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SYMMETRIC CROSSCAP NUMBER OF GROUPS OF ORDER LESS THAN OR EQUAL TO 63

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Abstract

Every finite group G acts on some nonorientable unbordered surfaces. The minimal topological genus of those surfaces is called the symmetric crosscap number of G. It is known that 3 is not the symmetric crosscap number of any group but it remains unknown whether there are other such values, called gaps. In this paper we obtain group presentations which allow one to find the actions realizing the symmetric crosscap number of groups of each group of order less than or equal to 63.

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1. Introduction

A Klein surface X is a compact surface endowed with a dianalytic structure [1]. Klein surfaces may be seen as a generalization of Riemann surfaces including bordered and nonorientable surfaces. An orientable unbordered Klein surface is a Riemann surface. Given a Klein surface X of topological genus g with k boundary components, the number $p = \eta g + k - 1$ is called the algebraic genus of X, where $\eta = 2$ if X is an orientable surface and $\eta = 1$ otherwise.

In the study of Klein surfaces and their automorphism groups the non-Euclidean crystallographic (NEC) groups play an essential role. An NEC group Γ is a discrete subgroup of $\mathcal{G}$ (the full group of isometries of the hyperbolic plane $\mathcal{H}$) with compact quotient $\mathcal{H}/\Gamma$. For a Klein surface X with $p \geq 2$, there exists an NEC group Γ such that $X = \mathcal{H}/\Gamma$ [18].

A finite group G of order N is an automorphism group of a Klein surface $X = \mathcal{H}/\Gamma$ if and only if there exists an NEC group Λ such that Γ is a normal subgroup of Λ with index N and $G = \Lambda/\Gamma$. Every finite group G may act as an automorphism group of nonorientable Klein surfaces without boundary. The minimum genus of these surfaces

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is called the symmetric crosscap number of G and it is denoted by $\tilde{\sigma}(G)$. Such a surface of topological genus $g \geq 3$ has at most $84(g-2)$ automorphisms. Hence, for each g there is a finite number of groups acting on surfaces of genus g. The systematic study of the symmetric crosscap number was begun by May in [17], although previous results from other authors are also to be noted; see for instance [5, 8, 14].

Four types of problems arise naturally when dealing with the symmetric crosscap number $\tilde{\sigma}(G)$.

First of all, to obtain $\tilde{\sigma}(G)$ for any given group G, and for the groups belonging to a given infinite family.

Second, to obtain $\tilde{\sigma}(G)$ for all groups G with $o(G) < n$ for a given (small) value of n.

Third, for a given value of g, to obtain all groups G such that $\tilde{\sigma}(G) = g$. Evidently this question is feasible only for low values of g.

Finally, to determine which values of g are in fact $\tilde{\sigma}(G) = g$ for a group G. The set of such values is called the symmetric crosscap spectrum and there exists a conjecture according to which $g = 3$ is the unique positive integer not belonging to the spectrum.

In this paper we deal with the second question. We will study the symmetric crosscap number of the groups with order less than or equal to 63. First, we will indicate all the results we know and then we will make a study of each order $n \leq 63$ that has not been studied in detail.

2. Preliminaries

An NEC group Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathcal{H}$, including orientation-reversing elements, with compact quotient $X = \mathcal{H}/\Gamma$. Each NEC group Γ has associated a signature [16]:

$$\sigma(\Gamma) = (g, \pm, [m_1, \ldots, m_r], \{(n_{i,1}, \ldots, n_{i,s_i}), i = 1, \ldots, k\}), \tag{2.1}$$

where $g, k, r, m_i, n_{i,j}$ are integers satisfying $g, k, r \geq 0, m_i \geq 2, n_{i,j} \geq 2$. We will denote by $[-]$, $(-)$ and $\{-\}$ the cases when $r = 0$, $s_i = 0$ and $k = 0$, respectively.

The signature determines a presentation of Γ, see [22], by generators x_i $(i = 1, \ldots, r)$; e_i $(i = 1, \ldots, k)$; $c_{i,j}$ $(i = 1, \ldots, k; j = 0, \ldots, s_i)$; a_i, b_i $(i = 1, \ldots, g)$ if σ has sign '+'; and d_i $(i = 1, \ldots, g)$ if σ has sign '–'. These generators satisfy the following relations:

$$x_i^{m_i} = 1; \quad c_{i,j-1}^2 = c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{i,j}} = 1; \quad e_i^{-1}c_{i,0}e_i c_{i,s_i} = 1$$

and

$$\prod_{i=1}^{r} x_i \prod_{i=1}^{k} e_i \prod_{i=1}^{g} (a_i b_i a_i^{-1} b_i^{-1}) = 1 \quad \text{if } \sigma \text{ has sign '+'},$$

$$\prod_{i=1}^{r} x_i \prod_{i=1}^{k} e_i \prod_{i=1}^{g} d_i^2 = 1 \quad \text{if } \sigma \text{ has sign '–'}.$$

The isometries x_i are elliptic, e_i, a_i, b_i are hyperbolic, $c_{i,j}$ are reflections and d_i are glide reflections.

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Every NEC group Γ with signature (2.1) has associated a fundamental region whose area $\mu(\Gamma)$, called the area of the group, is

$$\mu(\Gamma) = 2\pi\Big(\eta g + k - 2 + \sum_{i=1}^{r}\Big(1 - \frac{1}{m_i}\Big) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\Big(1 - \frac{1}{n_{i,j}}\Big)\Big), \tag{2.2}$$

with $\eta = 2$ or 1 depending on the sign '+' or '–' in the signature. An NEC group with signature (2.1) actually exists if and only if the right-hand side of (2.2) is greater than 0. We denote by $|\Gamma|$ the expression $\mu(\Gamma)/2\pi$ and call it the reduced area of Γ.

If Γ is a subgroup of an NEC group Λ of finite index N, then also Γ is an NEC group and the Riemann–Hurwitz formula holds, $|\Gamma| = N|\Lambda|$.

Let X be a nonorientable Klein surface of topological genus $g \geq 3$ without boundary. Then by [18] there exists an NEC group Γ with signature

$$\sigma(\Gamma) = (g, -, [-], \{-\})$$

such that $X = \mathcal{H}/\Gamma$.

Let $X = \mathcal{H}/\Gamma$ be a nonorientable unbordered Klein surface on which G acts as an automorphism group. Then there exists another NEC group Λ such that $G = \Lambda/\Gamma$. From the Riemann–Hurwitz relation, we have $g - 2 = o(G)|\Lambda|$, where $o(G)$ denotes the order of G. Then

$$\tilde{\sigma}(G) \leq g = 2 + o(G)|\Lambda|,$$

and so to obtain the symmetric crosscap number is equivalent to finding a group Λ and an epimorphism $\theta : \Lambda \to G$ such that $\Gamma = \ker\theta$ is a surface NEC group (and, so, without elements with finite order) and $G = \theta(\Lambda^+)$, where Λ^+ is the subgroup consisting of the orientation-preserving elements of Λ, see [19], and minimal $|\Lambda|$.

The symmetric crosscap numbers of groups belonging to the families $C_m \times D_n$, $D_m \times D_n$, $DC_3 \times C_n$ and $A_4 \times C_n$ have been obtained [9–11]. For Abelian groups of odd order they were calculated in [8] and this result was extended to all Abelian groups in [14]. May obtained in [17] the symmetric crosscap numbers of dicyclic groups and Hamiltonian groups without odd-order part. Also, the symmetric crosscap numbers of the groups of order less than 32 have been calculated in [12]. On the other hand, the groups having symmetric crosscap numbers 1 and 2 have been classified by Tucker [21]. The groups of genus 1 are C_n, D_n, A_4, S_4 and A_5. We have two families of groups of genus 2, $C_2 \times C_n$, $n > 2$ even, and $C_2 \times D_n$, n even.

Conder, at a conference in Castro-Urdiales in 2010, announced that using computing software, he had obtained the symmetric crosscap numbers of groups of order up to 127, in terms of their 'SmallGroupLibrary' description. The result of this research is available in his web page [7]. The list contains the GAP reference of each group, its symmetric crosscap number and the corresponding NEC group. However, this list gives information neither on the algebraic structure of the groups nor on the epimorphism θ which determines the action of the NEC group Λ on the group G. Throughout the paper, we use extensively this fundamental work by Conder, in order to study which are the concerned groups.

For each group G we have described its algebraic structure, its presentation and the corresponding epimorphism, but here we will only show the algebraic structure and its presentation. In the most complicated cases, we will show also the epimorphism. In the presentations we skip the Abelian relations. The full details are to be found in [3] and in [2]. The algebraic identification allows us to know the subgroup structure of the involved groups, and this is essential to determine all the groups that act on a surface of a given genus. For some groups we indicate the reference where its study can be found.

3. Symmetric crosscap number of groups of order 32

In order 32 we have 51 different groups, see Table 1. All of them have been studied using the classification made by Hall and Senior in [15], and their multiplication tables. We are going to illustrate this section with one example. Take the group [32,22]. This group has algebraic structure $C_2 \times ((C_4 \times C_2) \rtimes C_2)$, and notation $\Gamma_2 c_1$, by Hall and Senior. The symmetric crosscap number of this group is 18. This group has a presentation given by generators a, b, c, d and relations $a^4 = b^2 = c^2 = d^2 = 1$, $ab = ba, ac = ca^3 b, bc = cb$ and d commutes with the other generators. For this group we have three NEC groups.

- For the NEC group (0; +; [2]; {(–),(–)}), with reduced area 1/2, we have that an associated epimorphism $\theta : \Lambda \to G$ is given by
$$\theta(x_1) = cb, \quad \theta(e_1) = a, \quad \theta(e_2) = a^{-1}bc, \quad \theta(c_{1,0}) = b, \quad \theta(c_{2,0}) = bd.$$
We have that the element $e_1 x_1 e_1$ has as image the generator c, the element e_1 has as image the generator a, the element $c_{1,0}c_{2,0}$ has as image the generator d and the element $(e_1 x_1 c_{1,0})^2$ has as image the generator b, and all these elements are orientation preserving. So, it is a group that acts on a nonorientable surface.
- For the NEC group (0; +; [4]; {(2,2,2)}), with reduced area 1/2, we have that an associated epimorphism $\theta : \Lambda \to G$ is given by
$$\theta(x_1) = a, \quad \theta(e_1) = a^{-1}, \quad \theta(c_{1,0}) = cb, \quad \theta(c_{1,1}) = b,$$
$$\theta(c_{1,2}) = bd, \quad \theta(c_{1,3}) = ca^2.$$
We have that the element $c_{1,0}c_{1,1}$ has as image the generator c, the element x_1 has as image the generator a, the element $c_{1,1}c_{1,2}$ has as image the generator d and the element $(x_1 c_{1,0} c_{1,1})^2$ has as image the generator b, and all these elements are orientation preserving. So, it is a group that acts on a nonorientable surface.
- For the NEC group (0; +; [–]; {(2,2), (–)}), with reduced area 1/2, we have that an associated epimorphism $\theta : \Lambda \to G$ is given by
$$\theta(e_1) = a, \quad \theta(e_2) = a^{-1}, \quad \theta(c_{1,0}) = cb, \quad \theta(c_{1,1}) = bd,$$
$$\theta(c_{1,2}) = ca^2, \quad \theta(c_{2,0}) = b.$$
We have that the element $c_{1,0}c_{2,0}$ has as image the generator c, the element e_1 has as image the generator a, the element $c_{2,0}c_{1,1}$ has as image the generator d and the element $(e_1 c_{1,0} c_{2,0})^2$ has as image the generator b, and all these elements are orientation preserving. So, it is a group that acts on a nonorientable surface.

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TABLE 1. Groups of order 32.

GAP	G – H.S. notation	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[32,1]	C_{32}	a^{32}	1	[21]
[32,2]	$(C_4 \times C_2) \rtimes C_4 - \Gamma_2 h$	$a^2, b^4, c^4, bcb^{-1}ac^{-1}$	18	
[32,3]	$C_8 \times C_4$	a^8, b^4	22	[14]
[32,4]	$C_8 \rtimes C_4 - \Gamma_2 i$	a^8, b^4, aba^3b^{-1}	22	
[32,5]	$(C_8 \times C_2) \rtimes C_2 - \Gamma_2 j_1$	$a^2, b^8, c^2, bcb^{-1}ac$	10	
[32,6]	$((C_4 \times C_2) \rtimes C_2) \rtimes C_2 - \Gamma_7 a_1$	$a^2, b^2, c^2, d^4, bdbad^{-1}, cdcbad^{-1}$	10	
[32,7]	$(C_8 \rtimes C_2) \rtimes C_2 - \Gamma_7 a_2$	$a^8, b^2, c^2, aba^3b, aca^{-1}bc$	10	
[32,8]	$(C_2 \times C_2).(C_4 \times C_2) - \Gamma_7 a_3$	$a^8, b^2, c^2a^4, aba^3b, aca^{-1}bc^{-1}$	22	
[32,9]	$(C_8 \times C_2) \rtimes C_2 - \Gamma_3 c_1$	$a^2, b^8, c^2, bcbac$	10	
[32,10]	$Q \rtimes C_4 - \Gamma_3 c_2$	$a^2, b^8, c^2b^4, bcbac^{-1}$	18	
[32,11]	$(C_4 \times C_4) \rtimes C_2 - \Gamma_3 e$	$a^4, b^4, c^2, bcba^{-1}c$	10	
[32,12]	$C_4 \rtimes C_8 - \Gamma_2 j_2$	$a^4, b^8, abab^{-1}$	22	
[32,13]	$C_8 \rtimes C_4 - \Gamma_3 d_2$	a^8, b^4, aba^5b^{-1}	18	
[32,14]	$C_8 \rtimes C_4 - \Gamma_3 d_1$	$a^8, b^4, abab^{-1}$	18	
[32,15]	$C_4.(C_4 \times C_2) - \Gamma_3 f$	$a^8, b^4a^4, abab^{-1}$	26	
[32,16]	$C_{16} \times C_2$	$a^{16}, b^2, aba^{-1}b$	2	[21]
[32,17]	$C_{16} \rtimes C_2 - \Gamma_2 k$	a^{16}, b^2, aba^7b	10	
[32,18]	$D_{16} - \Gamma_8 a_1$	$a^2, b^2, (ab)^{16}$	1	[21]
[32,19]	$QD_{32} - \Gamma_8 a_2$	a^{16}, b^2, aba^9b	10	
[32,20]	$DC_8 - \Gamma_8 a_3$	$a^{16}, a^8b^2, abab^{-1}$	18	[17]
[32,21]	$C_4 \times C_4 \times C_2$	a^4, b^4, c^2	26	[14]
[32,22]	$C_2 \times ((C_4 \times C_2) \rtimes C_2) - \Gamma_2 c_1$	$a^4, b^2, c^2, d^2, acbac$	18	
[32,23]	$C_2 \times (C_4 \rtimes C_4) - \Gamma_2 c_2$	$a^4, b^4, c^2, abab^{-1}$	26	
[32,24]	$(C_4 \times C_4) \rtimes C_2 - \Gamma_2 f$	$a^4, b^4, c^2, bcb^{-1}a^2c$	26	
[32,25]	$C_4 \times D_4 - \Gamma_2 e_1$	$a^4, b^2, c^2, (bc)^4$	18	
[32,26]	$C_4 \times Q - \Gamma_2 e_2$	$a^4, b^4, a^2b^2, baba^{-1}$	42	
[32,27]	$(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2 - \Gamma_4 a_1$	$a^2, b^2, c^2, d^2, e^2, cecae, dedbe$	6	
[32,28]	$(C_4 \times C_2 \times C_2) \rtimes C_2 - \Gamma_4 b_1$	$a^2, b^2, c^4, d^2, bdbad, (cd)^2$	10	
[32,29]	$(C_2 \times Q) \rtimes C_2 - \Gamma_4 b_2$	$a^2, b^2, c^4, c^2d^2, bdbad^{-1}, cdcd^{-1}$	26	
[32,30]	$(C_4 \times C_2 \times C_2) \rtimes C_2 - \Gamma_4 c_1$	$a^4, b^2, c^2, d^2, adba^{-1}d, cdca^2d$	18	
[32,31]	$(C_4 \times C_4) \rtimes C_2 - \Gamma_4 c_1$	$a^4, b^4, c^2, (ac)^2, bcba^2c$	22	
[32,32]	$(C_2 \times C_2).(C_2 \times C_2 \times C_2) - \Gamma_4 c_3$	$a^4, b^4, c^2b^2a^2, acac^{-1}, bcba^2c^{-1}$	42	
[32,33]	$(C_4 \times C_4) \rtimes C_2 - \Gamma_4 d$	$a^4, b^4, c^2, acab^2c, bcb^{-1}a^2c$	34	
[32,34]	$(C_4 \times C_4) \rtimes C_2 - \Gamma_4 a_2$	$a^4, b^4, c^2, (ac)^2, (bc)^2$	10	
[32,35]	$C_4 \rtimes Q - \Gamma_4 a_3$	$a^4, b^4, c^2a^2, acac^{-1}$	42	
[32,36]	$C_8 \times C_2 \times C_2$	a^8, b^2, c^2	18	[14]
[32,37]	$C_2 \times (C_8 \rtimes C_2) - \Gamma_2 d$	a^8, b^2, c^2, aba^3b	18	
[32,38]	$(C_8 \times C_2) \rtimes C_2 - \Gamma_2 g$	$a^8, b^2, c^2, bcba^4c$	18	
[32,39]	$C_2 \times D_8 - \Gamma_3 a_1$	$a^2, b^2, c^2, (bc)^8$	2	[21]
[32,40]	$C_2 \times QD_8 - \Gamma_3 a_2$	a^8, b^2, c^2, aba^5b	18	
[32,41]	$C_2 \times DC_4 - \Gamma_3 a_3$	$a^8, c^2, b^2a^4, abab^{-1}$	26	
[32,42]	$(C_8 \times C_2) \rtimes C_2 - \Gamma_3 b$	$a^8, b^2, c^2, (ac)^2, bcba^4c$	10	
[32,43]	$(C_2 \times D_4) \rtimes C_2 - \Gamma_6 a_1$	$a^8, b^2, c^2, (ab)^2, aca^3c$	6	
[32,44]	$(C_2 \times Q) \rtimes C_2 - \Gamma_6 a_2$	$a^8, b^2, c^2a^4, aba^3b, acac^{-1}$	18	
[32,45]	$C_4 \times C_2 \times C_2 \times C_2$	a^4, b^2, c^2, d^2	26	[14]
[32,46]	$D_2 \times D_4 - \Gamma_2 a_1$	$a^2, b^2, c^2, d^2, (ab)^2, (cd)^4$	10	[10]
[32,47]	$C_2 \times C_2 \times Q - \Gamma_2 a_2$	$a^4, c^2, d^2, b^2a^2, abab^{-1}$	34	
[32,48]	$C_2 \times ((C_4 \times C_2) \rtimes C_2) - \Gamma_2 b$	$a^4, b^2, c^2, d^2, bcba^2c$	14	
[32,49]	$(C_2 \times D_4) \rtimes C_2 - \Gamma_5 a_1$	$a^4, b^2a^2, c^2a^2, d^2a^2, abab^{-1}, cdcd^{-1}$	10	
[32,50]	$(C_2 \times Q) \rtimes C_2 - \Gamma_5 a_2$	$a^2, b^4, c^2, d^2b^2, acab^2c, (bc)^2, adab^2d^{-1}$	22	
[32,51]	$C_2 \times C_2 \times C_2 \times C_2 \times C_2$	a^2, b^2, c^2, d^2, e^2	18	[14]

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TABLE 2. Groups of order 36.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[36,1]	DC_9	$a^4, b^9, baba^{-1}$	20	[17]
[36,2]	C_{36}	a^{36}	1	[21]
[36,3]	$(C_2 \times C_2) \rtimes C_9$	$a^2, b^2, c^9, [a,b], c^{-1}acb, c^{-1}bcba$	7	
[36,4]	D_{18}	$a^2, b^2, (ab)^{18}$	1	[21]
[36,5]	$C_{18} \times C_2$	a^{18}, b^2	2	[21]
[36,6]	$C_3 \times DC_3$	$a^{12}, b^3, baba^{-1}$	17	
[36,7]	$(C_3 \times C_3) \rtimes C_4$	$a^4, b^3, c^3, baba^{-1}, caca^{-1}$	41	
[36,8]	$C_{12} \times C_3$	a^{12}, b^3	23	[14]
[36,9]	$(4,4\|2,3)$	$a^4, b^4, (ab)^2, (ab^{-1})^3$	5	[6]
[36,10]	$D_3 \times D_3$	$a^2, b^2, c^2, d^2, (ab)^3, (cd)^3$	5	[10]
[36,11]	$C_3 \times A_4$	a^3, [+ (1 2)(3 4), (1 2 3)]	14	[11]
[36,12]	$C_6 \times D_3$	$a^3, b^2, c^2, (bc)^3$	14	[9]
[36,13]	$C_2 \times ((C_3 \times C_3) \rtimes C_2)$	$a^2, b^3, c^3d^2, (ba)^2, (ca)^2$	11	
[36,14]	$C_6 \times C_6$	a^6, b^6	26	[14]

TABLE 3. Groups of order 39.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[39,1]	$C_{13} \rtimes C_3$	$a^3, b^{13}, bab^{10}a^{-1}$	15	
[39,2]	C_{39}	a^{39}	1	[21]

4. Symmetric crosscap number of groups of order 33 to 47

In some orders, the only groups are the cyclic and the dihedral. The tables corresponding to those orders have been omitted. In orders 36, 40, 42 and 44 (Tables 2–7) we have a great variety of groups in which the study has been done in detail.

For example, in order 36 we have the group [36,7], which has algebraic structure $(C_3 \times C_3) \rtimes C_4$. This group has symmetric crosscap number 41. A presentation of this group is given by generators a, b, c and relations $a^4 = b^3 = c^3 = 1, ba = ab^2$ and $ca = ac^2$, given by GAP. For the NEC group (0; +; [3,4,4]; {(–)}), with reduced area $13/12$, we have that an associated epimorphism $\theta : \Lambda \to G$ is given by

$$\theta(x_1) = b, \quad \theta(x_2) = abc, \quad \theta(x_3) = a, \quad \theta(e_1) = a^{-1}c^{-1}a^{-1}, \quad \theta(c_{1,0}) = a^2.$$

Clearly, the elements x_3, x_1 and $x_1^2x_3^3x_2$ have as images the generators a, b, c, respectively. All of them are elements that preserve the orientation, so we have proved that it is a group that acts on a nonorientable surface.

5. Symmetric crosscap number of groups of order 48

There are 52 different groups of order 48 (Table 8), and the study of these groups is difficult in general. The real genus of these groups was obtained in [13], where a presentation of each group appears, but in some cases we needed to find another presentation.

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TABLE 4. Groups of order 40.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[40,1]	$C_5 \rtimes C_8$	$a^8, b^5, baba^{-1}$	29	
[40,2]	C_{40}	a^{40}	1	[21]
[40,3]	$C_5 \rtimes C_8$	a^8, b^5, bab^3a^{-1}	27	
[40,4]	DC_{10}	$a^4, b^4, c^5, b^2a^2, baba^{-1}, caca^{-1}$	22	[17]
[40,5]	$C_4 \times D_5$	$a^4, b^2, c^2, (bc)^5$	12	[9]
[40,6]	D_{20}	$a^2, b^2(ab)^{20}$	1	[21]
[40,7]	$C_2 \times (C_5 \rtimes C_4)$	$a^4, b^5, c^2, bab^3a^{-1}$	22	
[40,8]	$(C_{10} \times C_2) \rtimes C_2$	$a^{10}, b^2, (aba)^2, (a^{-1}b)^2(ab)^2$	12	
[40,9]	$C_{20} \times C_2$	a^{20}, b^2	2	[21]
[40,10]	$C_5 \times D_4$	$a^5, b^2, c^2, (bc)^4$	12	[9]
[40,11]	$C_5 \times Q$	$a^5, b^4, c^4, b^2c^2, cbcb^{-1}$	30	
[40,12]	$C_2 \times (C_5 \rtimes C_4)$	$a^5, b^4, bxb^{-1}a^3$	12	
[40,13]	$C_2 \times D_{10}$	$a^2, b^2, c^2, (bc)^{10}$	2	[21]
[40,14]	$C_{10} \times C_2 \times C_2$	a^{10}, b^2, c^2	22	[14]

TABLE 5. Groups of order 42.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[42,1]	$(C_7 \rtimes C_3) \rtimes C_2$	$a^7, b^6, b^{-1}aba^2$	9	[2]
[42,2]	$C_2 \times (C_7 \rtimes C_3)$	$a^3, b^7, c^2, bab^5a^{-1}$	23	
[42,3]	$C_7 \times D_3$	a^7, [+ (1 2 3), (1 2)]	13	[9]
[42,4]	$C_3 \times D_7$	$a^3, b^2, c^2, (bc)^7$	13	[9]
[42,5]	D_{21}	$a^2, b^2, (ab)^{21}$	1	[21]
[42,6]	C_{42}	a^{42}	1	[21]

TABLE 6. Groups of order 44.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[44,1]	DC_{11}	$a^4, b^{11}, baba^{-1}$	24	[17]
[44,2]	C_{44}	a^{44}	1	[21]
[44,3]	D_{22}	$a^2, b^2, (ab)^{22}$	1	[21]
[44,4]	$C_{22} \times C_2$	a^{22}, b^2	2	[21]

TABLE 7. Groups of order 45.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[45,1]	C_{45}	a^{45}	1	[21]
[45,2]	$C_{15} \times C_3$	a^{15}, b^3	29	[8]

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We have to indicate that for the group [48,30] the presentation is given in the table by generators (1 2 3) and (1 2 3 4)(5 6 7 8) belonging to $S_4 \times S_4$; see [20].

To illustrate the difficulty of this section, we are going to analyse the group [48,19]. This group has algebraic structure $(C_2 \times DC_3) \rtimes C_2$. The symmetric crosscap number of this group is 26. A presentation of this group is given by generators a, b and relations $a^4 = b^6 = a^{-1}b^{-1}a^2ba^{-1} = a^{-1}b^{-2}ab^{-2} = (a^{-1}b^{-1})^2(ab)^2 = 1$. This presentation is given by GAP, because if we see the presentation in [13] we can observe that there is a misprint in the fourth relation, as the last exponent should be a -2 instead of a 2. For the NEC group (0; +; [4,4]; {(–)}), with reduced area $1/2$, we have that an epimorphism $\theta : \Lambda \to G$ is given by

$$\theta(x_1) = a^{-1}, \quad \theta(x_2) = ab, \quad \theta(e_1) = b^{-1}, \quad \theta(c_{1,0}) = b^3.$$

Let us study the element ab. We have that $(ab)^4 = ab(ab)^2ab = ab(ba)^2ab = ab^2aba^2b = a^2b^{-2}ba^2b = a^2b^{-1}a^2b = a^2b^{-1}ba^2 = 1$; so that ab has order 4. Take the element x_1, whose image is the generator a^{-1}, and the element $x_1^2x_2$ that has as image the generator b. Both elements preserve the orientation. So, we have proved that the group acts on a nonorientable surface.

6. Symmetric crosscap number of groups of order 49 to 63

As in Section 4, some of the orders contain only Abelian or dihedral groups. As in the previous section, these tables have been omitted. The rest can be found in Tables 9–17. As an example, in order 54 we have the group [54,13], which has algebraic structure $C_3 \times ((C_3 \times C_3) \rtimes C_2)$. The symmetric crosscap number of this group is 29. A presentation of this group is given by generators a, b, c and d and relations $a^3 = b^3 = c^2 = d^3 = 1, ab = ba, cac^{-1} = a^2, cbc^{-1} = b^2$ and d commutes with the rest of the generators. We have two NEC groups.

(i) For the NEC group (0; +; [6]; {(3,3)}), with reduced area $1/2$, we have that an associated epimorphism $\theta : \Lambda \to G$ is given by

$$\theta(x_1) = dc, \quad \theta(e_1) = cd^{-1}, \quad \theta(c_{1,0}) = ca, \quad \theta(c_{1,1}) = cb, \quad \theta(c_{1,2}) = ca^{-1}.$$

We have that $x_1^3, x_1^4, c_{1,0}c_{1,2}$ and $c_{1,0}c_{1,2}c_{1,0}c_{1,1}$ have as images the generators c, d, a and b, respectively. So, we have that the group generated by images of elements that preserve the orientation is a group that acts on a nonorientable surface.

(ii) For the NEC group (0; +; [2,3]; {(3)}), with the same reduced area as before, we have that an associated epimorphism $\theta : \Lambda \to G$ is given by

$$\theta(x_1) = c, \quad \theta(x_2) = da, \quad \theta(e_1) = a^{-1}d^{-1}c, \quad \theta(c_{1,0}) = bac, \quad \theta(c_{1,1}) = cb.$$

We have that $\theta(x_1) = c$, $\theta(x_1c_{1,1}) = b$, $\theta((x_1c_{1,1})^2c_{1,0}c_{1,1}(x_1c_{1,1})^2) = a$ and $\theta(x_2((x_1c_{1,1})^2c_{1,0}c_{1,1}(x_1c_{1,1})^2)^2) = d$. So, all the generators are in the image of θ. On the other hand, the nonorientable element $(x_1c_{1,1})^3$ has as image the identity element. So, we have proved that it is a group that acts on a nonorientable surface.

Note that the algebraic structure of the group [60,1], $C_5 \times DC_3$, that was studied in [11] can be expressed as $C_3 \rtimes C_{20}$. Hence, this group belongs to the family $C_3 \rtimes C_{8k+12}$, whose symmetric crosscap numbers fill the numbers of the form $24k + 15$; see [4].

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Table 8. Groups of order 48.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[48,1]	$C_3 \rtimes C_{16}$	$a^{16}, b^3, baba^{-1}$	31	
[48,2]	C_{48}	a^{48}	1	[21]
[48,3]	$(C_4 \times C_4) \rtimes C_3$	$a^3, b^3, (ab)^3, (a^{-1}b)^4$	18	
[48,4]	$C_8 \times D_3$	$a^8, b^2, c^2, (bc)^3$	18	[9]
[48,5]	$C_{24} \rtimes C_2$	$a^2, b^8, abbab^6, (b^{-1}a)^3ba(b^{-1}a)^2$	20	
[48,6]	$C_{24} \rtimes C_2$	$a^{24}, b^2, baba^{13}$	14	
[48,7]	D_{24}	$a^2, b^2, (ab)^{24}$	1	[21]
[48,8]	DC_{12}	$a^{24}, a^{12}b^2, b^{-1}aba$	26	[17]
[48,9]	$C_2 \times (C_3 \rtimes C_8)$	$a^2, b^8, c^8, (b^3c)^3$	34	
[48,10]	$(C_3 \rtimes C_8) \rtimes C_2$	$a^8, b^{-1}a^2b^{-1}, (b^{-1}aba^{-1}b^{-1}a^{-1})^2$	36	
[48,11]	$C_4 \times DC_3$	$a^6, c^4, a^3b^2, b^{-1}aba$	26	[11]
[48,12]	$DC_3 \rtimes C_4$	$a^4, b^4, a^{-1}b^{-1}a^2ba^{-1}, a^{-1}b^2ab^2,$ $(a^{-1}b^{-1})^3(ab)^2a^{-1}b$	26	
[48,13]	$C_{12} \rtimes C_4$	$a^4, b^{12}, a^{-1}b^{-1}ab^{-1}$	26	
[48,14]	$(C_{12} \times C_2) \rtimes C_2$	$a^3, b^4, c^4, (bc)^2, (b^{-1}c)^2, c^{-1}aca$	14	
[48,19]	$(C_2 \times DC_3) \rtimes C_2$	$a^4, b^6, a^{-1}b^{-1}a^2ba^{-1}, a^{-1}b^{-2}ab^{-2}, (a^{-1}b^{-1})^2(ab)^2$	26	
[48,20]	$C_{12} \times C_4$	a^{12}, b^4	34	[14]
[48,21]	$C_3 \times (4, 4\|2, 2)$	$a^3, b^4, c^4, (bc)^2, (b^{-1}c)^2$	14	
[48,22]	$C_3 \times (C_4 \rtimes C_4)$	$a^3, b^4, c^4, c^{-1}bcb$	34	
[48,23]	$C_{24} \times C_2$	a^{24}, b^2	2	[21]
[48,24]	$C_3 \times (C_8 \rtimes C_2)$	a^8, b^2, c^3baba^3	14	
[48,25]	$C_3 \times D_8$	$a^3, b^2, c^2, (bc)^8$	18	[9]
[48,26]	$C_3 \times QD_8$	$a^8, b^2, c^3, baba^5$	20	
[48,27]	$C_3 \times DC_4$	$a^8, c^3, a^4b^2, abab^{-1}$	34	
[48,28]	$SL(2, 3).C_2$	$a^3, b^3, c^2, (ab)^2, cacb^{-1}$	22	
[48,29]	$GL(2, 3)$	$a^2, b^3, c^3, (bc)^4, (ab)^2, (ac)^2, [b, c](c^{-1}b^{-1})^2$	10	
[48,30]	$A_4 \rtimes C_4$	[+(1 2 3), (1 2 3 4)(5 6 7 8)]	22	
[48,31]	$C_4 \times A_4$	a^4, [+ (1 4)(3 2), (1 2 3)]	10	[11]
[48,32]	$C_2 \times <2, 3, 3>$	$a^3, b^3, c^2, abab^{-1}a^{-1}b^{-1}, (aba)^4$	22	
[48,33]	$SL(2, 3) \rtimes C_2$	$a^2, b^3, c^3, (bc)^4, abac, [b, c]^2(c^{-1}b^{-1})^2$	10	
[48,34]	$C_2 \times DC_6$	$a^{12}, c^2, a^6b^2, abab^{-1}$	38	
[48,35]	$C_4 \times D_6$	$a^4, b^2, c^2, (bc)^6$	26	[9]
[48,36]	$C_2 \times D_{12}$	$a^2, b^2, c^2, (bc)^{12}$	2	[21]
[48,37]	$(C_{12} \times C_2) \rtimes C_2$	$a^3, b^2, c^2, d^2, dcbcdb, dcbdbc, bdcdbc,$ $(ba)^2, (da)^2$	14	
[48,38]	$D_4 \times S_3$	$a^2, b^2, c^2, d^2, (ab)^4, (cd)^3$	8	
[48,39]	$(C_2 \times DC_3) \rtimes C_2$	$a^2, b^4, c^{-1}b^2c^{-1}, b^{-1}aba, b^{-1}c^{-1}abc^{-1}a,$ $(b^{-1}c)^2ac^{-1}b^{-1}c^{-1}bcac^{-1}$	26	
[48,40]	$Q \times D_3$	$a^2, b^2, c^4, d^4, (ab)^3, c^2d^2, dcdc^{-1}$	38	
[48,41]	$(C_4 \times D_3) \rtimes C_2$	$a^4, b^2, c^2, a^{-1}ba^2ba^{-1}, a^{-1}ca^2ca^{-1},$ $(a^{-1}cb)^2, (a^{-1}b)^2(ac)^2, c(ba^{-1})^2bcb$	18	
[48,42]	$C_2 \times C_2 \times DC_3$	$a^6, c^2, d^2, a^3b^2, abab^{-1}$	38	
[48,43]	$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	$a^4, b^6, c^2, (ab)^2, (a^{-1}b)^2$	14	
[48,44]	$C_{12} \times C_2 \times C_2$	a^{12}, b^2, c^2	26	[14]
[48,45]	$C_6 \times D_4$	$a^6, b^2, c^2, (bc)^4$	26	[9]
[48,46]	$C_6 \times Q$	$a^6, b^4, c^4, b^2c^2, cbcb^{-1}$	38	
[48,47]	$C_3 \times ((C_4 \times C_2) \rtimes C_2)$	$a^2, b^2, c^2, d^3, (ab)^4, (bc)^4, abcacb,$ $abcbac, bcabac$	26	
[48,48]	$C_2 \times S_4$	a^2, [+ (1 2), (1 2 3 4)]	4	[6]
[48,49]	$C_2 \times C_2 \times A_4$	a^2, b^2, [+ (1 2)(3 4), (1 2 3)]	14	
[48,50]	$(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3$	$a^2, b^3, c^3, (cb)^2, (ab^{-1})^3,$ $c^{-1}b^{-1}abca, cbab^{-1}c^{-1}a$	10	
[48,51]	$D_2 \times D_6$	$a^2, b^2, c^2, d^2, (ab)^2, (cd)^6$	14	[10]
[48,52]	$C_6 \times C_2 \times C_2 \times C_2$	a^6, b^2, c^2, d^2	38	[14]

TABLE 9. Groups of order 49.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[49,1]	C_{49}	a^{49}	1	[21]
[49,2]	$C_7 \times C_7$	a^7, b^7	37	[8]

TABLE 10. Groups of order 50.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[50,1]	D_{25}	$a^2, b^2, (ab)^{25}$	1	[21]
[50,2]	C_{50}	a^{50}	1	[21]
[50,3]	$C_5 \times D_5$	$a^5, b^2, c^2, (bc)^5$	17	[9]
[50,4]	$(C_5 \times C_5) \rtimes C_2$	$a^2, b^5, c^5, (ba)^2, (ca)^2$	17	
[50,5]	$C_5 \times C_{10}$	a^5, b^{10}	37	[14]

TABLE 11. Groups of order 52.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[52,1]	DC_{13}	$a^{26}, a^{13}b^2, b^{-1}aba$	28	[17]
[52,2]	C_{52}	a^{52}	1	[21]
[52,3]	$C_{13} \rtimes C_4$	a^4, b^{13}, bab^8a^{-1}	13	
[52,4]	D_{26}	$a^2, b^2, (ab)^{26}$	1	[21]
[52,5]	$C_{26} \times C_2$	a^{26}, b^2	2	[21]

TABLE 12. Groups of order 54.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[54,1]	D_{27}	$a^2, b^2, (ab)^{27}$	1	[21]
[54,2]	C_{54}	a^{54}	1	[21]
[54,3]	$C_3 \times D_9$	$a^3, b^2, c^2, (bc)^9$	17	[9]
[54,4]	$C_9 \times D_3$	$a^9, b^2, c^2, (bc)^3$	17	[9]
[54,5]	$(C_3 \times C_3) \rtimes C_3) \rtimes C_2$	$a^3, b^6, (ab)^2, (ba^{-1}b)^3$	11	
[54,6]	$(C_9 \rtimes C_3) \times C_2$	$a^2, b^9, c^3, (ba)^2, cbb^6c^{-1}b^{-1}$	17	
[54,7]	$(C_9 \times C_3) \rtimes C_2$	$a^2, b^3, c^9, (ba)^2, (ca)^2$	17	
[54,8]	$((C_3 \times C_3) \rtimes C_3) \rtimes C_2$	$a^2, b^3, c^2, (b^{-1}a)^2, (ca)^3, (b^{-1}c)^2(bc)^2,$ $(ab^{-1}c)^2bac$	11	
[54,9]	$C_{18} \times C_3$	a^{18}, b^3	35	[14]
[54,10]	$C_2 \times ((C_3 \times C_3) \rtimes C_3)$	$a^3, b^3, c^3, d^2, bac^{-1}b^{-1}a^{-1}$	29	
[54,11]	$C_2 \times (C_9 \rtimes C_3)$	$a^3, b^9, c^2, bab^5a^{-1}$	35	
[54,12]	$C_3 \times C_3 \times D_3$	$a^3, b^3, c^2, d^2, (cd)^3$	29	
[54,13]	$C_3 \times ((C_3 \times C_3) \rtimes C_2)$	$a^3, b^3, c^2, d^3, (ca)^2, (cb)^2$	29	
[54,14]	$(C_3 \times C_3 \times C_3) \rtimes C_2$	$a^2, b^3, c^3, d^3, (ba)^2, (ca)^2, (da)^2$	29	
[54,15]	$C_6 \times C_3 \times C_3$	a^6, b^3, c^3	65	[14]

TABLE 13. Groups of order 55.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[55,1]	$C_{11} \rtimes C_5$	a^5, b^{11}, bab^7a^{-1}	35	
[55,2]	C_{55}	a^{55}	1	[21]

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TABLE 14. Groups of order 56.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[56,1]	$C_7 \rtimes C_8$	$a^8, b^7, baba^{-1}$	43	
[56,2]	C_{56}	a^{56}	1	[21]
[56,3]	$C_7 \rtimes Q$	$a^4, b^4, c^7, b^2a^2, baba^{-1}, caca^{-1}$	30	
[56,4]	$C_4 \times D_7$	$a^4, b^2, c^2, (bc)^7$	16	[9]
[56,5]	D_{28}	$a^2, b^2(ab)^{28}$	1	[21]
[56,6]	$C_2 \times (C_7 \rtimes C_4)$	$a^4, b^7, c^2, baba^{-1}$	30	
[56,7]	$(C_{14} \times C_2) \rtimes C_2$	$a^2, b^{14}, (bab)^2, (b^{-1}a)^2(ba)^2$	16	
[56,8]	$C_{28} \times C_2$	a^{28}, b^2	2	[21]
[56,9]	$C_7 \times D_4$	$a^7, b^2, c^2, (bc)^4$	16	[9]
[56,10]	$C_7 \times Q$	$a^7, b^4, c^4, b^2c^2, cbcb^{-1}$	42	
[56,11]	$(C_2 \times C_2 \times C_2) \rtimes C_7$	$a^7, b^2, c^2, d^2, badca^{-1}, caba^{-1}, daca^{-1}$	8	
[56,12]	$C_2 \times D_{14}$	$a^2, b^2, c^2, (bc)^{14}$	2	[21]
[56,13]	$C_{14} \times C_2 \times C_2$	a^{14}, b^2, c^2	30	[14]

TABLE 15. Groups of order 57.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[57,1]	$C_{19} \rtimes C_3$	$a^3, b^{19}, bab^{12}a^{-1}$	21	
[57,2]	C_{57}	a^{57}	1	[21]

TABLE 16. Groups of order 60.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[60,1]	$C_5 \times DC_3$	$a^4, b^3, c^5, baba^{-1}$	39	[11]
[60,2]	$C_3 \times (C_5 \rtimes C_4)$	$a^4, b^5, c^3, baba^{-1}$	42	
[60,3]	$C_{15} \rtimes C_4$	$a^4, b^{15}, a^{-1}b^{-1}ab^{-1}$	32	
[60,4]	C_{60}	a^{60}	1	[21]
[60,5]	A_5		1	[21]
[60,6]	$C_3 \times (C_5 \rtimes C_4)$	$a^4, b^5, c^3, bab^3a^{-1}$	21	
[60,7]	$C_{15} \rtimes C_4$	$a^4, b^6, a^{-1}b^{-2}ab^{-2}, ab^{-1}a^{-2}b^{-1}a^{-1}b$	32	
[60,8]	$D_3 \times D_5$	$a^2, b^2, c^2, d^2, (ab)^3, (cd)^5$	9	
[60,9]	$C_5 \times A_4$	a^5, [+ (1 2 3), (1 4)(2 3)]	13	[11]
[60,10]	$C_6 \times D_5$	$a^6, b^2, c^2, (bc)^5$	22	[9]
[60,11]	$C_5 \times D_6$	$a^5, b^2, c^2, (bc)^6$	22	[9]
[60,12]	D_{30}	$a^2, b^2(ab)^{30}$	1	[21]
[60,13]	$C_{30} \times C_2$	a^{30}, b^2	2	[21]

TABLE 17. Groups of order 63.

GAP	G	Relations [+ Generators]	$\tilde{\sigma}(G)$	Reference
[63,1]	$C_7 \rtimes C_9$	$a^7, b^9, b^{-1}aba^5$	49	
[63,2]	C_{63}	a^{63}	1	[21]
[63,3]	$C_3 \times (C_7 \rtimes C_3)$	$a^3, b^7, c^3, bab^5a^{-1}$	23	
[63,4]	$C_{21} \times C_3$	a^{21}, b^3	41	[8]

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