FRACTIONAL INTEGRAL OPERATORS IN NONHOMOGENEOUS SPACES

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Abstract

We discuss here the boundedness of the fractional integral operator I_α and its generalized version on generalized nonhomogeneous Morrey spaces. To prove the boundedness of I_α, we employ the boundedness of the so-called maximal fractional integral operator $I_{a,\kappa}^*$. In addition, we prove an Olsen-type inequality, which is analogous to that in the case of homogeneous type.

1. Introduction

Let $\mathbb{R}^d$ be equipped with a (positive) Radon measure μ satisfying the *growth condition*:

$$\mu(Q(x, \ell_Q)) \le C \cdot (\ell_Q)^n, \quad x \in \operatorname{supp} \mu, \ \ell_Q > 0,$$

for some constants C and n, where $0 < n \le d$. Here $Q(x, \ell_Q)$ denotes the cube with centre $x \in \mathbb{R}^d$ and side length ℓ_Q. Many authors call such μ a *nondoubling measure*, since it does not necessarily satisfy the so-called *doubling condition*, a condition which is usually required when one studies the properties of an operator in a measure space. In this case, the space $(\mathbb{R}^d, \mu)$ is known as a *nonhomogeneous space*. See, for instance, the works of [11, 12, 18, 20].

In this paper, we study the fractional integral operator $I_\alpha = I_\alpha(n, \mu)$, given by the formula

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{n-\alpha}} \, d\mu(y),$$

where $0 < \alpha < n$, and its generalized version (which we define later). Note that if $n = d$ and μ is the usual Lebesgue measure on $\mathbb{R}^d$, then I_α is the classical fractional integral operator which is known to be bounded from L^p to L^q for $1 < p < d/\alpha$

and $1/q = 1/p - \alpha/d$; this is due to Hardy and Littlewood [6, 7] and Sobolev [16]. Some extensions of this result may be found in, for instance, [1, 9], or in [17, 19], and the references therein.

In the nonhomogeneous setting, the boundedness of I_α on the Lebesgue spaces $L^p(\mu)$ has been studied recently in [4, 15]. Our goal here is to prove the boundedness of I_α, and its generalized version, on the *generalized nonhomogeneous Morrey space* $\mathcal{M}_{p,\phi}(k, \mu)$. In addition, we also prove an Olsen-type inequality involving I_α and a multiplicator W (see [13]).

Before we go into details, let us fix some notation. By $\mathcal{Q}(\mu)$ we mean the set of all cubes with positive μ-measure. Given $Q \in \mathcal{Q}(\mu)$ and $k > 1$, let us set $k\,Q$ as the cube concentric to Q and having k-times as long side length as Q. Denoting by ℓ_Q the side length of $Q \in \mathcal{Q}(\mu)$, we have $\ell_{k\,Q} = k\ell_Q$. Next, we write $\| f : X \|$ to denote the norm of f in the space X. The generalized nonhomogeneous Morrey spaces $\mathcal{M}_{p,\phi}(k, \mu)$ is the set of all μ-locally integrable functions f on $\mathbb{R}^d$ for which

$$\| f : \mathcal{M}_{p,\phi}(k, \mu) \| := \sup_Q \frac{1}{\phi(\mu(k\,Q))} \left(\frac{1}{\mu(k\,Q)} \int_Q |f(y)|^p \, d\mu(y) \right)^{1/p} < \infty.$$

Here $1 < p < \infty$ and we always assume tacitly that the function $\phi : (0, \infty) \to (0, \infty)$ is *almost decreasing*, that is, there exists $C_1 > 0$ such that $\phi(s) \geq C_1\phi(t)$ for $s < t$; and also satisfies the *doubling condition*, that is, there exists $C_2 > 0$ such that

$$\frac{1}{C_2} \leq \frac{\phi(s)}{\phi(t)} \leq C_2,$$

whenever $1 \leq s/t \leq 2$. Meanwhile, $k > 1$ serves as an auxiliary parameter.

As in [14], we have the following fact.

FACT 1. *For $k_1, k_2 > 1$, the spaces $\mathcal{M}_{p,\phi}(k_1, \mu)$ and $\mathcal{M}_{p,\phi}(k_2, \mu)$ coincide as a set and their norms are mutually equivalent.*

In view of this fact, we set $\mathcal{M}_{p,\phi}(\mu) := \mathcal{M}_{p,\phi}(2, \mu)$. One may observe that if $1 \leq p_1 \leq p_2 < \infty$, then $\mathcal{M}_{p_2,\phi}(\mu) \subseteq \mathcal{M}_{p_1,\phi}(\mu)$ for we have $\| f : \mathcal{M}_{p_1,\phi} \| \leq \| f : \mathcal{M}_{p_2,\phi} \|$ whenever $f \in \mathcal{M}_{p_2,\phi}$.

To prove the boundedness of I_α from $\mathcal{M}_{p,\phi}(\mu)$ to $\mathcal{M}_{q,\psi}(\mu)$, we need to employ the (modified) *maximal operator* M_k $(k > 1)$, which is defined by

$$M_k f(x) := \sup_{Q \ni x} \frac{1}{\mu(k\,Q)} \int_Q f(y) \, d\mu(y),$$

for all μ-locally integrable functions f on $\mathbb{R}^d$.

THEOREM 1.1 (Sawano [14]). *For $p > 1$, M_k is a bounded operator on $\mathcal{M}_{p,\phi}(\mu)$.*

2. The boundedness of I_α

To study the boundedness of I_α on the generalized Morrey spaces, we define the *maximal fractional integral operator* $I^*_{a,\kappa}$, given by

$$I^*_{a,\kappa} f(x) := \int_{\mathbb{R}^d} K^*_{a,\kappa}(x, y) f(y) \, d\mu(y),$$

where the kernel $K^*_{a,\kappa}$ is given by

$$K^*_{a,\kappa}(x, y) := \sup_{Q \ni x, y} \mu(\kappa Q)^{a-1}.$$

Here $0 < a < 1$ and $\kappa > 1$; both of them serve as parameters. Since we are not concerned with keeping track of the precise value of various constants, we denote by C various constants independent of functions f. Using the following fact, it suffices for us to study the boundedness of $I^*_{\alpha/n,\kappa}$.

FACT 2 (Sawano *et al.* [15]). *We have* $I_\alpha f \leq C \, I^*_{\alpha/n,\kappa} f$ *for all positive μ-measurable functions f.*

Our result is the following theorem, which is in some sense stronger than the result obtained in [14].

THEOREM 2.1. *Let* $a = \alpha/n \in (0, 1)$. *If the function ϕ is surjective and satisfies the inequality* $\phi(t) \leq C \, t^b$ *with* $-1/p \leq b < -a < 0$, *then* $I^*_{a,\kappa}$ *is bounded from* $\mathcal{M}_{p,\phi}(\mu)$ *to* $\mathcal{M}_{q,\phi^{p/q}}(\mu)$, *where* $p > 1$ *and* $q = bp/(a + b)$.

A helpful remark may be in order. In the above theorem, the hypothesis $b \geq -1/p$ is needed for $\|f : \mathcal{M}_{p,\phi}(\mu)\|$ to make sense. Meanwhile, the hypothesis $b < -a$ together with $p/q = (a + b)/b < 1$ implies that $p < q$. When μ is the usual Lebesgue measure on $\mathbb{R}^d$ and $\phi(t) = t^{(\lambda-n)/(np)}$ with $0 \leq \lambda < n - \alpha p$ and $1 < p < n/\alpha$, the above results recover the boundedness of I_α from the classical Morrey space $L^{p,\lambda}$ to $L^{q,\lambda}$ where $1/q = 1/p - \alpha/(n - \lambda)$ (see [1]). For related results in the generalized 'homogeneous' Morrey spaces, see [3, 5].

PROOF OF THEOREM 2.1. Without loss of generality, we assume that $f \geq 0$, not identically zero, and a usual truncation procedure allows us to assume even that $M_{\sqrt{\kappa}} f(x)$ is finite everywhere. For $x \in \text{supp } \mu$ and $\tau > 0$, we write

$$A_j := \left\{ y \in \mathbb{R}^d \setminus \{x\} : 2^{j-1}\tau < \inf_{Q \ni x, y} \mu(\kappa Q) \leq 2^j \tau \right\},$$

for every $j \in \mathbb{Z}$. As in [15], there exists a collection of cubes $Q^1_j, \ldots, Q^{N_\kappa}_j$ containing x, with the covering constant N_κ being independent of x, τ, and j, such that

$$A_j \subseteq \sqrt{\kappa} Q^1_j \cup \cdots \cup \sqrt{\kappa} Q^{N_\kappa}_j$$

and

$$\mu(\kappa Q^l_j) \leq 2^{j+1}\tau, \quad 1 \leq l \leq N_\kappa,$$

for every $j \in \mathbb{Z}$. As a special case we accept $Q^1_j = \mathbb{R}^d$, although $\mathbb{R}^d$ is not compact.

In the following, for the sake of simplicity, we assume that the Q_j^l are all compact. (If this is not the case, a minor modification suffices.)

Now, for each $y \in A_j$, we have $K_{a,\kappa}^*(x, y) < C(2^j \tau)^{a-1}$, and so

$$
\begin{aligned}
I_{a,\kappa}^* f(x) &\leq C \sum_{j \in \mathbb{Z}} (2^j \tau)^{a-1} \int_{A_j} f(y) \, d\mu(y) \\
&\leq C \sum_{j \in \mathbb{Z}} \sum_{l=1}^{N_\kappa} (2^j \tau)^{a-1} \int_{\sqrt{\kappa} Q_j^l} f(y) \, d\mu(y) \\
&\leq C \left[\sum_{j=-\infty}^{0} \sum_{l=1}^{N_\kappa} \cdots + \sum_{j=1}^{\infty} \sum_{l=1}^{N_\kappa} \cdots \right] \\
&=: C[I + II].
\end{aligned}
$$

For the first summation, we use the definition of $M_{\sqrt{\kappa}} f$ to have

$$
\begin{aligned}
I &\leq C \sum_{j=-\infty}^{0} \sum_{l=1}^{N_\kappa} (2^j \tau)^a \left(\frac{1}{\mu(\kappa Q_j^l)} \int_{\sqrt{\kappa} Q_j^l} f(y) \, d\mu(y) \right) \\
&\leq C \tau^a M_{\sqrt{\kappa}} f(x).
\end{aligned}
$$

For the second summation, we make use of Hölder's inequality to obtain

$$
\begin{aligned}
II &\leq \sum_{j=1}^{\infty} \sum_{l=1}^{N_\kappa} (2^j \tau)^{a-1} (\mu(\kappa Q_j^l))^{1-(1/p)} \left(\int_{\sqrt{\kappa} Q_j^l} f(y)^p \, d\mu(y) \right)^{1/p} \\
&\leq \sum_{j=1}^{\infty} \sum_{l=1}^{N_\kappa} (2^j \tau)^a \left(\frac{1}{\mu(\kappa Q_j^l)} \int_{\sqrt{\kappa} Q_j^l} f(y)^p \, d\mu(y) \right)^{1/p} \\
&\leq \sum_{j=1}^{\infty} \sum_{l=1}^{N_\kappa} (2^j \tau)^a \phi(\mu(\kappa Q_j^l)) \| f \, : \, \mathcal{M}_{p,\phi}(\mu) \| \\
&\leq C \| f \, : \, \mathcal{M}_{p,\phi}(\mu) \| \sum_{j=1}^{\infty} (2^j \tau)^a (\mu(\kappa Q_j^l))^b \\
&\leq C \| f \, : \, \mathcal{M}_{p,\phi}(\mu) \| \sum_{j=1}^{\infty} (2^j \tau)^{a+b} \\
&\leq C \tau^{a+b} \| f \, : \, \mathcal{M}_{p,\phi}(\mu) \|
\end{aligned}
$$

since $b < -a$.

Combining the two estimates and choosing $\tau = (M_{\sqrt{\kappa}} f(x) / \| f \, : \, \mathcal{M}_{p,\phi}(\mu) \|)^{1/b}$, we obtain

$$
\begin{aligned}
I_{a,\kappa}^* f(x) &\leq C \left(M_{\sqrt{\kappa}} f(x) \right)^{(a/b)+1} \| f \, : \, \mathcal{M}_{p,\phi}(\mu) \|^{-(a/b)} \\
&\leq C \left(M_{\sqrt{\kappa}} f(x) \right)^{p/q} \| f \, : \, \mathcal{M}_{p,\phi}(\mu) \|^{1-(p/q)}.
\end{aligned}
$$

Since $M_{\sqrt{\kappa}}$ is bounded on $\mathcal{M}_{p,\phi}(\mu)$, we obtain

$$\|I_{a,\kappa}^* f \ : \ \mathcal{M}_{q,\phi^{p/q}}(\mu)\| \leq C \, \|f \ : \ \mathcal{M}_{p,\phi}(\mu)\|,$$

that is, $I_{a,\kappa}^*$ is bounded from $\mathcal{M}_{p,\phi}(\mu)$ to $\mathcal{M}_{q,\phi^{p/q}}(\mu)$, as desired. $\qquad\square$

COROLLARY 2.2. *If ϕ, p and q are as in Theorem 2.1, then I_α is bounded from $\mathcal{M}_{p,\phi}(\mu)$ to $\mathcal{M}_{q,\phi^{p/q}}(\mu)$.*

REMARK 2.3. As a consequence of our result, we have that if T is a kernel operator with kernel K satisfying

$$K(x, y) \leq C \, |x - y|^{\alpha-n},$$

then T is bounded from $\mathcal{M}_{p,\phi}(\mu)$ to $\mathcal{M}_{q,\phi^{p/q}}(\mu)$, where ϕ, p and q are as in Theorem 2.1. With this result, one may also obtain estimates for fractional powers of operators that generate semigroups with the Gaussian bounds for the associated heat kernels (see [2] about heat kernels).

2.1. An Olsen-type inequality Following Corollary 2.2, we have the Olsen-type inequality below. (For its origin, see the work of Olsen [13]. For an analogous result in the 'homogeneous' case, see [5]. The 'Olsen inequality' is useful in understanding the behaviour of the solution to a perturbed Schrödinger equation with a small perturbed potential W; see [8].)

COROLLARY 2.4. *If ϕ, p and q are as in Theorem 2.1, then $W \cdot I_\alpha$ is bounded on $\mathcal{M}_{p,\phi}(\mu)$ provided that $W \in \mathcal{M}_{s,\phi^{p/s}}(\mu)$, where $1/s = 1/p - 1/q$. Precisely, we have*

$$\|W \cdot I_\alpha f \ : \ \mathcal{M}_{p,\phi}(\mu)\| \leq C \, \|W \ : \ \mathcal{M}_{s,\phi^{p/s}}(\mu)\| \cdot \|f \ : \ \mathcal{M}_{p,\phi}(\mu)\|,$$

where $1/s = 1/p - 1/q$.

The following theorem provides an Olsen-type inequality for a multiplication operator involving the fractional integral operator $I_{a,\kappa}^*$.

THEOREM 2.5. *Suppose that $t^{1/p}\phi(t)$ is almost increasing and satisfies the inequality*

$$\int_r^\infty t^{a-1}\phi(t) \, dt \leq C \, r^a \phi(r) \tag{2.1}$$

for all $r > 0$ with some $a < 1/p$ and $C > 0$ independent of r. Then

$$\|W \cdot I_{a,\kappa}^* f \ : \ \mathcal{M}_{p,\phi}(\mu)\| \leq C \, \|W \ : \ L^{1/a}(\mu)\| \cdot \|f \ : \ \mathcal{M}_{p,\phi}(\mu)\|.$$

PROOF 1. Let $Q \in \mathcal{Q}(\mu)$ be fixed. We shall establish

$$I \leq C \, \|W \ : \ L^{1/a}(\mu)\| \cdot \|f \ : \ \mathcal{M}_{p,\phi}(\mu)\| \tag{2.2}$$

with K sufficiently large. We decompose f according to $K\,Q$, where

$$I := \frac{1}{\phi(\mu(2K\,Q))}\left(\frac{1}{\mu(2K\,Q)}\int_Q |W \cdot I_{a,\kappa}^* f(y)|^p \, d\mu(y)\right)^{1/p}.$$

Set $f_1 = f \cdot \chi_{K\,Q}$ and $f_2 = f - f_1$. Along this decomposition, (2.2) can be split into

$$II \le C \, \|W \,:\, L^{1/a}(\mu)\| \cdot \|f \,:\, \mathcal{M}_{p,\phi}(\mu)\| \tag{2.3}$$

$$III \le C \, \|W \,:\, L^{1/a}(\mu)\| \cdot \|f \,:\, \mathcal{M}_{p,\phi}(\mu)\|, \tag{2.4}$$

where

$$II := \frac{1}{\phi(\mu(2K\,Q))}\left(\frac{1}{\mu(2K\,Q)}\int_Q |W \cdot I_{a,\kappa}^* f_1(y)|^p \, d\mu(y)\right)^{1/p}$$

$$III := \frac{1}{\phi(\mu(2K\,Q))}\left(\frac{1}{\mu(2K\,Q)}\int_Q |W \cdot I_{a,\kappa}^* f_2(y)|^p \, d\mu(y)\right)^{1/p}.$$

As for (2.3) we use Hölder's inequality to obtain

$$\begin{aligned}
II &\le \frac{1}{\phi(\mu(2K\,Q))}\left(\frac{1}{\mu(2K\,Q)}\int_{\mathbb{R}^d} |W \cdot I_{a,\kappa}^* f_1(y)|^p \, d\mu(y)\right)^{1/p} \\
&\le \frac{C}{\phi(\mu(2K\,Q))}\left(\frac{1}{\mu(2K\,Q)}\right)^{1/p} \|W \,:\, L^{1/a}(\mu)\| \cdot \|f_1 \,:\, L^p(\mu)\| \\
&\le C \, \|W \,:\, L^{1/a}(\mu)\| \cdot \|f \,:\, \mathcal{M}_{p,\phi}(\mu)\|.
\end{aligned}$$

Let us denote by c_Q the centre of Q. To obtain (2.4), we observe that

$$K_{a,\kappa}^*(x,\,y) \le \sup_{c_Q,\,y\in R\in\mathcal{Q}(\mu)} \mu(\sqrt{\kappa}\,R)^{-1+a}$$

for all $x \in Q$ and $y \in \mathbb{R}^d \setminus K\,Q$, provided K is sufficiently large. Therefore, it suffices to show

$$IV \le C \, \|W \,:\, L^{1/a}(\mu)\| \cdot \|f \,:\, \mathcal{M}_{p,\phi}(\mu)\|,$$

where

$$IV := \frac{1}{\phi(\mu(2K\,Q))}\left(\frac{1}{\mu(2K\,Q)}\int_Q |W \cdot \tilde{I}_a^L f_2|^p \, d\mu\right)^{1/p}$$

$$\tilde{I}_a^L g(x) := \int_{\mathbb{R}^d} \tilde{k}_a^L(x,\,y) f(y) \, d\mu(y)$$

$$\tilde{k}_a^L(x,\,y) := \sup_{\substack{Q\subset R\in\mathcal{Q}(\mu) \\ \ell_R\le L,\ y\in R}} \mu(\sqrt{\kappa}\,R)^{-1+a}.$$

Let us set

$$A_j := \left\{ y \in \operatorname{supp} \mu \ : \ 2^{j-1} < \inf_{\substack{Q \subset R \in \mathcal{Q}(\mu) \\ \ell_R \leq L, \ y \in R}} \mu(\sqrt{\kappa}\, R) \leq 2^j \right\}.$$

Then from the definition of $\tilde{k}_a^L$ for each $j \in \mathbb{Z}$, a geometric observation yields a finite collection of cubes $Q_j^1, \ldots, Q_j^{N_\kappa} \in \mathcal{Q}(\mu)$ such that

$$A_j \subset \bigcup_{l=1}^{N_\kappa} \sqrt[4]{\kappa}\, Q_j^l$$

and that

$$2^{j-1} \leq \mu(\sqrt{\kappa}\, Q_j^l) \leq 2^j \quad \text{for each } l = 1, \ldots, N_\kappa.$$

Here the number N_κ is again a covering constant which may depend only on κ and d. Again it can happen that the cube Q_j^1 equals $\mathbb{R}^d$. In this case we only need to make a minor modification of the proof below.

With the help of this covering we obtain

$$\tilde{I}_a^L f_2(x) \leq C \sum_{\substack{j \in \mathbb{Z} \\ A_j \neq \emptyset}} \sum_{l=1}^{N_\kappa} 2^{-(1-a)j} \int_{\sqrt[4]{\kappa}\, Q_j^l} f(y)\, d\mu(y)$$

$$\leq C \sum_{\substack{j \in \mathbb{Z} \\ A_j \neq \emptyset}} 2^{aj} \phi(2^j) \| f : \mathcal{M}_{1,\phi}(\mu) \|.$$

Note that if $A_j \neq \emptyset$, then we have $\mu(Q) \leq 2^j$. As a consequence, we have by virtue of (2.1)

$$\sum_{\substack{j \in \mathbb{Z} \\ A_j \neq \emptyset}} 2^{aj} \phi(2^j) \leq C \sum_{\substack{j \in \mathbb{Z} \\ A_j \neq \emptyset}} \int_{2^j}^{2^{j+1}} t^{a-1} \phi(t)\, dt$$

$$\leq C \int_{\mu(Q)}^{\infty} t^{a-1} \phi(t)\, dt$$

$$\leq C\, \mu(Q)^a \phi(\mu(Q)).$$

From this inequality and the fact that $\| f : \mathcal{M}_{1,\phi}(\mu) \| \leq \| f : \mathcal{M}_{p,\phi}(\mu) \|$, we now have

$$\tilde{I}_a^L f_2(x) \leq C\, \mu(Q)^a \phi(\mu(Q)) \| f : \mathcal{M}_{p,\phi}(\mu) \|.$$

for all $x \in Q$ and, hence,

$$
\begin{aligned}
IV &\leq C \, \frac{\mu(Q)^a \phi(\mu(Q)) \left(\int_Q |W(y)|^p \, d\mu(y) \right)^{1/p}}{\mu(2K\,Q)^{1/p} \phi(\mu(2K\,Q))} \| f \, : \, \mathcal{M}_{p,\phi}(\mu) \| \\
&\leq C \, \frac{\mu(Q)^{1/p} \phi(\mu(Q)) \left(\int_Q |W(y)|^{1/a} \, d\mu(y) \right)^a}{\mu(2K\,Q)^{1/p} \phi(\mu(2K\,Q))} \| f \, : \, \mathcal{M}_{p,\phi}(\mu) \| \\
&\leq C \left(\int_{\mathbb{R}^d} |W(y)|^{1/a} \, d\mu(y) \right)^a \| f \, : \, \mathcal{M}_{p,\phi}(\mu) \|,
\end{aligned}
$$

since $t^{1/p} \phi(t)$ is almost increasing. This is the desired result. □

3. A generalized version

The generalized fractional integral operator T_ρ, defined for a given function $\rho : (0, \infty) \to (0, \infty)$ by

$$
T_\rho f(x) = \int_{\mathbb{R}^d} \frac{\rho(|x-y|)}{|x-y|^d} f(y) \, dy,
$$

was first studied by [9, 10], in the homogeneous setting. For $\rho(t) = t^\alpha$, $0 < \alpha < d$, the operator T_ρ is equal to I_α, the classical fractional integral operator.

Now, we discuss an analogue of T_ρ in the nonhomogeneous setting. Write

$$
D(x, y) := \inf_{x, y \in Q \in \mathcal{Q}(\mu)} \mu(\kappa Q)
$$

and

$$
l_\rho(x, y) := \frac{\rho(D(x, y))}{D(x, y)}.
$$

Given a positive function f, we define T_ρ by

$$
T_\rho f(x) := \int_{\mathbb{R}^d} l_\rho(x, y) f(y) \, d\mu(y).
$$

Note that for $\rho(t) = t^\alpha$, the operator T_ρ takes the form of $I_{a,\kappa}^*$.

THEOREM 3.1. *Let $1 < p < q < \infty$. Assume that ϕ is surjective and ρ satisfies the doubling condition. Suppose further that there exists $C > 0$ such that*

$$
\phi(r) \int_0^r \frac{\rho(t)}{t} \, dt + \int_r^\infty \frac{\phi(t)\rho(t)}{t} \, dt \leq C \, \phi(r)^{p/q}
$$

for every $r > 0$. Then, there exists $C > 0$ such that

$$
\| T_\rho f \, : \, \mathcal{M}_{q,\phi^{p/q}}(\mu) \| \leq C \, \| f \, : \, \mathcal{M}_{p,\phi}(\mu) \|
$$

for all positive $f \in \mathcal{M}_{p,\phi}(\mu)$.

PROOF 2. The kernel being positive, we can truncate the kernel with respect to the side length of cubes. Let $L > 0$ and define

$$l_\rho^L(x, y) := \sup_{\substack{x, y \in Q \in \mathcal{Q}(\mu) \\ \ell_Q \leq L}} \frac{\rho(\mu(\kappa Q))}{\mu(\kappa Q)}.$$

Given a positive function f, define

$$T_\rho^L f(x) := \int_{\mathbb{R}^d} l_\rho^L(x, y) f(y) \, d\mu(y).$$

It suffices to prove the assertion for T_ρ^L instead of T_ρ with constants independent of L. Fix $x \in \mathrm{supp}\, \mu$ and $\tau > 0$. For $j \in \mathbb{Z}$ we define

$$A_j := \left\{ y \in \mathbb{R}^d \ : \ 2^{j-1}\tau < \inf_{\substack{x, y \in Q \in \mathcal{Q}(\mu) \\ \ell_Q \leq L}} \mu(\kappa Q) \leq 2^j \tau \right\}.$$

Then

$$T_\rho^L f(x) \leq C \sum_{j \in \mathbb{Z}} \frac{\rho(2^j \tau)}{2^j \tau} \int_{A_j} f(y) \, d\mu(y).$$

As before, there exists N_κ with the following properties

$$A_j \subset \sqrt{\kappa} Q_j^1 \cup \cdots \cup \sqrt{\kappa} Q_j^{N_\kappa}.$$

Therefore,

$$
\begin{aligned}
T_\rho^L f(x) &\leq C \sum_{j \in \mathbb{Z}} \sum_{l=1}^{N_\kappa} \frac{\rho(2^j \tau)}{2^j \tau} \int_{\sqrt{\kappa} Q_j^l} f(y) \, d\mu(y) \\
&\leq C \sum_{j \in \mathbb{Z} \backslash \mathbb{N}} \sum_{l=1}^{N_\kappa} \rho(2^j \tau) \cdot M_{\sqrt{\kappa}} f(x) \\
&\quad + C \sum_{j \in \mathbb{N}} \sum_{l=1}^{N_\kappa} \frac{\rho(2^j \tau)}{(2^j \tau)^{1/p}} \left(\int_{\sqrt{\kappa} Q_j^l} f(y)^p \, d\mu(y) \right)^{1/p} \\
&\leq C \int_0^\tau \frac{\rho(t)}{t} \, dt \cdot M_{\sqrt{\kappa}} f(x) + C \int_\tau^\infty \frac{\rho(t)\phi(t)}{t} \, dt \cdot \|f \ : \ \mathcal{M}_{p,\phi}(\mu)\| \\
&\leq C \, \phi(\tau)^{\frac{p}{q}-1} M_{\sqrt{\kappa}} f(x) + \phi(\tau)^{p/q} \|f \ : \ \mathcal{M}_{p,\phi}(\mu)\|.
\end{aligned}
$$

Assuming ϕ surjective and that $1 < p < q < \infty$, we can optimize the above inequality. As a consequence

$$T_\rho^L f(x) \leq C \, (M_{\sqrt{\kappa}} f(x))^{p/q} \cdot \|f \ : \ \mathcal{M}_{p,\phi}(\mu)\|^{1-p/q}.$$

Now that the maximal operator $M_{\sqrt{\kappa}}$ is bounded on $\mathcal{M}_{p,\phi}(\mu)$, we have the desired result. □

COROLLARY 3.2. *Suppose that ϕ is surjective. If ρ satisfies the doubling condition and*

$$\phi(r) \int_0^r \frac{\rho(t)}{t}\, dt + \int_r^\infty \frac{\phi(t)\rho(t)}{t}\, dt \le C\, \phi(r)^{p/q}$$

for $1 < p < q < \infty$, then

$$\|W \cdot T_\rho f \,:\, \mathcal{M}_{p,\phi}(\mu)\| \le C\, \|W \,:\, \mathcal{M}_{s,\phi^{p/s}}(\mu)\| \cdot \|f \,:\, \mathcal{M}_{p,\phi}(\mu)\|$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $1/s = 1/p - 1/q$.

Acknowledgements

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