ON CERTAIN SUBGROUPS OF A JOIN OF SUBNORMAL SUBGROUPS

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1. Introduction. Suppose the group G is generated by subnormal subgroups H and K, and that A, B are normal subgroups of finite index in H, K respectively. The following question has been asked by J. C. Lennox: *Under what circumstances is the subgroup* $J=\langle A, B\rangle$ *subnormal in* G? In particular, it is of interest to know when J has finite index in G, for, if this is the case, we may factor out by the normal core of J in G and apply Wielandt's theorem on joins of subnormal subgroups of finite groups [**11**] to deduce that J is subnormal in G. Here we prove the following result.

THEOREM 1. *Let* $G=\langle H, K\rangle$, *with* H, K *subnormal in* G, *and suppose that* A, B *are normal subgroups of finite index in* H, K *respectively. Then, if* G/G' *has finite rank,* $J=\langle A, B\rangle$ *has finite index in* G.

It is not possible to dispense with either the hypothesis of finite rank or that of subnormality: If $W=E \operatorname{wr} C$ denotes the wreath product of an infinite, elementary abelian p-group E by a cyclic group C of order p, then W is nilpotent [**2**] and so C is subnormal in W, but E does not have finite index in W.

If, on the other hand, $Z=H \operatorname{wr} C$, where H is infinite cyclic, then H does not have finite index in Z although Z has rank $p+1$.

Further, it is not difficult to find an example of a group G^* satisfying all of the hypotheses of Theorem 1 apart from that of finite rank, such that the resulting join is not even subnormal in G^*. The group G constructed by P. Hall and described in detail in Theorem 6.1 of [**5**] is a split extension $M]J$, where $J=\langle H, K\rangle$ is a self-normalising subgroup of G generated by abelian subgroups H and K, each subnormal in G, and $M=A\times B$ is an elementary abelian 2-group, where $[A, H]=[B, K]=1$. If x is any element of A, then H has index 2 in $\langle x\rangle H=L$, say, which is subnormal in G since HA is. However, J is a proper, self-normalising subgroup of $\langle L, K\rangle$. Thus we have proved the following theorem.

THEOREM 2. *There is a group* G^* *generated by subnormal abelian subgroups* L *and* K, *such that* L *has a subgroup* H *of index* 2 *and* $\langle H, K\rangle$ *is not subnormal in* G^*.

As an application of Theorem 1, the following partial solution to a similar problem will be provided.

THEOREM 3. *Suppose* G *is a finitely generated group, generated by subnormal subgroups* H *and* K. *Let* A, B *be normal subgroups of* H, K *respectively such that* H/A, K/B *satisfy* max-*sn* (*the maximal condition for subnormal subgroups*). *Then* $J=\langle A, B\rangle$ *is subnormal in* G.

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2. Proofs. The main ingredient in the proof of Theorem 1 is Lemma 1 below, the first part of which is Theorem B of [**4**], while the second part may be deduced from this result in the same way as "Corollary B1" is deduced from "Theorem B" in [**8**]. (The *permutizer* of K in H, denoted by $P_H(K)$, is the largest subgroup P of H such that $PK = KP$.)

LEMMA 1. *Let G be a group, generated by subnormal subgroups H and K, such that G/G' has finite rank. Then, given integers a, b, there exists an integer c such that $\gamma_c(G) \leq \gamma_a(H)\gamma_b(K)$. Also, for some integer d, $\gamma_d(H) \leq P_H(K)$.*

We shall also require a further lemma.

LEMMA 2. *Suppose H, K are subnormal in $G = HK$, and that A, B are subgroups of finite index in H, K respectively. Then $\langle A, B\rangle$ has finite index in G.*

Actually this result remains true if the hypothesis of subnormality is removed, being a consequence of a theorem of B. H. Neumann on coverings by finitely many cosets (see Amberg [**1**, Lemma 5.1]). However, it may be quite easily proved by induction on the defect of H in G, thus avoiding appeal to the more general result. The proof is omitted.

Finally, we require a lemma which is no doubt well-known and which may be seen to be the "nilpotent case" of Theorem 1.

LEMMA 3. *Suppose $G = \langle H, K\rangle$ is a nilpotent group of finite rank and that A, B are (normal) subgroups of finite index in H, K respectively. Then $J = \langle A, B\rangle$ has finite index in G.*

Proof. We proceed by induction on c, the nilpotency class of G. We may clearly suppose $c \geq 2$ and that the appropriate inductive hypothesis holds. Then $J\gamma_c(G)$ has finite index in G, and it is enough to show that $|\gamma_c(G) : J \cap \gamma_c(G)|$ is finite. Now, for some integer d, $x^d \in J$ for all $x \in H \cup K$. Further, $\gamma_c(G)$ is generated by commutators of the form $[x_1, \ldots, x_c]$, where the x_i belong to $H \cup K$. Writing $y_i = x_i^d$, a simple inductive argument shows that, for each integer $j \leq c$, $[x_1, \ldots, x_j]^{d^j} \equiv [y_1, \ldots, y_j]$ modulo $\gamma_{j+1}(G)$. Thus $\gamma_c(G)/\gamma_c(G) \cap J$ has exponent at most d^c and is therefore finite, as required.

Proof of Theorem 1. Let $P = P_H(K)$. Then $A \cap P$ has finite index in P, which is subnormal in G [**9**, Lemma 3] and hence in PK. By Lemma 2, $\langle A \cap P, B\rangle$ has finite index in PK. Thus $|PK : J \cap PK|$ is also finite. By Lemma 1, there is an integer a such that $\gamma_a(H) \leq P$ and therefore an integer c such that $\gamma_c(G) \leq PK$. Thus $|\gamma_c(G) : J \cap \gamma_c(G)|$ is finite. Since G/G' has finite rank, so has $G/\gamma_c(G)$ [**6**]. Factoring by $\gamma_c(G)$, we may apply Lemma 3 to obtain the desired result.

Proof of Theorem 3. We first reduce to the case where H/A and K/B are soluble. Let $P=P_K(A)$. Then P normalises B and permutes with A, and so B^A is normal in $\langle A, B, P\rangle$. $J=B^AA$ is thus subnormal in $\langle A, B, P\rangle$. By Corollary B.1 of [**8**], we know that there is an integer λ such that $K^{(\lambda)}\le P$, and so J is subnormal in $\langle A, BK^{(\lambda)}\rangle$. Hence we may replace B by $BK^{(\lambda)}$, i.e. we may suppose K/B is soluble. Similarly H/A may be assumed soluble. Since they satisfy max-*sn*, H/A and K/B are thus polycyclic, and it follows from a well-known theorem of Mal'cev (see e.g. Theorem 3.25 of [**7**]) that they are nilpotent-by-abelian-by-finite. Let L, M be normal subgroups of finite index in H, K such that $(L/A)'$, $(M/B)'$ are nilpotent of class a, b respectively. By Theorem 1 we may suppose $L=H$ and $M=K$, since $\langle L, M\rangle$ is also finitely generated and subnormal in G. Now, by Lemma 1, there is an integer c such that $\gamma_c(G)\le H'K'$. So $\gamma_c(G)K'=(\gamma_c(G)K'\cap H')K'=RK'$, say, a product of two subnormal subgroups. We may thus apply Lemma 2 of [**10**] to deduce that, for some integer d, $\gamma_d(\gamma_c(G)K')\le\gamma_{a+1}(R)\gamma_{b+1}(K')\le AB$. Hence $\gamma_d(\gamma_c(G))\le J$ and, factoring, we may suppose that G is metanilpotent. But in a finitely generated metanilpotent group, the join of any two subnormal subgroups is subnormal [**5**, Corollary 1 to Theorem 5.2] and so the theorem is proved.

REMARK. The conclusion that J is subnormal in G also follows if "max-*sn*" is replaced by "min-*sn*" in Theorem 3. For, again reducing to the soluble case, we may this time assume that H/A and K/B are Černikov groups and hence abelian-by-finite.

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