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SOME PROPERTIES OF THE LATTICE OF SUBALGEBRAS OF A BOOLEAN ALGEBRA

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We investigate the structure of the lattice of subalgebras of an infinite Boolean algebra; in particular, we make a contribution to the question as to when such a lattice is simple.

0. Introduction

For a Boolean algebra $(D, +, \circ, \bar{\ }, 0, 1)$, the set Sub D of all subalgebras is an algebraic lattice under set inclusion with least element $2 = \{0,1\}$ and greatest element D . If $A , B \leq D$, then $A \wedge B$ is just $A \cap B$, and $A \vee B$ is the subalgebra of D generated by $A \cup B$.

One of the earliest results in the study of Sub D is the fact that, if D is finite, then Sub D is dually isomorphic to a finite partition lattice, the base set being $At(D)$, the set of all atoms of D , see [1]. Subsequently it was shown by D. Sachs that, for an arbitrary Boolean algebra D , Sub D is dually isomorphic to a sublattice of a partition lattice, and that Sub D characterizes D. Birkhoff's result cited above implies that Sub D is simple, if D is finite.

In this note, the structure of Sub D is investigated further; in particular we make a contribution to the question when Sub D is simple for

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infinite D.

1. Notation

For any lattice L, we set $[x,y) = \{z \in L | x \le z < y\}$, $[x) = \{z \in L | x \le z\}$; other intervals are defined analogously.

Let D be a Boolean algebra; for $M \subseteq D$ we define $M^+ = \{x \in M | x > 0\}$, $-M = \{\overline{x} | x \in M\}$, and $[M]$ to be the subalgebra of D generated by M. If $M = \{x\}$, we just write $[x]$ instead of $[\{x\}]$. For $A \le D$, $x \in D \setminus A$, we call $[A \cup \{x\}]$ a simple extension of A, and denote it by $A(x)$. Note that every element of $A(x)$ is of the form $u \circ x + v \circ \overline{x}$ for some $u, v \in A$.

For $d \in D^+$, $D|d = \{x \in D | x \le d\}$ is the relative algebra of d in D. Note that $D|d$ also is the principal ideal of D generated by d, and we sometimes alternatively write $(d]$ for $D|d$, if we want to emphasize this fact. It is well known that D is isomorphic to $D|d \times D|\overline{d}$; conversely, if D is isomorphic to $A \times B$, then there exists a $d \in D$ such that $A \cong D|d$, and $B \cong D|\overline{d}$.

If C is a linearly ordered set with least element, the set of all finite unions of right closed, left open intervals is a Boolean subalgebra of the power set of C, and denoted by $I(C)$; this algebra is called the interval algebra of C. In an unpublished paper, S. Todorčević [6] has shown that for an interval algebra D, Sub D is sectionally complemented, that is Sub A is complemented for every $A \le D$.

For the remaining unexplained notation and terminology the reader is referred to Grätzer's book [3].

2. General structure of Sub D

A lattice L is called semimodular if, for $x, y \in L$, the fact that x covers $x \wedge y$ implies that $x \vee y$ covers y. Note that every modular lattice is semimodular.

PROPOSITION 2.1. *If* $|D| = 8$, *then* Sub D *is modular; if* $|D| \ge 16$, *then* Sub D *is not semimodular.*

Proof. If $|D| = 8$, then Sub D is easily seen to be a diamond, so it is modular. If $|D| \ge 16$, then D has a subalgebra with four atoms, so, let without loss of generality D be generated by its atoms $\{a, b, c, d\}$, and set $A = [a+c]$, $B = [c+d]$. Then A covers $A \cap B = 2$, but $A \vee B = D$ does not cover B. □

In contrast to this, Sachs [5] has remarked that $(\text{Sub } D)^d$, the dual lattice of Sub D, is semimodular; however, $(\text{Sub } D)^d$ usually is not algebraic.

PROPOSITION 2.2. *If* D *is infinite, then* $(\text{Sub } D)^d$ *is not algebraic.*

Proof. We show that no dual atom of Sub D is dually compact. As noted by Sachs, the dual atoms of Sub D are of the form $I \cup -I$, where I is the intersection of two different prime ideals P_1 and P_2 of D, so, let $A \leq D$ have this form.

Assume that $\{a, b, c\} \subseteq D \backslash A$, and a, b, c are pairwise disjoint. Since $a \circ b = 0$, we suppose, without loss of generality, that $a \in P_1 \backslash P_2$; then $a \circ c = 0$ implies $c \in P_2 \backslash P_1$, and from $a \circ b = c \circ b = 0$ we get $b \in P_1 \cap P_2 \leq A$, a contradiction. Now we choose an element u from $D \backslash A$, and a set $\{u_i \mid i < \omega\}$ of pairwise disjoint elements such that $u_0 = u$, and $u_i \in I$ for $0 < i < \omega$. For each $i < \omega$, let $m_i = u_0 + \ldots + u_i$, and, for $j < \omega$, let $M_j \leq D$ be generated by $\{m_i \mid j \leq i < \omega\}$; then, $\{M_j \mid j < \omega\}$ is a decreasing chain of subalgebras of D with $\cap\{M_j \mid j < \omega\} = 2 \leq A$. Since no m_i is in A, we have $M_j \nleq A$ for all $j < \omega$, which implies that A is not dually compact. □

The next result shows that Sub D is in fact far from being distributive. Call an element a of a lattice L

(i) distributive, if $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$ for all $x, y \in L$

(ii) prime, if $x \wedge y \leq a$ implies $x \leq a$ or $y \leq a$

(iii) irreducible, if $x \wedge y = a$ implies $x = a$ or $y = a$.

If a is prime, it is irreducible, and if L is distributive, the converse also holds.

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PROPOSITION 2.3.

1. Sub D *has no proper distributive or prime elements.*

2. $A \leq D$ *is irreducible if and only if* A *is a dual atom of* Sub D.

Proof.

1. We may suppose that $|D| \geq 8$; let $2 < A < D$, $b \in D\backslash A$, and $c \in A(b)\backslash A$, $c \neq b$. Then $A = A \vee ([b] \cap [c])$, but $c \in A(c) \cap A(b)$, showing that A is not distributive. Since $[c] \cap [b] = 2 \leq A$, and $[c]$, $[b] \nleq A$, A is not prime.

2. The if-part is obvious, so, let $A < D$ be irreducible. First, assume that A is not of the form $I \cup -I$ for some ideal I of D. Then there exist $a \in A$, b_1, $b_2 \in D\backslash A$, such that $b_1 \circ \overline{a} = b_2 \circ a = 0$. Let $x \in A(b_1) \cap A(b_2)$; then there exist $s_1, s_2, t_1, t_2 \in A$, such that $x = s_1 \circ b_1 + s_2 \circ \overline{b_1} = t_1 \circ b_2 + t_2 \circ \overline{b_2}$. So, $a \circ x = a \circ t_2 \circ \overline{b_2}$ $= a \circ t_2 \in A$, and $\overline{a} \circ x = \overline{a} \circ s_2 \circ \overline{b_1} = \overline{a} \circ s_2 \in A$, which together imply that $x \in A$; it follows that $A = A(b_1) \cap A(b_2)$, contradicting the fact that A is irreducible. Thus, let $A = I \cup -I$ for some ideal I of D, and assume that I is not the intersection of two prime ideals; then there exists a $B \leq D$ such that $A \cap B = 2$, and B is generated by its atoms b_1, b_2, b_3. If $x \in A(b_1) \cap A(b_2) \cap A(b_3)$, then there exist r_i, s_i, $t_i \in A$, $1 \leq i \leq 2$, such that

$$x = r_1 \circ b_1 + r_2 \circ b_2 + r_2 \circ b_3 \in A(b_1)$$

$$= s_1 \circ b_2 + s_2 \circ b_3 + s_2 \circ b_1 \in A(b_2)$$

$$= t_1 \circ b_3 + t_2 \circ b_1 + t_2 \circ b_2 \in A(b_3) .$$

Using $A = I \cup -I$, it is straightforward, if somewhat cumbersome, to show that $x \in A$, again contradicting the irreducibility of A. □

The question of the existence of prime ideals in Sub D will be touched on later.

3. Congruences on Sub D

We start with the following easy observation:

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LEMMA 3.1. *Let* θ *be a nontrivial congruence on* Sub D ; *then* $A \equiv 2\ (\theta)$ *for every finite* $A \le D$.

Proof. It is enough to show that $[u] \equiv 2\ (\theta)$ for every $u \in D$. Since θ is nontrivial, there exist A , $B \le D$, such that $A \subsetneq B$, $A \ne B$, and $A \equiv B\ (\theta)$. Let $b \in B \backslash A$; then

$$2 = [b] \cap A \equiv_{\theta} [b] \cap B = [b] .$$

If $u \in D \backslash [b]$, let C be generated by $\{b \circ u, \overline{b} \circ \overline{u}\}$; then

$$2 = [u] \cap C \equiv_{\theta} [u] \cap C(b) = [u] . \quad \square$$

COROLLARY 3.2. Sub D *is subdirectly irreducible, weakly modular, and weakly complemented.*

In the sequel, we shall just write $A \equiv B$, if A is congruent to B modulo the smallest nontrivial congruence on Sub D . Note that 3.2 implies that, if $A \le D$ and Sub A is simple, then $A \equiv 2$; consequently, if $D = D_o \times D_1 \times \ldots \times D_n$, and Sub D_i is simple for each $i \le n$, then Sub D is simple.

Next, we want to give some simple conditions for Sub D to be simple.

PROPOSITION 3.3. *If* $D = A \times A$, *then* Sub D *is simple.*

Proof. Let $E = \{(a,a) \mid a \in A\} \le D$, $u = (0,1) \in D$, $B = A \times 2$, $C = 2 \times A$; then $E(u) = D$, and $B \cap E = C \cap E = 2$, implying B , $C \equiv 2$. On the other hand, $D = B \vee C$, thus, $D \equiv 2$. $\square$

In particular, every homogeneous D has Sub D simple.

PROPOSITION 3.4. *If* $|D| = \lambda \ge \omega$, *and* D *contains a free subalgebra with* λ *generators, then* Sub D *is simple.*

Proof. Choose some $u \in D$, such that both $(u]$ and $(\overline{u}]$ contain an independent set of cardinality λ. Let $\{m_i \mid i < \lambda\}$ be an enumeration of $(u]^+$, $\{b_i \mid i < \lambda\}$ an independent set of elements below $\overline{u}$, and let $F \le D$ be generated by this set; furthermore, let $B_1 \le D$ be generated by $(u]$, that is $B = (u] \cup [\overline{u})$. For each $i < \lambda$ set $c_i = m_i + b_i$, and let $C \le D$ be generated by the c_i . The independence of the b_i then implies that $B_1 \cap C = 2$. Also, $m_i = c_i \circ \overline{b}_i$ and $b_i = c_i \circ \overline{m}_i$, so we have

$B_1 \vee F = B_1 \vee C = F \vee C$. Since F is free, $F \equiv 2$ by the preceding proposition, and therefore $C \equiv F \vee C$. Hence, $2 = B_1 \cap C \equiv B_1 \cap (F \vee C) = B_1 \cap (B_1 \vee C) = B_1$. By symmetry, we find that $B_2 = (\bar{u}] \cup [u)$ also is congruent to 2 . Since $D = B_1 \vee B_2$, Sub D is simple. □

In particular, if D is complete, Sub D is simple by the theorem of Balćar and Fraňek.

If $A \leq D$, $u \in D$, call u independent of A , if $a \circ u > 0$ and $a \circ \bar{u} > 0$ for all $a \in A^+$. Note that this is equivalent to $A \cap (u] = A \cap (\bar{u}] = \{0\}$.

PROPOSITION 3.5. *If D is the free product of A and B , then* Sub D *is simple.*

Proof. This follows from the simple fact, that $C \leq D$, $u \in D$ independent of C , imply $C \equiv 2$: Indeed, independence implies that for $E_1 = (u] \cup [\bar{u})$, $E_2 = (\bar{u}] \cup [u)$, we have $C \cap E_1 = C \cap E_2 \equiv 2$; since $C \equiv C(u)$, this gives us $C(u) \cap E_1 \equiv 2$ and $C(u) \cap E_2 = 2$. On the other hand, $C \leq (C(u) \cap E_1) \vee (C(u) \cap E_2)$. If $c \in C$, then $c \circ u \in C(u) \cap E_1$, and $c \circ \bar{u} \in C(u) \cap E_2$. This shows $C \equiv 2$. For the rest, observe that each $b \in B \setminus 2$ is independent of A and vice versa. □

Now, let us turn to conditions which ensure us that Sub D is not simple. Each ideal I of Sub D induces an equivalence relation θ_I on Sub D , if we let $A \equiv B$ (θ_I) if there exists a $C \in I$ such that $A \vee C = B \vee C$. Clearly, θ_I is a $\vee$-congruence on Sub D . If I is a distributive element of the lattice of ideals of Sub D , then θ_I is a lattice congruence on Sub D , see [3] III.3.4. For each cardinal γ , $\omega \leq \gamma \leq \lambda = |D|$, let $I_\gamma = \{A \leq D \mid |A| < \gamma\}$, and θ_γ be the relation defined above. The following lemma simplifies later considerations.

LEMMA 3.6. *Let $|D| = \lambda$; for $\omega \leq \gamma \leq \lambda$, θ_γ is a congruence if and only if the following condition holds:*

If A , B , $C \leq D$, $|C| < \gamma$, and $B \leq A \vee C$, then there exists an

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$S \in I_\gamma$, $(A \cap B) \vee S = B \vee S$.

Moreover, if γ *is regular,* C *can be assumed to have only four elements.*

Proof. We only show sufficiency, and it is enough to prove that $A \equiv B\ (\theta_\gamma)$ and $Q \leq D$ imply $Q \cap A \equiv Q \cap B\ (\theta_\gamma)$. Let, for some $C \in I_\gamma$, $A \vee C = B \vee C$, and set $Q_1 = Q \cap (A \vee C) = Q \cap (B \vee C)$; then $Q_1 \cap A = Q \cap A$, and $Q_1 \cap B = Q \cap B$, so we can suppose, without loss of generality, that $Q \leq A \vee C = B \vee C$. By the condition, there exist S_1, $S_2 \in I_\gamma$, such that $(Q \cap A) \vee S_1 = Q \vee S_1$, and $(Q \cap B) \vee S_2 = Q \vee S_2$; hence, $(Q \cap A) \vee S_1 \vee S_2 = (Q \cap B) \vee S_1 \vee S_2$, and $S_1 \vee S_2 \in I_\gamma$.

For the second part, let $C = \{c_i | i<\delta\}$, $\delta < \gamma$, and $Q \leq A \vee C$. Set $A_o = A$, $A_{\alpha+1} = A_\alpha(c_\alpha)$, and $A_\alpha = \cup\{A_\beta | \beta<\alpha\}$, if α is a limit. Then $A \vee C = \cup\{A_\alpha | \alpha<\delta\}$. For $i < \delta$, set $Q_i = Q \cap A_i$; then, $Q = \cup\{Q_i | i<\delta\}$. It suffices to show that for each $i < \delta$ there exists an $S_i \in I_\gamma$, such that $Q_i \vee S_i = (Q \cap A) \vee S_i$. Set $S = [\cup\{S_i | i<\delta\}]$; then $Q \vee S = (Q \cap A) \vee S$, and $S \in I_\gamma$ by the regularity of γ. Let $i = 0$; then $Q_o = Q \cap A_o = Q \cap A$, and we set $S_o = 2$. Suppose that for all $\alpha < \beta < \delta$ we have $Q_\alpha \vee S_\alpha = (Q \cap A) \vee S_\alpha$, $S_\alpha \in I_\gamma$. If β is a limit, set $S_\beta = [\cup\{S_\alpha | \alpha<\beta\}]$, and note that $Q_\beta = \cup\{Q_\alpha | \alpha<\beta\}$. So, let $\beta = \alpha + 1$; then, $Q_\beta = Q \cap A_{\alpha+1} = Q \cap A_\alpha(c_\alpha)$; thus, by our hypothesis, there exists a $T \in I_\gamma$ which satisfies $(Q \cap A_\alpha) \vee T = (Q \cap A_\alpha(c_\alpha)) \vee T$. By the induction hypothesis there exists an $S_\alpha \in I_\gamma$ satisfying $(Q \cap A_\alpha) \vee S_\alpha = (Q \cap A) \vee S_\alpha$; now set $S_\beta = S_\alpha \vee T$. □

The proof of the following easy lemma is left to the reader.

LEMMA 3.7. *Let* D *be infinite and* Sub D *be sectionally complemented; if* $\omega \leq \gamma \leq |D|$, *then* θ_γ *is a congruence if and only if the following condition holds:*

If $C \in I_\gamma$, *then every* $Q \leq A \vee C$ *disjoint from* A *has cardinality less than* γ.

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Our next aim is to describe the congruence lattice of Sub D for $D = FC(\alpha)$, the finite-cofinite algebra with α atoms. Note that $FC(\alpha)$ is a subalgebra of an interval algebra, hence its lattice of subalgebras is sectionally complemented.

PROPOSITION 3.8. *Let* $D = FC(\alpha)$ *, where* $\alpha = \aleph_\gamma$ *. Then the congruences of* Sub D *form a chain of type* $\gamma + 3$ *, if* $\gamma < \omega$ *, and of type* $\gamma + 2$ *otherwise.*

Proof. We first show that for all β , $\omega \le \beta \le \alpha$, the ideal I_β has the property of 3.7; so, together with the two improper congruences, the θ_β form a chain of the desired type. Afterwards we proceed to show that every proper congruence on Sub D is of the form θ_β for some infinite $\beta \le \alpha$.

Let $C \in I_\beta$ and assume the existence of A , $B \le D$, such that $B \le A \vee C$, $|B| = \beta$, and $A \cap B = 2$. Let $\mathrm{At}(B) = \{b_i | i<\beta\}$ be the set of atoms of B , and suppose $b_i = a_1^i \circ c_1^i + \ldots + a_{\tau(i)}^i \circ c_{\tau(i)}^i$, where $a_j^i \in A$ and $c_j^i \in C$. If c_j^i is cofinite, then $a \circ c_j^i \notin A$ for only finitely many atoms of A . Since there are only less than β elements c_j^i , we may suppose, without loss of generality, that each b_i has the form $b_i = a_1^i + a_2^i \circ c_2^i + \ldots + a_{s(i)}^i \circ c_{s(i)}^i$, where each c_j^i is finite.

Now we set $s_i = a_2^i \circ c_2^i + \ldots + a_{\tau(i)}^i \circ c_{\tau(i)}^i$; since, for each $i < \beta$, b_i is not an element of A , we must have $s_i > 0$. On the other hand, each c_k^i is finite, and there are only less than β such c_k^i , so, we must have $s_i = s_j$ for some i , $j < \beta$. This contradicts $b_i \circ b_j = 0$.

Next, let ψ be a nontrivial congruence on Sub D , and consider the property

(*) If $A \le D$, $|A| = \lambda \ge \omega$, and $A \equiv 2$ (ψ) , then $B \equiv 2$ (ψ) for all $B \in I_{\lambda^+}$.

We show this by induction. Call an atom m of A proper, if m is an atom of D ; otherwise call m improper.

(a) Let $|A| = \omega$, and suppose that A has infinitely many proper atoms

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$\{m_i | i<\omega\}$; since $A \equiv 2$ (ψ) , we may assume that A is generated by these atoms. Let $|B| = \omega$; then all atoms of B are finite. Let $\{c_i | i<\omega\}$ be the set of atoms of D which are $\leq$ some atom of B , but not in A , and let C be generated by the c_i . If C is finite, then $B = A \vee Q$ for some finite $Q \leq D$, and thus $B \equiv 2$ (ψ) . Thus, let $c_i \neq c_j$ for $i \neq j$. Clearly, $A \cap C = 2$ and $A \vee B \leq A \vee C$. For each $i < \omega$, let $q_i = m_i + c_i$, and let Q be generated by the q_i . Then, $Q \cap A = Q \cap C = 2$, and $Q \vee A = Q \vee C = A \vee C$. This implies $Q \equiv C$ (ψ) and it follows that $C \equiv 2$ (ψ) , observing that $Q \cap C = 2$. Since $B \leq A \vee C$, we have $B \equiv 2$ (ψ) .

(b) Let $|A| = \omega$, and A be generated by the improper atoms $\{m_i | i<\omega\}$ For each $i < \omega$, let $m_i = x_i + y_i$, where x_i is an atom of D , and $y_i = \bar{x}_i \circ m_i$. Let Q be generated by the x_i , and R be generated by the y_i ; then, as before, $A \cap Q = A \cap R = R \cap Q = 2$, and $A \vee Q = A \vee R = Q \vee R$. This implies $Q \equiv 2$ (ψ) , and we can proceed as in (a) with Q instead of A , noting that all atoms of Q are proper.

Now suppose that (*) holds for all $\kappa < \lambda \leq \alpha$, and let A , $B \leq D$ such that $|A| = |B| = \lambda$, and $A \equiv 2$ (ψ) .

(c) A is generated by the λ proper atoms $\{m_i | i<\lambda\}$. Let $\{c_i | i<\delta\}$ be the set of all atoms of D which are below some atom of B , but not in A , and let C be generated by the c_i . If $\delta < \lambda$, then $C \equiv 2$ (ψ) by our induction hypothesis, and thus $B \equiv 2$ (ψ) , since $B \leq A \vee C$. If $\delta = \lambda$, proceed as in (a).

(d) A has less than λ proper atoms. Construct an algebra Q with λ proper atoms and $Q \equiv 2$ (ψ) similar to (b); then proceed as in (c). This proves that (*) holds for all $\lambda \leq \alpha$.

Now let λ be the smallest cardinal such that $|E| = \lambda$ implies $E \not\equiv 2$ (ψ) for all $E \leq D$. Let A , $B \leq D$, $A \subseteq B$, and $A \equiv B$ (ψ) ; let C be a complement of A in Sub B ; then $2 = A \cap C \equiv_\psi B \cap C = C$, and thus, $|C| < \lambda$ by our definition of λ . This implies $A \equiv B$ (θ_λ) .

For the converse, let $A \equiv B$ (θ_λ) , $A \subseteq B$, and C a complement of A in Sub B . Since $A \equiv B$ (θ_λ) , we have $|C| < \lambda$, and hence, $C \equiv 2(\psi)$ by

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(*) and our choice of λ . It follows that $A \equiv_\psi A \vee C = B$. □

Call a Boolean algebra D λ-like, if for all $d \in D$, $D|d$ or $D|\bar{d}$ has cardinality less than λ , that is the set $\{d \in D \mid D|d$ has cardinality less than $\lambda\}$ is a prime ideal of D . If, for example, D is the interval algebra of an infinite cardinal λ , then D is λ-like. The only countable ω-like algebra is $FC(\omega)$; more generally, it can be shown that an infinite Boolean algebra D is ω-like if and only if D is a finite-cofinite algebra.

PROPOSITION 3.9. *Let* $|D| = \lambda \geq \omega$, λ *regular, and* D *be* λ*-like. Then* Sub D *is not simple.*

Proof. By 3.6, it suffices to show that D has the following property:

(*) If A , $B \leq D$, $u \in D$ such that $|A| = |B| = \lambda$, and $B \leq A(u)$, then there exists a $C \leq D$ with $|C| < \lambda$, such that $(A \cap B) \vee C = B \vee C$.

So, let A , B , and u be as described above, and suppose, without loss of generality, that $(\bar{u}]$ has cardinality less than λ . Using this fact and the regularity of λ , we may suppose, after a simple thinning process, that there exists a $q \in A$, and, if B is generated by $\{b_i | i<\lambda\}$, for each $i < \lambda$ there exists an $a_i \in A$ satisfying $b_i = a_i \circ u + q \circ \bar{u}$; furthermore, we may assume that for all i , $j < \lambda$, $a_i \circ \bar{u} = a_j \circ \bar{u}$. Then, for i , $j < \lambda$,

$$\begin{aligned} b_i \circ \bar{b}_j &= (a_i \circ u + q \circ \bar{u}) \circ (\overline{a_j} + \bar{u}) \circ (\bar{q} + u) \\ &= a_i \circ \overline{a_j} \circ u \\ &= a_i \circ \overline{a_j} \in A \text{ , since } a_i \circ \bar{u} = a_j \circ \bar{u} . \end{aligned}$$

This in turn implies that also $\overline{b_i} + b_j \in A$.

If $(b_o]$ has cardinality less than λ , then so has the set $\{b_o \circ b_i | i<\lambda\}$, and in this case we set $M = (b_o]$. Since $b_i = b_o \circ b_i + \overline{b_o} \circ b_i$, and $\overline{b_o} \circ b_i \in A$, we have $(A \cap B) \vee [M] = B \vee [M]$.

If $(\overline{b_o}]$ has cardinality less than λ , then so has $[b_o)$; in this case, we set $M = [b_o)$, observing that $b_i = (b_o + b_i) \circ (\overline{b_o} + b_i)$, and $\overline{b_o} + b_i \in A$. □

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As we shall see later, the hypothesis that λ is regular, is essential.

Proposition 3.9 also implies a partial answer to problem 29 of [2]:

Call an algebra D almost Jonsson, if for each $B \leq D$ with $|B| = |D| = \lambda$, there exists an $A \leq D$ such that $|A| < \lambda$ and $A \vee B = D$. Call D packed, if $A, B \leq D$, $|A| = |B| = |D| = \lambda$ imply $|A \cap B| = \lambda$. Note that an almost Jonsson or packed Boolean algebra is $|D|$-like. The question mentioned above asks if there is an almost Jonsson algebra which is not packed, and vice versa.

PROPOSITION 3.10. *Let D be an infinite Boolean algebra which is almost Jonsson and has regular cardinality λ; then D is packed.*

Proof. Let $A, B \leq D$, $|A| = |B| = \lambda$; since D is almost Jonsson, there exists a $C \leq D$ with $|C| < \lambda$ which satisfies $A \vee C = D$. Since $B \leq A \vee C$, and D is λ-like, 3.9 implies the existence of a $Q \leq D$, such that $|Q| < \lambda$, and $(A \cap B) \vee Q = B \vee Q$. Thus, $A \cap B$ must have cardinality λ. □

Next we turn our attention to the interval algebras of well-ordered sets.

If D is the interval algebra of a chain C, then each $d \in D^+$ has a unique representation $d = [x_o, y_o) \cup \ldots \cup [x_n, y_n)$, where $x_o < y_o < x_1 < \ldots < x_n < y_n$, and possibly $y_n = \infty$, that is $[x_n, y_n) = [x_n)$. If $d \in D$ has this form, we set $I(d) = \{x_i | i \leq n\} \cup \{y_i | i \leq n\}$.

PROPOSITION 3.11. *Let $\lambda \geq \omega$ be an ordinal and D its interval algebra. Then* Sub D *is not simple if and only if λ is a regular cardinal.*

Proof. One direction follows immediately from Proposition 3.9, thus, let us suppose that λ is not a regular cardinal. In what follows, we shall use the symbols $+$ and $\circ$ both for ordinal addition and multiplication, and for the operations on D; the meaning will be clear from the context.

If $\lambda = \beta + \gamma$, with $\beta > \gamma$, then $D \cong I(\beta) \times I(\gamma) \cong I(\gamma) \times I(\beta) \cong I(\gamma+\beta) \cong I(\beta)$, so we can assume that in particular λ is a limit ordinal. If $\lambda = \beta \circ n$ for some $n < \omega$, then D

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is isomorphic to the product of n copies of $I(\beta)$, and it follows from 3.3 that Sub D is simple. Thus, let us suppose that λ is not of the form $\beta \circ n$, and that $cf\ \lambda = \gamma < \lambda$; then there exists a γ-termed sequence $\{\alpha_\xi | \xi<\gamma\}$ of limit ordinals with supremum λ , such that $\alpha_o = \gamma$, and $\alpha_\xi \circ 3 < \alpha_{\xi+1}$ for all $\xi < \gamma$.

Our goal is to construct a finite number of subalgebras of D , each of which is congruent to 2 , and whose supremum is D .

The crucial observation is the following: Let $A_1 \leq D$ be generated by $\{[\alpha_\xi,\alpha_\rho) | \xi<\rho<\gamma\}$, $A_2 \leq D$ be generated by $\{[\xi,\rho) | \xi<\rho<\gamma\}$, $A_3 \leq D$ be generated by $\{[\xi,\rho) \cup [\alpha_\xi,\alpha_\rho) | \xi<\rho<\gamma\}$. Then, $A_1 \cap A_3 = A_2 \cap A_3 = 2$, and $A_1, A_2 \leq A_3([0,\alpha_o))$; it follows that A_1 and A_2 are congruent to 2 , and so is $A = A_1 \vee A_2$. Next, let $B \leq D$ be generated by $\{[\alpha_\xi,\alpha_\xi \circ 2) | \xi<\gamma\}$, and $B_1 \leq D$ be generated by $\{[\xi,\xi+1) \cup [\alpha_\xi,\alpha_\xi \circ 2) | \xi<\gamma\}$. Note that B is isomorphic to $FC(\gamma)$; as before, $B \cap B_1 = 2$, and $B \leq B_1([0,\alpha_o))$, hence, $B \equiv 2$.

Let $C \leq D$ be generated by $\{[\alpha_i+\xi,\alpha_i+\rho) | i<\gamma, \xi<\rho<\alpha_i\}$, and $R \leq D$ be generated by $\{[\alpha_i+\xi,\alpha_i+\rho) \cup [\alpha_i \circ 2+\xi,\alpha_i \circ 2+\rho) | i<\gamma, \xi<\rho<\alpha\}$. Let $c \in C^+$ such that $\lambda \notin I(c)$; then, for each $z \in I(c)$ there exists a $\xi < \gamma$, such that $\alpha_\xi \leq z < \alpha_\xi \circ 2$; if $r \in R^+$ such that $\lambda \notin I(r)$, there exists a $z \in I(r)$ and a $\xi < \gamma$ such that $\alpha_\xi \circ 2 < z < \alpha_\xi \circ 3$. It follows that $R \cap C = 2$; since $C \leq R \vee B$ and $B \equiv 2$, this implies $C \equiv 2$.

For each $i < \gamma$, partition $[\alpha_{i+1},\alpha_{i+1} \circ 2)$ into faithfully enumerated subsets $I_1^i = \{m_\delta | \delta<\alpha_{i+1}\}$, and $I_2^i = \{n_\delta | \delta<\alpha_{i+1}\}$, and set

$$P_i = \left[\{[\alpha_i \circ 2+\xi,\alpha_i \circ 2+\rho) \cup [m_\xi,m_\rho) | \xi<\rho<\alpha_{i+1}\}\right]$$

$$Q_i = \left[\{[\alpha_i \circ 2+\xi,\alpha_i \circ 2+\rho) \cup [n_\xi,n_\rho) | \xi<\rho<\alpha_{i+1}\}\right] .$$

Let P be generated by $\cup\{P_i | i<\gamma\}$, Q be generated by $\cup\{Q_i | i<\gamma\}$.

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Similarly as before, it is shown that $P \cap Q = 2$. Also,

$$P \vee A \vee B \vee C = Q \vee A \vee B \vee C = D .$$

This implies $P \equiv Q \equiv D$, and it follows from $P \cap Q = 2$ that $D \equiv 2$. □

Now we can describe the congruences of Sub D if D is countable.

PROPOSITION 3.12. *Let* D *be countable; then* Sub D *is not simple if and only if* D *is isomorphic to* $FC(\omega)$.

Proof. If $D \cong FC(\omega)$, then Sub D is not simple by 3.8. If $D \not\cong FC(\omega)$, there are two cases:

(a) D contains an infinite free subalgebra: then Sub D is simple by 3.4.

(b) D does not contain an infinite free subalgebra: then D is superatomic, and it is well known that in this case D is the interval algebra of $\omega^\beta \circ n$, where $0 < n < \omega$, and $0 < \beta < \omega_1$. So, Sub D is simple by the preceding proposition. □

Thus far, all the proper congruences we have exhibited on Sub D were of the form θ_γ , and in all cases D was $|D|$-like. We would like to conclude this section with an example which shows two things:

1. There exists a Boolean algebra D such that $|D| = \omega_1$, D is not ω_1-like, and Sub D is not simple.
2. θ_{ω_1} is not a congruence on Sub D .

EXAMPLE 3.13. Let M be a subset of the real numbers, such that M has a smallest element and $|M| = \omega_1$, and let $E = I(M) \times I(M)$; then $|E| = \omega_1$, E is an interval algebra, and Sub E is simple. Now set $D = E \times FC(\omega_1)$; then

1. D is not ω_1-like;
2. Sub D is sectionally complemented;
3. θ_{ω_1} is not a congruence on Sub D ;
4. E and $FC(\omega_1)$ have no isomorphic uncountable subalgebras.

For (2), observe that D is a subalgebra of an interval algebra, and to see (3), note that the canonical copy of E in Sub D is congruent to 2 , since Sub E is simple.

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Let $u \in D$ such that $D|u \cong E$ and $D|\bar{u} \cong FC(\omega_1)$; let P , $Q \le D$ be canonical copies of $D|u$ and $D|\bar{u}$ respectively. Then, $P(u) = (u] \cup [\bar{u})$, and $Q = (\bar{u}] \cup [u)$.

Let I be the ideal of Sub D which is generated by $\{P\} \cup \{S \le D \,\big|\, |S| \le \omega\}$. Consider the following condition on I :

$$(*) \quad (A] \cap \big(I \vee (B]\big) = \big((A] \cap I\big) \vee (A \cap B] \text{ , for all } A \text{ , } B \le D \text{ .}$$

(Here, the appearing intervals and $\vee$ are to be taken in the lattice of ideals of Sub D .)

If I satisfies $(*)$, that is if I is standard, then it induces a proper congruence on Sub D , see [3], III.3.5. Since $\supseteq$ holds in any lattice, we only have to show $\subseteq$. So, let A , B , $C \le D$, $C \le A$, and $C \le P(u) \vee T_1 \vee B$ for some countable $T_1 \le D$. We have to show the existence of an $S \le D$ such that $S \le A$, $S \in I$, and $C \le S \vee (A \cap B)$.

If A , B , or C are countable, there is nothing to show, so let us suppose that they are all uncountable.

Let C_1 be a complement of $C \cap B$ in Sub C ; then $C_1' = C \cap B \le A \cap B$.

Let C_2 be a complement of $C_1 \cap P(u)$ in Sub C_1 ; then $C_2' = C_1 \cap P(u) \in I$, and $C_2' \le A$.

Let $C_3 = Q(u) \cap C_2$ and T_2 be a complement of C_3 in Sub C_2 . Then, $T_2 \cap D|u = T_2 \cap D|\bar{u} = \{0\}$, which implies that u is independent of T_2 . Assume that T_2 is uncountable; let $h : T_2 \to D|u$ be the canonical projection $h(t) = t \circ u$; it is not hard to see that h is an embedding, so $D|u$ has a subalgebra isomorphic to T_2 ; likewise, $D|\bar{u}$ has a subalgebra isomorphic to T_2 ; since T_2 is uncountable, this contradicts (4).

So, $C_2 = C_3 \vee T_2$, where $T_2 \in I$ and $T_2 \le A$, $C_3 \le Q(u)$. Since each complement of $C_3 \cap Q$ in Sub C_3 is finite, we may as well suppose that $C_3 \le Q$.

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Let us pause for a moment to recapitulate what we have so far:

(a) $C = C_1' \vee C_2' \vee T_2 \vee C_3$;

(b) $C_1' \leq A \cap B$;

(c) $C_2' \vee T_2 \leq A$, $C_2' \vee T_2 \in I$;

(d) $C_3 \leq Q$, $C_3 \cap B = 2$.

So we are finished, if we can show that C_3 is an element of I .

Now let us look at B ; let B_1 be a complement of $B \cap P(u)$ in Sub B . If B_1 is countable, then $B \in I$, and we are done, so, let us suppose that B_1 is uncountable.

Let $B_2 = B_1 \cap Q(u)$ and B_2' a complement of B_2 in Sub B_1 . Since $B_2' \cap D]u = B_2' \cap D]\bar{u} = \{0\}$, we conclude as in a previous argument that B_2' is countable. Also as before, we suppose that $B_2 \leq Q$.

We now have

$$C_3 \leq P(u) \vee T_1 \vee B_2' \vee B_2 .$$

Let $R = (T_1 \vee B_2') \cap Q(u)$ and suppose, without loss of generality, that $u \in R$. Then,

$$C_3 \leq P \vee R \vee B_2 .$$

Now it is not hard to show that $C_3 \leq R \vee B_2$. We also note that R is countable as a subalgebra of $T_1 \vee B_2'$.

Our final aim is to show that there is a countable $U \leq Q$ such that $C_3 \leq U \vee B_3$.

Let $c \in C_3^+$, such that, without loss of generality, $c < \bar{u}$, and let $c = r_o \circ b_o + \ldots + r_n \circ b_n$ for some $r_i \in R$, $b_i \in B_2$, and $r_i \circ b_i > 0$ for $i \leq n$.

Let M be the free prime ideal of $D|\bar{u}$ generated by the atoms of $D|\bar{u}$; then, $Q = M \cup -M$. Since $c \in M$, observe that each $r_i \circ b_i$ is an element of M . Suppose, without loss of generality, that r_o is not an

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element of Q ; then either $r_o \leq \bar{u}$, or $r_o = u + x$ for some $x < \bar{u}$ with $x \in M$.

(a) $r_o \leq \bar{u}$: then $u + r_o \in Q$. Assume that $b_o \notin M$, hence, $b_o = u + y$ for some $y \notin M$. We now have $r_o \circ b_o = r_o \circ (u+y)$ $= r_o \circ y \in M$, which implies $r_o \in M$ or $y \in M$, a contradiction.

Thus, $b_o \in M$, in particular, $b_o < \bar{u}$; then, $(u+r_o) \circ b_o = r_o \circ b_o$ and $u + r_o \in Q$.

(b) $r_o = u + x$ for some $x \in M$; then clearly $b_o \leq \bar{u}$, $r_o \circ b_o = x \circ b_o$, and $x \in Q$.

If we replace each r_i if necessary by one of the elements of Q as described above, and then let $U \leq D$ be generated by these elements and $R \cap Q$, then U is countable, since R is countable, U is a subalgebra of Q , and $C_3 \leq U \vee B_2$. Since $C_3 \cap B_2 = 2$, it follows from 3.7 and 3.8 that C_3 is countable, hence, $C_3 \in I$. □

4. Concluding remarks

Just looking briefly at prime ideals of Sub D , we state the following theorem without proof, since it would involve too much new notation and preliminary results which do not seem to be justified.

PROPOSITION 4.1. *If* P *is a prime ideal in* Sub D , *then* $A \equiv B$ *and* $A \in P$ *imply* $B \in P$. *It follows that* Sub D *is not simple.*

If Sub D is not simple, it need not have a prime ideal. Let $D = FC(\lambda)$, and partition the set of atoms of D into $\{x_i | i<\lambda\}$, and $\{y_i | i<\lambda\}$; then set $A = [\{x_i | i<\lambda\}]$, $B = [\{y_i | i<\lambda\}]$, and $C = [\{x_i+y_i | i<\lambda\}]$. These algebras generate a 0,1-diamond in Sub D , so it cannot have a prime ideal. Incidentally, this shows that for no countable D Sub D has a prime ideal. Indeed, the only algebra D we know where Sub D has a prime ideal is the packed algebra constructed by M. Rubin [4] under $\Diamond_{\omega_1}$.

PROBLEM 1. Find an algebra D such that D is not packed, and Sub D has a prime ideal.

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The results of the preceding chapter seem to suggest that a nice characterization of those D having Sub D (not) simple is hard to come by. All the congruences that we have been able to exhibit on Sub D arose from a distributive ideal; this suggests

PROBLEM 2. Find an algebra D and a congruence on Sub D which is not induced by a distributive ideal.

Note that such an algebra cannot have Sub D sectionally complemented, in particular, D is not a subalgebra of an interval algebra. Finally, it might be worthy of mention, that the facts that $D|d$ has cardinality $\lambda \geq \omega_1$ for all $d \in D^+$ and θ_λ is a congruence on Sub D, imply that D is Bonnet-rigid in the sense of [2].

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