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ANNIHILATOR GRAPHS AND SEMIGROUPS OF MATRICES

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Matrices provide essential tools in many branches of mathematics and matrix semigroups have applications in various areas. In this paper we give a complete description of all infinite matrix semigroups satisfying a certain combinatorial property defined in terms of annihilator graphs.

1. INTRODUCTION

Research on combinatorial properties of words in groups originates from the following well-known theorem due to Bernhard Neumann [**11**], which was obtained as an answer to a question of Paul Erdős: a group is centre-by-finite if and only if every infinite sequence contains a pair of elements that commute. Combinatorial properties of groups and semigroups with all infinite subsets containing certain special properties have been considered by many authors, including de Luca, Hall, Justin, Kelarev, Okniński, Pirillo and Varricchio (see [**2, 3, 6, 7, 10**]).

The following combinatorial property was introduced in [**9**] using annihilator graphs. The *annihilator graph* $\mathrm{Ann}(S)$ of a semigroup S has all elements of S as vertices and it has edges (u, v) for all $u, v \in S$ such that $uv = 0$ and $u \neq v$. Let D be a directed graph. We say that an infinite semigroup S is *annihilator D-saturated* if and only if, for every infinite subset T of S, the annihilator graph of S has a subgraph isomorphic to D with all vertices in T. In [**8**] a complete description was given of all commutative semigroups that were annihilator D-saturated. A natural question that arises concerns the structure of other classes of semigroups that satisfy this combinatorial property. In this paper we describe all pairs (D, S), where D is a directed graph and S is a matrix semigroup, such that S is annihilator D-saturated.

2. PRELIMINARIES

The reader is referred to [**1, 4**] and [**13**] for standard graph, semigroup and group theoretic terminology, respectively. By the word 'graph' we mean a directed graph without loops or multiple edges. A *complete symmetric graph* K_∞ is a graph such that

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$(u,v) \in E(G)$ and $(v,u) \in E(G)$ for all distinct $u, v \in G$. An *infinite ascending (*respecively *descending) chain* A_∞ (respectively D_∞) is the graph with the set of all positive integers as vertices and with edges (i,j) such that $i < j$ (respectively $i > j$).

We refer to [**12, 14**] for preliminaries on matrix semigroups and fields, respectively. For a skew field K, the set of all $n \times n$ matrices with entries in K is denoted by $M_n(K)$. A *linear semigroup* is a subsemigroup of $M_n(K)$, for some n, K. The semigroup structure of $M_n(K)$ is given in the next lemma, where $GL_j(K)$ is the maximal group of matrices of rank j over K, and $M_j = \{a \in M_n(K) \mid \operatorname{rank}(a) \leqslant j\}$ for $0 \leqslant j \leqslant n$.

Lemma 1. ([**12**, Theorem 2.3].) *The sets*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M_n(K)$$

are the only ideals of the monoid $M_n(K)$. Each Rees factor M_j/M_{j-1} is isomorphic to the completely 0-simple semigroup $\mathcal{M}(GL_j(K), X_j, Y_j, Q_j)$, where the matrix $Q_j = (q_{yx})$ is defined for $x \in X_j, y \in Y_j$, by $q_{yx} = yx$ if yx is of rank j and 0 otherwise.

A matrix is said to be *monomial* if every row and column contains at most one nonzero entry. If G is a group, then the set of all $n \times n$ monomial matrices over $G^0 = G \cup \{0\}$ forms an inverse semigroup denoted by $M_n(G)$. The structure of monomial matrix semigroups is well known. Put

$$M_j = \{s \in M_n(G) \mid s \text{ has at most } j \text{ nonzero entries}\}.$$

Lemma 2. ([**5**].) *Let G be a group. Then $M_n(G)$ is an inverse semigroup with the only ideals*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M_n(G),$$

where $M_j = \{s \mid s \text{ has at most } j \text{ nonzero entries}\}$. Moreover,

$$M_j/M_{j-1} \cong \mathcal{M}\left(G_j, \binom{n}{j}, \binom{n}{j}, \Delta\right),$$

where G_j is an extension of $G^j = G \times \cdots \times G$ by the symmetric group S_j and Δ is the identity matrix. All idempotents of $M_n(G)$ are diagonal and a power of every element is diagonal.

The next result is also needed in the proofs of the main theorems.

Lemma 3. ([**8**, Lemma 4.3].) *Every infinite graph contains an infinite set of vertices which induces a null subgraph, an infinite ascending chain, an infinite descending chain or an infinite complete symmetric graph.*

For a subset T of a semigroup S, let

$$T^* = \{uv \in S \mid u, v \in T, u \neq v\}.$$

3. Main Theorems

Theorem 4. *Let S be an infinite matrix subsemigroup of $M_n(G)$, where $M_n(G)$ has ideal series*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M_n(G)$$

and each factor M_j/M_{j-1} is isomorphic to the completely 0-simple semigroup $\mathcal{M}^0(G_j; I_j, \Lambda_j; P_j)$, for all $1 \leqslant j \leqslant n$. Let G_S be the set of all elements contained in subgroups of S and let D be a finite graph with at least one edge. Then the following statements are equivalent:

(i) *S is annihilator D-saturated;*

(ii) *S is periodic, G_S is finite and $s^2 = 0$, for all but a finite number of elements of S;*

(iii) *for every infinite subset T of S, $0 \in T^\star$.*

The following example shows that for a linear semigroup S the conditions in Theorem 4(ii) are not sufficent to imply that S is annihilator D-saturated.

Example 5. Let $S = \{u_x \mid x \in \mathbb{R}\} \cup \{v_x \mid x \in \mathbb{R}\}$, where

$$u_x = \begin{pmatrix} 0 & x & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } v_x = \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $s^2 = 0$ for all $s \in S$ it is clear that S is periodic and that $G_S = \emptyset$. However, S is not annihilator D-saturated since $u_x u_y \neq 0$ for $x \neq y$.

This leads to the following theorem.

Theorem 6. *Let S be an infinite matrix subsemigroup of $M_n(K)$, where $M_n(K)$ has ideal series*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M_n(K).$$

Let G_S be the set of all elements contained in subgroups of S and let D be a finite graph with at least one edge. Then S is annihilator D-saturated if and only if the following conditions hold:

(i) *S is periodic;*

(ii) *$s^2 = 0$, for all but a finite number of elements of S;*

(iii) *G_S is finite;*

(iv) *for every infinite subset T of S, $0 \in T^\star$.*

4. Proofs

Proof of Theorem 4: (i) $\Rightarrow$ (ii): If S contains an element s of infinite order, then the vertices $s, s^2, s^3, \ldots$ are not adjacent in $\mathrm{Ann}(S)$. Since D has edges, we see that the subgraph induced by the vertices $s, s^2, s^3, \ldots$ does not contain a subgraph isomorphic to D, a contradiction. Thus S is periodic.

Suppose to the contrary that G_S is infinite. Each Rees factor M_j/M_{j-1} contains a finite number of $\mathcal{H}$-classes and hence a finite number of maximal subgroups. Therefore G_S contains an infinite maximal subgroup. The subgraph induced by this subgroup is null and so D does not embed in this subgraph. Therefore S is not annihilator D-saturated, contradicting (i). Thus G_S is finite.

Suppose that the set $T = \{s \in S \mid s^2 \neq 0\}$ is infinite. Then T contains an infinite subset U whose elements all belong to the same $\mathcal{H}$-class of $M_n(G)$. Pick two elements $v, w \in U$. There exists $t \in M_n(G)$ such that $v = wt$. Therefore the element $vwt = v^2 \neq 0$ and so $vw \neq 0$. Similarly, wv is nonzero and so the elements of U induce an infinite null subgraph in $\mathrm{Ann}(S)$. This contradicts (i) again and so we conclude that T is finite.

(ii) $\Rightarrow$ (iii): Take any infinite subset T of S. There exists an infinite subset U of T whose elements all belong to the same $\mathcal{H}$-class of $M_n(G)$. Since the set $\{s \in S \mid s^2 \neq 0\}$ is finite, we can find two elements $v, w \in U$ such that $v^2 = w^2 = 0$. There exists $t \in M_n(G)$ such that $u = wt$. Thus $wv = w^2a = 0$ and so $0 \in T^\star$.

(iii) $\Rightarrow$ (i): Take any infinite subset T of S and consider the infinite subset U of T in (ii) as before. Since $0 \in U^\star$, there exist two distinct elements $v, w \in U$ such that $vw = 0$. For any other pair of elements $x, y \in U$, there exist $t, s \in M_n(G)$ such that $x = tv$ and $y = ws$. Therefore $xy = tvws = 0$. Similarly, $yx = 0$ and so the elements of U induce an infinite complete symmetric subgraph in $\mathrm{Ann}(S)$. The graph D embeds in this subgraph and hence S is annihilator D-saturated. ∎

Proof of Theorem 6: The 'only if' part. Suppose that S is annihilator D-saturated.

The argument used in the proof of Theorem 4(i) demonstrates that S is periodic. Thus (i) holds.

Suppose to the contrary that there exists an infinite subset T of S such that $t^2 \neq 0$, for all $t \in T$. By Lemma 3, there exists a countably infinite subset $U = u_1, u_2, \ldots$ of T such that the subgraph H of $\mathrm{Ann}(S)$ with all vertices of U is null, or isomorphic to either A_∞, D_∞ or K_∞.

If H is null, then S is not annihilator D-saturated, a contradiction.

If $H \cong A_\infty$ or $H \cong K_\infty$, then we may assume that

$$u_i u_j = 0, \tag{1}$$

for all $u_i, u_j \in U$ and $1 \leqslant i < j$. The elements of U are contained in a vectorspace (see [**14**, Section 4.1]) of dimension n^2. Therefore there exists a set $u_{s_1}, u_{s_2}, \ldots, u_{s_{n^2}}$ that

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spans U, where $s_i < s_j$ for $1 \leqslant i < j \leqslant n^2$. Then

$$u_{s_{n^2}+1} = a_1 u_{s_1} + \cdots + a_{n^2} u_{s_{n^2}},$$

where $a_i \in K$. Hence

$$u^2_{s_{n^2}+1} = (a_1 u_{s_1} + \cdots + a_{n^2} u_{s_{n^2}}) u_{s_{n^2}+1}. \tag{2}$$

The left-hand side of (2) is nonzero, since $u_{s_{n^2}+1} \in T$. The right-hand side of (2) is zero, by (1). This contradiction shows that $H \ncong A_\infty$ and $H \ncong K_\infty$.

If $H \cong D_\infty$, then we may assume that $u_i u_j = 0$, for all $1 \leqslant j < i$ and that $u_{s_1}, u_{s_2}, \ldots, u_{s_{n^2}}$ spans U as before. Therefore

$$u_{s_{n^2}+1} = a_1 u_{s_1} + \cdots + a_{n^2} u_{s_{n^2}},$$

for some $a_i \in K$. Hence

$$u^2_{s_{n^2}+1} = u_{s_{n^2}+1}(a_1 u_{s_1} + \cdots + a_{n^2} u_{s_{n^2}}),$$

which yields the same contradiction.

This contradicts Lemma 3 and so we conclude that T is finite. Therefore (ii) holds.

It follows immediately that G_S is finite, since $s^2 \neq 0$, for all $s \in G_S$. Thus (iii) holds.

Take any infinite subset T of S. Again, Lemma 3 implies that T contains an infinite subset W such that the subgraph H of $\mathrm{Ann}(S)$ induced by the vertices of W is null, or isomorphic to A_∞, D_∞ or K_∞. The first possibility is impossible by our hypothesis. In the remainining cases H contains edges. If (a, b) is an edge in H, then $ab = 0$ and so $0 \in T^\star$. Hence (iv) holds.

The 'if' part. Take any infinite subset T of S. Applying Lemma 3, T contains a countably infinite subset $U = u_1, u_2, \ldots$ such that the subgraph H of $\mathrm{Ann}(S)$ induced by the elements of U is null, or isomorphic to A_∞, D_∞ or K_∞.

The first case implies that S is not annihilator D-saturated and so is impossible.

If $H \cong A_\infty$, then we may assume that the elements of U have been indexed such that $u_i u_j = 0$ (and $u_j u_i \neq 0$), for all $1 \leqslant i < j$. Choose a spanning subset $u_{s_1}, u_{s_2}, \ldots, u_{s_{n^2}}$ of U, where $s_i < s_j$ for $1 \leqslant i < j \leqslant n^2$. We get

$$u_{s_{n^2}+2} = a_1 u_{s_1} + \cdots + a_{n^2} u_{s_{n^2}},$$

for some $a_i \in K$. Hence

$$u_{s_{n^2}+2} u_{s_{n^2}+1} = (a_1 u_{s_1} + \cdots + a_{n^2} u_{s_{n^2}}) u_{s_{n^2}+1}. \tag{3}$$

The left-hand side of (3) is nonzero. The right-hand side of (3) is zero. Therefore $H \ncong A_\infty$.

If $H \cong D_\infty$, then a similar argument yields the same contradiction and shows that $H \ncong D_\infty$.

We deduce from Lemma 3 that $H \cong K_\infty$. The graph D embeds in this subgraph. Hence D embeds in $\mathrm{Ann}(S)$ and S is annihilator D-saturated in this case, too. ∎

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