSE in content if not in notational sophistication. Chapter 1, called 'Preliminaries Notions' [*sic*], covers a miscellany: sets, functions, number theory, simple combinatorics. Groups are introduced in Chapter 3, starting with the usual axioms (closure being a property of a binary operation), and this is followed by chapters on cyclic groups and permutation groups. There is an 'optional' chapter on groups of arithmetic functions, then the main flow continues with matrix groups, cosets and Lagrange's Theorem, and normal and quotient ('factor') groups as far as Cauchy's Theorem. Chapter 10 is called 'Some special groups' and includes commutators, derived subgroups and maximal subgroups; and the final chapter is on homomorphisms as far as Cayley's Theorem and

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