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STRATIFICATION OF THE MODULI SPACE OF FOUR-GONAL CURVES

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Abstract Let X be a smooth irreducible projective curve of genus g and gonality 4. We show that the canonical model of X is contained in a uniquely defined surface, ruled by conics, whose geometry is deeply related to that of X. This surface allows us to define four invariants of X and, hence, to stratify the moduli space of four-gonal curves by means of closed irreducible subvarieties, whose dimensions we compute.

Keywords: four-gonal curves; moduli space; ruled surfaces

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1. Introduction

Let X be a smooth irreducible curve of genus g and gonality γ, i.e. γ is the minimal degree of a base-point-free linear series on X. Let $\mathcal{M}_g$ denote the moduli space of curves of genus g and let $\mathcal{M}_{g,\gamma} \subset \mathcal{M}_g$ denote the variety parametrizing the γ-gonal curves; it is well known that $\mathcal{M}_{g,\gamma}$ is an irreducible variety of dimension $2g + 2\gamma - 5$, as long as $2 \leqslant \gamma \leqslant \frac{1}{2}g + 1$ (see [**1**, **13**]).

The structure of $\mathcal{M}_{g,\gamma}$ is completely understood in the cases $\gamma = 2$ (hyperelliptic curves) and $\gamma = 3$ (trigonal curves). In this paper we are interested in the study of four-gonal curves. We briefly recall the setting in the trigonal case.

Let K denote the canonical divisor on X and let $X_K \subset \mathbb{P}^{g-1}$ be the canonical model of X. From the geometric Riemann–Roch theorem, any trigonal divisor spans a line in $\mathbb{P}^{g-1}$; therefore, X_K is contained in a rational normal ruled surface, R say. It is clear that R is of the form $\mathbb{P}(\mathcal{O}(m) \oplus \mathcal{O}(g-2-m))$; assuming that $m \leqslant g-2-m$, the integer m is uniquely determined and it is called the *Maroni invariant* of X.

Set $\mathcal{M}_{g,3}(m)$ to be the variety parametrizing the trigonal curves of Maroni invariant not bigger than m. The following fact holds.

Theorem 1.1. *If* $(g-4)/3 \leqslant m < (g-2)/2$ *(respectively,* $m = (g-2)/2$*), then* $\mathcal{M}_{g,3}(m)$ *is a locally closed subset of* $\mathcal{M}_{g,3}$ *of dimension* $g + 2m + 4$ *(respectively,* $2g+1$*). (See [**14**, Proposition 1.2].)*

One can see that for each curve of genus $g \geqslant 5$ of Maroni invariant m there exists a unique linear series g^1_λ, where λ is the minimum integer bigger than 3 and $\lambda = g - m - 1$. Hence, λ is uniquely determined by m, and the above filtration of $\mathcal{M}_{g,3}$ given by the varieties $\mathcal{M}_{g,3}(m)$ can be rewritten in terms of λ.

In general, it seems interesting to find 'good invariants' arising from the geometric properties of γ-gonal canonical curves, in order to obtain an analogous stratification of the moduli space $\mathcal{M}_{g,\gamma}$.

As in the trigonal case, one can introduce the rational normal scroll V, whose fibres are the $(\gamma-2)$-planes spanned by the γ-gonal divisor on X. Clearly, $V = \mathbb{P}(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{\gamma-1}))$, where $a_1 + \cdots + a_{\gamma-1} = g - \gamma + 1$; in this way the integers $a_1, \ldots, a_{\gamma-2}$ play the role of the Maroni invariant m in the trigonal case.

In this paper we focus on four-gonal curves. We show that in the volume $V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$ there exists an (almost always) uniquely determined 'minimal' surface, ruled by conics, containing X_K.

Such a surface S gives rise to two other invariants: on the one hand, one defines the number t, which is the uniquely determined invariant of a suitable geometrically ruled surface birationally equivalent to S; on the other hand, analysing the embedding of X in S, we obtain another number $\lambda > 4$, which turns out to be the minimum degree of a linear series on X different from the gonal one.

Comparing the configuration $X_K \subset S \subset V$ in the four-gonal case with the analogous situation $X_K \subset R$ of the trigonal case, it is clear that the invariant m has been replaced, in some sense, by a, b and t. Finally, one can prove that λ is now independent of a, b and t; so, a four-gonal curve is determined by the four invariants a, b, λ and t.

In § 8 we describe the geometric meaning of λ, while, in §§ 7 and 9 we find the ranges for the above invariants λ, t and a, b, respectively.

If $t = 0$ the cited ranges become

$$\frac{g+3}{3} \leqslant \lambda \leqslant \frac{g+3}{2}, \tag{R$_1$}$$

$$a_{\min} \leqslant a \leqslant \frac{g-3}{3}, \tag{R$_2$}$$

$$g - \lambda - 1 \leqslant a + b \leqslant \frac{2(g-3)}{3}, \tag{R$_3$}$$

where

$$a_{\min} = \begin{cases} \left\lceil \dfrac{\lambda-4}{2} \right\rceil & \text{if } \lambda \geqslant \dfrac{2g+6}{5}, \\ g - 2\lambda + 1 & \text{if } \lambda \leqslant \dfrac{2g+6}{5}. \end{cases}$$

In § 10 (see Theorem 10.6) we then show that, if (R$_1$), (R$_2$), (R$_3$) are satisfied, there exists a four-gonal curve of genus g and invariants a, b, λ and $t = 0$.

Finally, in § 12 we study the moduli spaces $\mathcal{M}_{g,4}$ of four-gonal curves with $t = 0$. Let $\mathcal{M}^\lambda_g \subset \mathcal{M}_{g,4}$ be the variety parametrizing the four-gonal curves of invariant λ and let $\mathcal{M}^\lambda_g(a,b) \subset \mathcal{M}^\lambda_g$ be the subvariety parametrizing the curves of further invariants a and b. We prove the following.

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Main Theorem. Let g, λ, a, b be positive integers satisfying (R_1), (R_2), (R_3) and $g \geqslant 10$. We then have the following.

(i) There exists a stratification of the moduli space $\mathcal{M}_{g,4}$ of four-gonal curves given by

$$\mathcal{M}_{g,4} = \bar{\mathcal{M}}_g^{\lceil (g+2)/2 \rceil} \supset \bar{\mathcal{M}}_g^{\lceil g/2 \rceil} \supset \cdots \supset \bar{\mathcal{M}}_g^{\lambda} \supset \cdots \supset \bar{\mathcal{M}}_g^{\lceil (g+3)/3 \rceil},$$

and $\bar{\mathcal{M}}_g^{\lambda}$ are irreducible locally closed subsets of dimension $g + 2\lambda + 1$, if $\lambda < \lceil (g+2)/2 \rceil$.

(ii) For each admissible λ, we can write that

$$\bar{\mathcal{M}}_g^{\lambda} = \bigcup_{a,b} \bar{\mathcal{M}}_g^{\lambda}(a,b),$$

where $\bar{\mathcal{M}}_g^{\lambda}(a,b)$ is a non-empty, irreducible subvariety whose dimension is

$$\dim(\mathcal{M}_g^{\lambda}(a,b)) = \begin{cases} 2(2a+b+\lambda) + 10 - g - \epsilon - \tau - \xi & \text{if } a \geqslant \dfrac{g-\lambda-1}{2}, \\ 2(a+b) + \lambda + 8 - \epsilon - \xi & \text{if } a < \dfrac{g-\lambda-1}{2}, \end{cases}$$

where

$$\epsilon := \begin{cases} 0 & \text{if } b < c, \\ 1 & \text{if } a < b = c, \\ 2 & \text{if } a = b = c, \end{cases} \qquad \tau := \begin{cases} 0 & \text{if } a < b, \\ 1 & \text{if } a = b \end{cases} \qquad \text{and} \qquad \xi := \begin{cases} 1 & \text{if } \lambda = \dfrac{g+3}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

In § 13 we briefly describe the moduli space of four-gonal curves of invariant $t \geqslant 1$, computing also in this case the dimension of each stratum. It turns out that such a dimension (for chosen g and λ in the good range) is strictly smaller than that of the corresponding stratum in the case $t = 0$. In other words, the $t = 0$ case describes the general and most significant situation.

2. Preliminaries

We say that a curve is four-gonal if it has a linear series g^1_4 but no g^1_d for any $d \leqslant 3$. We also assume that such a curve is not bihyperelliptic (i.e. the degree 4 map on $\mathbb{P}^1$ does not factorize through a hyperelliptic curve), in particular, that it is not bielliptic.

Let X be a four-gonal curve of genus g. In order to have a unique g^1_4 on X, we assume that $g \geqslant 10$.

Denote by $\varphi_K \colon X \to X_K \subset \mathbb{P}^{g-1}$ the canonical map associated with X and denote by X_K the canonical model of X. In general, if Y is a variety and D is a divisor on Y, we denote by $\varphi_D \colon Y \to \varphi_D(Y) \subset \mathbb{P}(H^0(Y, \mathcal{O}_Y(D)))$ the morphism associated with D.

If $\varPhi \in g^1_4$ is a four-gonal divisor, by the geometric Riemann–Roch theorem (see [**2**, Chapter I, § 2]) we have that $\dim\langle \varphi_K(\varPhi) \rangle = \deg(\varPhi) - h^0(\mathcal{O}_X(\varPhi)) = 2$; therefore,

$$V := \bigcup_{\varPhi \in g^1_4} \langle \varphi_K(\varPhi) \rangle \subset \mathbb{P}^{g-1}$$

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is a scroll, ruled by planes on $\mathbb{P}^1$, containing X_K. Denote by $\pi\colon V \to \mathbb{P}^1$ the natural projection.

Recall that a non-degenerate variety $W \subset \mathbb{P}^r$ is said to be *projectively normal* if it is normal and, for any $k \in \mathbb{N}$, the homomorphism

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(W, \mathcal{O}_W(k))$$

induced by the exact sequence of sheaves

$$0 \to \mathcal{I}_W \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_W \to 0$$

is surjective.

We say that W is *linearly normal* if the homomorphism above is surjective for $k = 1$. In particular, if W is a non-degenerate curve, then it is linearly normal if and only if $h^0(W, \mathcal{O}_W(1)) = h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) = r + 1$.

It is well known that X_K is projectively normal, so V is a rational normal scroll (hence, projectively normal as well). We then set $V = \mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a vector bundle of rank 3 on $\mathbb{P}^1$, i.e. $\mathcal{F} = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)$ for suitable non-negative integers $a \leqslant b \leqslant c$. It is also well known that, for any k, it holds that

$$h^0(V, \mathcal{O}_V(k)) = h^0(\mathbb{P}^1, \pi_* \mathcal{O}_V(k)) = h^0(\mathbb{P}^1, \operatorname{Sym}^k \mathcal{F}), \tag{2.1}$$

and that the Riemann–Roch theorem for any vector bundle $\mathcal{G}$ on $\mathbb{P}^1$ with non-negative splitting type gives that $h^0(\mathbb{P}^1, \mathcal{G}) = \deg(\mathcal{G}) + \operatorname{rk}(\mathcal{G})$.

From the above two relations, since $a, b, c \geqslant 0$, we then have that $h^0(V, \mathcal{O}_V(1)) = h^0(\mathbb{P}^1, \mathcal{F}) = \deg(\mathcal{F}) + \operatorname{rk}(\mathcal{F})$. Taking into account that $h^0(V, \mathcal{O}_V(1)) = g$, we finally obtain that

$$a + b + c = g - 3. \tag{2.2}$$

In the following we need some basic notation and facts about ruled surfaces.

We denote by $\mathbb{F}_t$ (where $t \geqslant 0$) the *Hirzebruch surface* of *invariant* t, i.e. the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ associated with the sheaf $\mathcal{O}(-t) \oplus \mathcal{O}$ (here $\mathcal{O}$ means $\mathcal{O}_{\mathbb{P}^1}$).

If $1 \leqslant a \leqslant b$, a *rational ruled surface* $R_{a,b}$ is $\mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b))$, naturally embedded in $\mathbb{P}^{a+b+1}$. Clearly, setting $t := b - a$, we have that $R_{a,b} \cong \mathbb{F}_t$, so t is the *invariant* of $R_{a,b}$.

We recall the following well-known facts (see [**11**, Chapter V, Proposition 2.9, Theorem 2.17 and Proposition 2.3]).

Lemma 2.1. *Let $\mathbb{F}_t$ be as before, let f be its generic fibre and let $C_0 = \mathbb{P}(\mathcal{O}(-t)) \subset \mathbb{F}_t$. We then have the following.*

(i) $C_0^2 = -t$.

(ii) *If U is any directrix (i.e. an irreducible unisecant curve) of $\mathbb{F}_t$, different from C_0, then $U^2 \geqslant t$.*

(iii) *If there exists a directrix U of R such that $U^2 = 0$, then $t = 0$, i.e. $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, $t > 0$ if and only if $\mathbb{F}_t$ has exactly one unisecant curve (namely, C_0) having negative self-intersection.*

(iv) $\operatorname{Num}(\mathbb{F}_t) = \mathbb{Z}\langle C_0 \rangle \times \mathbb{Z}\langle f \rangle$.

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Finally, we recall three classical formulae concerning ruled surfaces and scrolls, due to Segre.

Unisecants formula (UF)

Let $R \subset \mathbb{P}^{r+1}$ be a ruled surface of degree r and invariant t, and let $\mathrm{Un}^d(R)$ be the variety of the unisecant curves on R having degree d and self-intersection bigger than t. The general element of $\mathrm{Un}^d(R)$ is then irreducible and $\dim(\mathrm{Un}^d(R)) = 2d + 1 - r$.

Genus formula (GF)

If Y is a q-secant curve on a ruled surface $R \subset \mathbb{P}^r$, then its arithmetic genus is $p_a(Y) = ((q-1)/2)[2(\deg(Y) - 1) - q\deg(R)]$.

The following relation (IF), generalizing the analogous property for ruled surfaces, comes from the intersection law on a scroll (see [**8**, Example 8.3.14]).

Intersection formula (IF)

Let W be a rational scroll ruled by n-planes and let C_1 and C_2 be two subschemes of W meeting properly and such that C_i is m_i-secant for $i = 1, 2$ (i.e. C_i meets the general fibre of W in a variety of degree m_i). Then, $\deg(C_1 \cdot C_2) = m_1 \deg(C_2) + m_2 \deg(C_1) - m_1 m_2 \deg(W)$.

We also recall the following notions and the result that classifies the degenerate fibres of a surface ruled by conics given in [**6**, Proposition 1.13 and Theorem 2.4].

Definition 2.2. Let D be a very ample bisecant divisor on a Hirzebruch surface $\mathbb{F}$; the surface $S_0 := \varphi_D(\mathbb{F})$ is then said to be *geometrically ruled by conics* (over $\mathbb{P}^1$). Equivalently, a projective surface $S_0 \subset \mathbb{P}^N$ is geometrically ruled by conics if there exists a surjective morphism $\pi\colon S_0 \to \mathbb{P}^1$ such that the fibre $\pi^{-1}(y)$ is a smooth rational curve of degree 2 for every point $y \in \mathbb{P}^1$, and such that π admits a section.

We say that a projective surface $S \subset \mathbb{P}^N$ is *ruled by conics* (over $\mathbb{P}^1$) if it is birational to a surface geometrically ruled by conics. Equivalently, S is ruled by conics if there exists a surjective morphism $\pi\colon S \to \mathbb{P}^1$ and an open subset $U \subseteq \mathbb{P}^1$ such that the fibre $\pi^{-1}(y)$ is a curve of degree 2 and arithmetic genus 0 for every point $y \in \mathbb{P}^1$, and the fibre $\pi^{-1}(y)$ is smooth for every point $y \in U$.

Theorem 2.3. *Let $S \subset \mathbb{P}^N$ be a projective surface ruled by conics over a smooth irreducible curve. The degenerate fibres of S are then of one of the following types.*

- *Type F_1. The union of two distinct lines; in this case S is smooth in each point of the fibre.*
- *Type $F_2(A)$. The union of two distinct lines, meeting in an ordinary double point of S.*
- *Type $F_2(D)$. The union of two coincident lines, containing exactly two ordinary double points of S.*

- *Type $F_n(A)$ ($n \geqslant 3$). The union of two distinct lines, meeting in a rational double point of type (A_{n-1}).*
- *Type $F_n(D)$ ($n \geqslant 3$). The union of two coincident lines, containing exactly one rational double point of S; in particular, this point is of type (A_3) if $n = 3$, and of type (D_n) if $n \geqslant 4$.*

A deep and detailed overview of the rational double points can be found in [**3**, Chapter 3].

Since any surface S ruled by conics is birational to a surface S_0 geometrically ruled by conics, S can be obtained from a suitable S_0 by a finite number of monoidal transformations. In particular, each singular fibre of S (as described in Theorem 2.3) arises in this way. Again in [**6**], we studied this situation, as summarized below.

Let $\mathbb{F}$ and D be as before, and let $S_0 = \varphi_D(\mathbb{F})$ be a surface geometrically ruled by conics via the morphism $\pi\colon S_0 \to \mathbb{P}^1$. Consider a point $P_1 \in S_0$ and let $f_0 := \pi^{-1}(y)$ be the fibre of S_0 containing P_1. Consider the blow-up σ_{P_1} of S_0 at P_1 and the corresponding projection on $\mathbb{P}^1$, π_1 say:

$$\begin{array}{ccc} \mathrm{Bl}_{P_1}(S_0) := S_1 & \xrightarrow{\sigma_{P_1}} & S_0 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{array}$$

Denote also by $f_1 := \pi_1^{-1}(y)$ the total transform of f_0 via σ_{P_1}. Now, take $P_2 \in f_1$ and the blow-up $\sigma_{P_2}\colon S_2 \to S_1$. With obvious notation, we can iterate this construction and obtain a sequence of blow-ups:

$$\begin{array}{ccccccc} \tilde{S}_0 := S_n & \xrightarrow{\sigma_{P_n}} \cdots \longrightarrow & S_2 & \xrightarrow{\sigma_{P_2}} & S_1 & \xrightarrow{\sigma_{P_1}} & S_0 \\ \cup & & \cup & & \cup & & \cup \\ \tilde{f}_0 := f_n & & f_2 & & P_2 \in f_1 & & P_1 \in f_0 \end{array}$$

where, for any $i = 1, \ldots, n$, we define $P_i \in f_{i-1}$, $f_i := \pi_i^{-1}(y)$, and $\pi_i\colon S_i := \mathrm{Bl}_{P_i}(S_{i-1}) \to \mathbb{P}^1$ is the natural projection.

Definition 2.4. With the above notation, we say that $f_n = \tilde{f}_0 \subset \tilde{S}_0$ is a fibre of *level n over f_0*.

Denoting by σ the sequence of blow-ups of S_0 defined above, setting $\tilde{D}$ to be the strict transform of D (very ample bisecant divisor on S_0) via σ, and setting B to be the base locus of $\tilde{D}$, S can be obtained in the following way:

$$\begin{array}{ccc} \tilde{S}_0 & \xrightarrow{\sigma} & S_0 \\ {\scriptstyle \varphi_{\tilde{D}-B}}\downarrow & \nearrow{\scriptstyle \rho} & \\ S & & \end{array}$$

where ρ is defined as the birational map such that the diagram is commutative.

Definition 2.5. We say that the fibre $f \subset S$ is an *embedded fibre of level* n if

$$n = \min_i \{\text{there exists a blow-up } \sigma\colon \tilde{S}_0 \to S_0 \text{ and a fibre } f_i \subset \tilde{S}_0 \text{ of level } i \text{ such that } f = \varphi_{\tilde{D}-B}(f_i)\}.$$

Again in [**6**], we noted that each fibre $f \subset S$ of type $F_n(A)$ or $F_n(D)$ is an embedded fibre of level n. We also gave the following definition in [**6**].

Definition 2.6. Let $f^{(1)}, \dots, f^{(p)}$ be the degenerate fibres of S, and let l_i be the level of $f^{(i)}$ for $i = 1, \dots, p$. If $\sum_{i=1}^{p} l_i = L$, we say that S is of *level* L.

Moreover, we proved that all the surfaces geometrically ruled by conics (g.r.c.) and giving rise, by a minimal number of elementary transformations, to a surface S ruled by conics of level L are exactly the elements of the following set:

$$\mathrm{GRC}_L(S) := \{S_0 \mid S_0 \text{ is a g.r.c. surface and } S \text{ can be obtained from it by a sequence of } L \text{ blow-ups and contractions}\}.$$

3. The surface S of minimum degree, ruled by conics and containing X_K

Starting from the situation $X_K \subset V \subset \mathbb{P}^{g-1}$, described at the beginning of the previous section, we try to 'canonically' define a surface (ruled by conics) containing X_K and contained in V.

Notation. As usual, if n is a rational number, $[n]$ denotes the greatest integer smaller than or equal to n, while $\lceil n \rceil$ denotes the smallest integer bigger than or equal to n.

Theorem 3.1. *There exists a surface S ruled by conics such that $X_K \subset S \subset V$ and* $\deg(S) \leqslant \lceil (3g-8)/2 \rceil$*. Moreover, S is unique unless* $\deg(S) = (3g-7)/2$*; in this case, S varies in a pencil.*

Proof. We consider the vector space $\mathcal{H} := H^0(\mathbb{P}^{g-1}, \mathcal{I}_{X_K}(2))/H^0(\mathbb{P}^{g-1}, \mathcal{I}_V(2))$ and set $N := \dim(\mathcal{H})$; clearly, $\varSigma := \mathbb{P}(\mathcal{H})$ parametrizes the hyperquadrics of $\mathbb{P}^{g-1}$ containing X_K but not containing V.

We recall that, if W is a projectively normal subvariety of $\mathbb{P}^{g-1}$, we get the cohomology exact sequence (see § 2)

$$0 \to H^0(\mathcal{I}_W(2)) \to H^0(\mathcal{O}_{\mathbb{P}^{g-1}}(2)) \to H^0(\mathcal{O}_W(2)) \to 0.$$

Hence, $h^0(\mathcal{O}_{\mathbb{P}^{g-1}}(2)) = h^0(\mathcal{I}_W(2)) + h^0(\mathcal{O}_W(2))$. Rewriting this equality for both X_K and V, we get that

$$h^0(\mathcal{I}_{X_K}(2)) + h^0(\mathcal{O}_{X_K}(2)) = h^0(\mathcal{O}_{\mathbb{P}^{g-1}}(2)) = h^0(\mathcal{I}_V(2)) + h^0(\mathcal{O}_V(2)),$$

so $N = h^0(\mathcal{I}_{X_K}(2)) - h^0(\mathcal{I}_V(2)) = h^0(\mathcal{O}_V(2)) - h^0(\mathcal{O}_{X_K}(2))$.

In order to compute N, recall (2.1) and the Riemann–Roch theorem on the scroll $V = \mathbb{P}(\mathcal{F})$:

$$h^0(V, \mathcal{O}_V(2)) = h^0(\mathbb{P}^1, \mathrm{Sym}^2(\mathcal{F})) = \deg(\mathrm{Sym}^2(\mathcal{F})) + \mathrm{rk}(\mathrm{Sym}^2(\mathcal{F})).$$

Clearly, $\mathrm{Sym}^2(\mathcal{F})$ is a free bundle of degree $4(a+b+c)$ and rank 6; therefore, from (2.2) we get $h^0(\mathcal{O}_V(2)) = 4g-6$. On the other hand, by the Riemann–Roch theorem $h^0(\mathcal{O}_{X_K}(2)) = 3(g-1)$. Hence, the above space Σ of hyperquadrics is a projective space of dimension $N-1 = h^0(\mathcal{O}_V(2)) - h^0(\mathcal{O}_{X_K}(2)) - 1 = g-4$.

For each $Q \in \Sigma \cong \mathbb{P}^{g-4}$, consider the scheme-theoretic intersection

$$Q \cdot V = \left(\bigcup_{i=1,\dots,h_Q} F_i \right) \cup S_Q,$$

where the F_i are the fibres of V entirely contained in Q, $h_Q \geqslant 0$, and S_Q is a surface that is ruled by conics (since Q intersects the general fibre F of V in a conic passing through the four points of the divisor $\Phi \subset F$) and contains X_K.

Note that S_Q is irreducible; if not, $S_Q = S_1 \cup S_2$, where the S_i are ruled surfaces. But, $X_K \subset S_Q$ and it cannot be contained in a ruled surface since each four-gonal divisor spans a plane.

In order to find a quadric $\bar{Q} \in \Sigma$ such that $\deg(S_{\bar{Q}})$ is the minimum, it is enough to require that the number $h_{\bar{Q}}$ is the maximum. Note that a fibre F is contained in a quadric $Q \in \Sigma$ if Q contains two points, say P_1 and P_2, belonging to F and such that the 0-cycle of V of degree 6 given by $\Phi + P_1 + P_2$ does not lie on a conic.

Since $\dim(\Sigma) = g-4$, we can impose that the space Σ contains $[(g-4)/2]$ pairs of points. If each such pair of points belongs to the same fibre (and satisfies the above conditions), then we can find a $\bar{Q} \in \Sigma$ containing $[(g-4)/2]$ fibres. Clearly, $\bar{Q}$ could contain further fibres; hence,

$$\deg(S_{\bar{Q}}) \leqslant \deg(\bar{Q} \cap V) - \left[\frac{g-4}{2}\right] \leqslant 2(g-3) - \left[\frac{g-4}{2}\right] = \left\lceil \frac{3g-8}{2} \right\rceil.$$

This proves the existence of the required surface $S := S_{\bar{Q}}$.

Concerning the uniqueness, we assume that there are two such surfaces, say S_1 and S_2. Since $X_K \subset (S_1 \cap S_2)$, from (IF) we get that

$$2g-2 = \deg(X_K) \leqslant \int (S_1 \cdot S_2) = 2\deg(S_1) + 2\deg(S_2) - 4\deg(V).$$

This relation is verified if and only if $\deg(S_1) = \deg(S_2) = (3g-7)/2$. To complete the proof, just observe that the linear system of the quadrics $\bar{Q} \in \Sigma$ containing $[(g-4)/2]$ fibres has dimension

$$\dim \Sigma - 2\left[\frac{g-4}{2}\right] = g-4-2\left(\frac{g-5}{2}\right) = 1;$$

therefore, there exists a pencil of distinct surfaces $S_{\bar{Q}}$. □

The existence of such a surface S has also been proved, using a different method, by Schreyer [**12**].

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Notation. From now on, f denotes the general fibre of S, so f is a conic lying on a plane $F = \langle f \rangle$. Moreover, if T is a surface ruled by conics, we denote by V_T the scroll whose fibres are the planes spanned by these conics. For example, if S is the surface defined in Theorem 3.1, the scroll V_S is exactly V.

Remark 3.2. The fibres of the ruled surface S defined in Theorem 3.1 cannot all be singular. Otherwise, from [**5**, Proposition 1.2], the surface S would be ruled by lines on a hyperelliptic curve, Y say, via $\alpha\colon S \to Y$, and the ruling $\pi\colon S \to \mathbb{P}^1$ would factorize through α. Hence, taking into account that the restriction $X_K \to Y$ of α has degree 2, we obtain that X_K is bihyperelliptic, contrary to the assumption made on X.

Remark 3.3. The surface S introduced in Theorem 3.1 is then ruled by conics in the sense of § 2.

4. Birational models of $X_K \subset S$

In this section we study a surface S (not necessarily of minimum degree, as for that defined in Theorem 3.1) such that S is ruled by conics and $X_K \subset S \subset V$, where V as usual denotes the three-dimensional scroll spanned by the four-gonal divisors on X_K.

Note that, since X_K is linearly normal, $S \subset \mathbb{P}^{g-1}$ is linearly normal. Moreover, the scroll $V = V_S$ is not a cone (see Corollary 9.9); Theorem 2.3 then holds, so the classification of the degenerate fibres of the surface S is the one described there.

In § 2 we have also summarized the results (contained in [**6**]) that allow us to associate with a surface S, ruled by conics and of a certain level L, the set $\mathrm{GRC}_L(S)$ consisting of all the g.r.c. surfaces linked to S via a sequence of L monoidal transformations.

Here, we are looking for the inverse procedure: how to recover the surface S (and the curve X_K) starting from a g.r.c. surface $S_0 \in \mathrm{GRC}_L(S)$.

Notation. Since each surface $S_0 \in \mathrm{GRC}_L(S)$ is geometrically ruled by conics, each one admits an invariant $\tau_0 := t(S_0)$, in the sense that $S_0 \cong \mathbb{F}_{\tau_0}$. We denote by $X_{\tau_0} \subset \mathbb{F}_{\tau_0} \cong S_0$ the corresponding model of $X_K \subset S$.

Since $X_{\tau_0} \subset \mathbb{F}_{\tau_0}$ is a four-secant curve, we have

$$X_{\tau_0} \sim 4C_0 + (\lambda_0 + \tau_0)f, \tag{4.1}$$

where C_0 and f are the generators of $\mathrm{Num}(\mathbb{F}_{\tau_0})$ (see Lemma 2.1) and λ_0 is a suitable integer. Moreover, denoting by $p_a(C)$ the arithmetic genus of a curve C, we set

$$\delta_{\tau_0} := p_a(X_{\tau_0}) - g. \tag{4.2}$$

Note that, if all the singularities of X_{τ_0} are ordinary double points, $\delta_{\tau_0} = \deg(\mathrm{Sing}(X_{\tau_0}))$.

Remark 4.1. We recall that the adjunction formula for the dualizing sheaf ω_{X_R} of a curve X_R on a smooth surface R is (see [**7**, Chapter 1, (1.5)])

$$\omega_{X_R} = \mathcal{K}_R \otimes \mathcal{O}_R(X_R)|_{X_R}, \tag{4.3}$$

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where $\mathcal{K}_R = \mathcal{O}_R(K_R)$ denotes the canonical sheaf of R. Taking the degrees, we then obtain that

$$2p_a(X_R) - 2 = X_R \cdot (X_R + K_R). \tag{4.4}$$

In our situation, $R = \mathbb{F}_{\tau_0}$ and $X_R = X_{\tau_0}$. Then, $\mathcal{K}_{\mathbb{F}_{\tau_0}} = \mathcal{O}_{\mathbb{F}_{\tau_0}}(-2C_0 - (\tau_0 + 2)f)$, so, using (4.1), we obtain that $\mathcal{K}_{\mathbb{F}_{\tau_0}} \otimes \mathcal{O}_{\mathbb{F}_{\tau_0}}(X_{\tau_0}) = \mathcal{O}_{\mathbb{F}_{\tau_0}}(2C_0 + (\lambda_0 - 2)f)$. Hence, from (4.3) we can obtain that $\omega_{X_{\tau_0}} = \mathcal{O}_{\mathbb{F}_{\tau_0}}(2C_0 + (\lambda_0 - 2)f)|_{X_{\tau_0}}$.

Finally, since $K_{\mathbb{F}_{\tau_0}} \sim -2C_0 - (\tau_0 + 2)f$, from (4.4) and (4.1) we find that $2p_a(X_{\tau_0}) - 2 = 6\lambda_0 - 6\tau_0 - 8$.

Proposition 4.2. *The following properties hold:*

(i) *the arithmetic genus of X_{τ_0} is $p_a(X_{\tau_0}) = 3(\lambda_0 - \tau_0 - 1)$,*

(ii) $\lambda_0 \geqslant \max\{3\tau_0, \tau_0 + 5\}$,

(iii) $\delta_{\tau_0} = 3(\lambda_0 - \tau_0 - 1) - g$.

Proof. (i) This is immediate from the last relation of Remark 4.1.

(ii) From [**11**, Chapter V, Theorem 2.18], since X_{τ_0} is irreducible, $\lambda_0 + \tau_0 \geqslant 4\tau_0$. Therefore, $\lambda_0 \geqslant 3\tau_0$. On the other hand, $p_a(X_{\tau_0}) \geqslant g \geqslant 10$ by assumption. Then, using (i), we obtain that $\lambda_0 \geqslant \tau_0 + 5$.

(iii) This follows from (4.2) and from (i). □

We wish to describe how to recover the canonical model X_K starting from the chosen birational model $X_{\tau_0} \subset \mathbb{F}_{\tau_0} \cong S_0 \in \mathrm{GRC}_L(S)$.

Since X_0 is the embedded model of X_{τ_0} obtained via the dualizing sheaf $\omega_{X_{\tau_0}}$ (described above), then, in order to obtain X_0, we have to embed $\mathbb{F}_{\tau_0}$ by the sheaf $\mathcal{O}_{\mathbb{F}_{\tau_0}}(2C_0 + (\lambda_0 - 2)f)$ (see Remark 4.1). Finally, we project the obtained curve X_0 from its singular points.

Remark 4.3. Note first that $\lambda_0 - 2 > 2\tau_0$. In fact, if $\tau_0 \leqslant 2$, then $\lambda_0 > \tau_0 + 4 \geqslant 2\tau_0 + 2$. If $\tau_0 \geqslant 3$, then $\lambda_0 \geqslant 3\tau_0 > 2\tau_0 + 2$ (both arguments follow from Proposition 4.2 (ii)).

Therefore (using [**11**, Chapter V, Theorem 2.18]), the linear system $|2C_0 + (\lambda_0 - 2)f|$ is very ample on $\mathbb{F}_{\tau_0}$. Moreover, from [**4**, Proposition 1.8], and from Proposition 4.2 (iii), we get that $h^0(\mathbb{F}_{\tau_0}, \mathcal{O}_{\mathbb{F}_{\tau_0}}(2C_0 + (\lambda_0 - 2)f)) = g + \delta_{\tau_0}$. Hence, there exists an isomorphism $\varphi\colon \mathbb{F}_{\tau_0} \xrightarrow{\cong} S_0 \subset \mathbb{P}^{g-1+\delta_{\tau_0}}$, where $\varphi = \varphi_{2C_0+(\lambda_0-2)f}$ and $S_0 := \varphi(\mathbb{F}_{\tau_0})$.

Clearly, S_0 is a projective ruled surface, whose fibres are all smooth conics and $X_0 = \varphi(X_{\tau_0}) \subset S_0$, so we have the commutative diagrams:

$$\begin{array}{ccc} \mathbb{F}_{\tau_0} & \xrightarrow[\cong]{\varphi} & S_0 \subset \mathbb{P}^{g-1+\delta_{\tau_0}} \\ & & \rho\uparrow\downarrow\pi \qquad \downarrow\pi \\ & & S \subset \mathbb{P}^{g-1} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X_{\tau_0} & \xrightarrow[\cong]{\varphi|_{X_{\tau_0}}} & X_0 \subset S_0 \\ & & \rho\uparrow\downarrow\pi \qquad \downarrow\pi \\ & & X_K \subset S \end{array}$$

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where π (which is the inverse of the map ρ) is exactly the desingularization morphism of X_0, or, equivalently, the linear projection centred in $\langle \Sigma \rangle$ is generated by the singular points of X_0 (possibly infinitely near).

Remark 4.4. Since there are at most two singular points on each fibre, $\langle \Sigma \rangle$ meets S_0 in a zero-dimensional variety of degree δ_{τ_0}. It is then clear that $\delta_{\tau_0} = L$ and $\deg(S) = \deg(S_0) - \delta_{\tau_0}$.

5. Singularities of a birational model X_0

The purpose of this section is to describe all the possible singularities of X_0.

Recall that, from Remark 4.3, the projection $\pi \colon X_0 \subset S_0 \to X_K \subset S$ is centred in the singular points of X_0, and the singular fibres of S correspond to the fibres of S_0 containing the singular points of X_0. Therefore, it is enough to examine the singular fibres of S and the four-gonal divisor on each of them.

In order to do this, we focus on one singular fibre f of S and the corresponding fibre $f_0 \subset S_0$.

Remark 5.1. Note that the curve $X_K \subset S$ intersects each fibre of S in four points (the four-gonal divisor $\Phi \in g^1_4$). In particular, X_K also meets each singular fibre f in four points. If $f = l \cup m$ and $l \neq m$, then two of them belong to the line l and two are on the other line m (possibly coinciding); where this not the case, X_K has a trisecant line, hence a trigonal series (from the geometric Riemann–Roch theorem). On the other hand, if $l = m$, then the support of $\Phi = X_K \cap f$ consists of two points (possibly coinciding).

Example 5.2. Let $f \subset S$ be an embedded fibre of level 1. Then, π is the projection centred at the point $P_0 \in f_0$, where $P_0 \in \mathrm{Sing}(X_0)$. Clearly, $f = f_0 + E$, where E is the exceptional divisor and f_0 still denotes the other component of f. Setting $A := f_0 \cdot E$, $P_i \in f_0$ and $Q_i \in E$ (where $P_i \neq A \neq Q_i$ and $P_i \neq Q_i$ for $i = 1, 2$), the possible cases are the following:

(a) $\Phi = P_1 + P_2 + Q_1 + Q_2$,

(b) $\Phi = P_1 + P_2 + 2Q_1$,

(c) $\Phi = 2P_1 + Q_1 + Q_2$,

(d) $\Phi = 2P_1 + 2Q_1$,

(e) $\Phi = P_1 + 2A + Q_1$,

(f) $\Phi = P_1 + 3A$ (where $X_K \cdot f_0 = P_1 + A$ and $X_K \cdot E = 2A$),

(g) $\Phi = 3A + Q_1$ (where $X_K \cdot f_0 = 2A$ and $X_K \cdot E = A + Q_1$).

Figure 1 illustrates the corresponding singularities of X_0.

It is clear that, in all the cases above, X_0 has a double point: more precisely, either a node, in cases (a), (c), (e), (g), or an ordinary cusp, in cases (b), (d), (f).

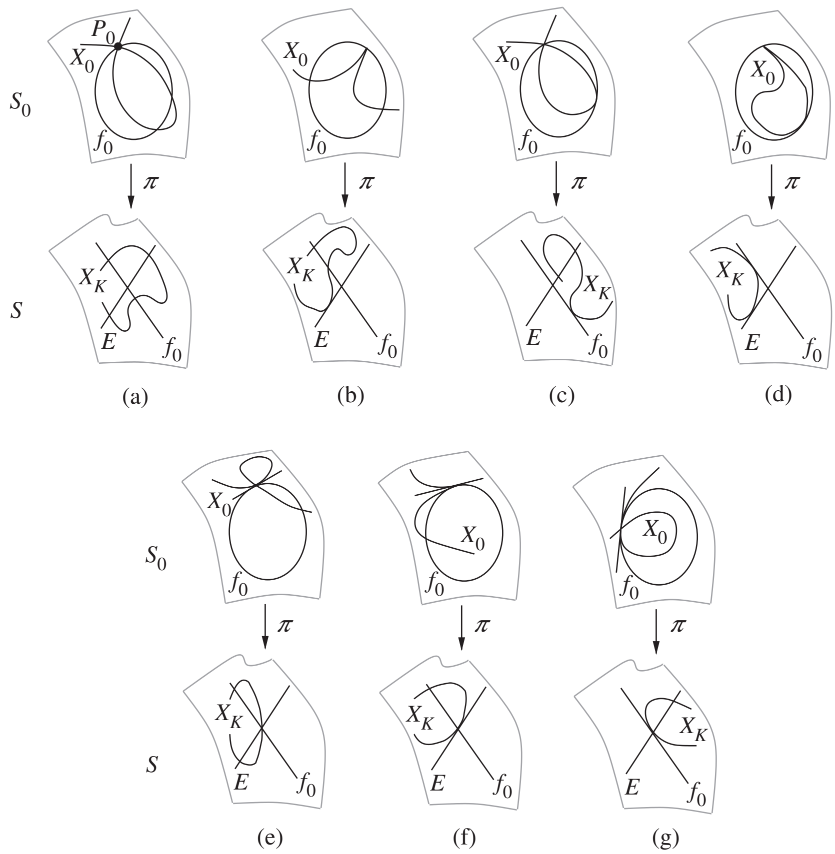

Figure 1. Level 1: possible singularities of X_0.

A description of the double points of an algebraic curve can be found, for instance, in [**10**, Lecture 20].

Here, we just recall that a *node of the nth kind* is a double point analytically equivalent to $y^2 - x^{2n} = 0$. In particular, if $n = 1, 2, 3$, it is called an *(ordinary) node*, *tacnode* or *oscnode*, respectively.

Moreover, a *cusp of the nth kind* is a double point analytically equivalent to $y^2 - x^{2n+1} = 0$. In particular, if $n = 1, 2$, it is called an *(ordinary) cusp* or *ramphoid cusp*, respectively.

Definition 5.3. We say, for short, that a double point P_0 of X_0 is *transversal* if the tangent line to the fibre f_0 at P_0 does not coincide with any of the tangent lines to X_0 at P_0; it is *tangent* otherwise.

Example 5.4. Assume that S is a surface ruled by conics having a fibre f of type (2A), as defined in Theorem 2.3. Clearly (see [**6**, § 3]), this fibre arises from a fibre $f_0 \subset S_0$ by projecting it from two points. More precisely, the projection $\pi\colon S_0 \to S$ can be factorized

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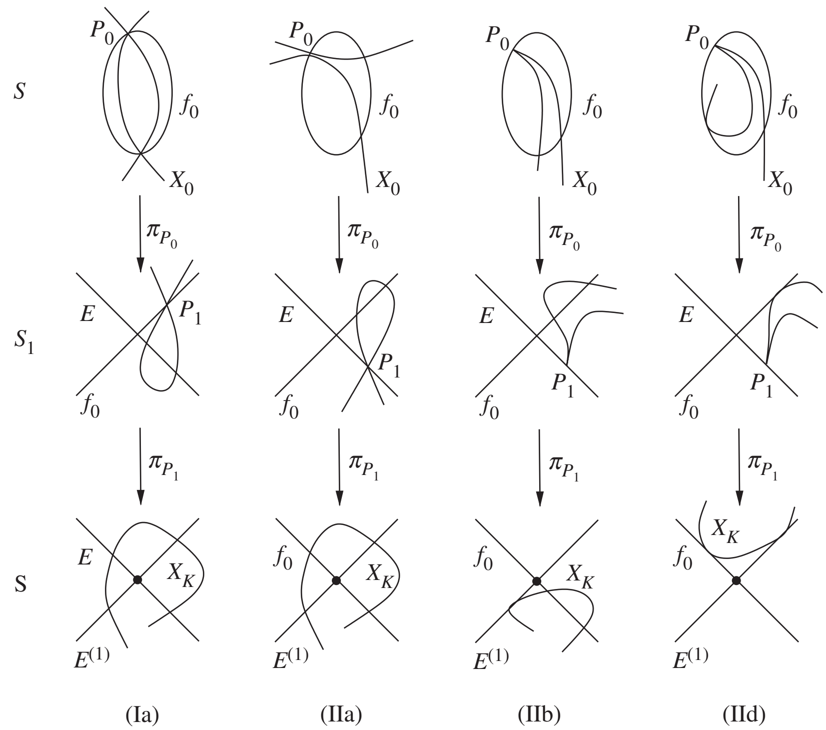

Figure 2. Type (2A): possible singularities of X_0.

by $\pi = \pi_{P_1} \circ \pi_{P_0}$, where $P_0 \in f_0$ and $P_1 \in f_1 := f_0 + E \subset \pi_{P_0}(S_0)$ and $P_1 \neq f_0 \cdot E$. There are two possibilities: either $P_1 \in f_0$ or $P_1 \in E$.

In the first case, $f = E + E^{(1)}$, while in the second one, where P_1 is infinitely near to P_0, we have $f = f_0 + E^{(1)}$ (in both cases $E^{(1)}$ denotes the exceptional divisor of the blow-up centred at P_1). Moreover, in both configurations, f turns out to be a union of two lines meeting in an ordinary double point for the surface S.

We start, in Figure 2, by sketching the situations corresponding to the configuration (a) (in both cases $f = E + E^{(1)}$ and $f = f_0 + E^{(1)}$) and the configurations (b) and (d) (both in case $f = f_0 + E^{(1)}$).

The construction (Ia) gives that X_0 has two nodes on the fibre f_0; in (IIa) the curve X_0 has a tacnode, while in (IIb) and (IId) it has a ramphoid cusp. Finally, one can easily see that the cases related to (e), (f), (g) do not occur.

Remark 5.5. The two examples above lead us to a general pattern. If X_0 has only one singular point $P_0 \in f_0$ and f is of type (nA), then $f = f_0 + E^{(n-1)}$ and π can be factorized by $\pi = \pi_{P_{n-1}} \circ \cdots \circ \pi_{P_1} \circ \pi_{P_0}$, where $P_{i+1} \in E^{(i)}$ for all i. Moreover, the type of the singularity of P_0 depends only on the intersection $X_K \cdot E^{(n-1)}$ on S, so we can always assume that the two points given by $X_K \cdot f_0$ on S are distinct.

We can now complete Example 5.4: if X_0 has one singular point on f_0, then the significant cases are (IIa) and (IIb). Here X_K is tangent (respectively, transversal) to $E^{(1)}$ on S.

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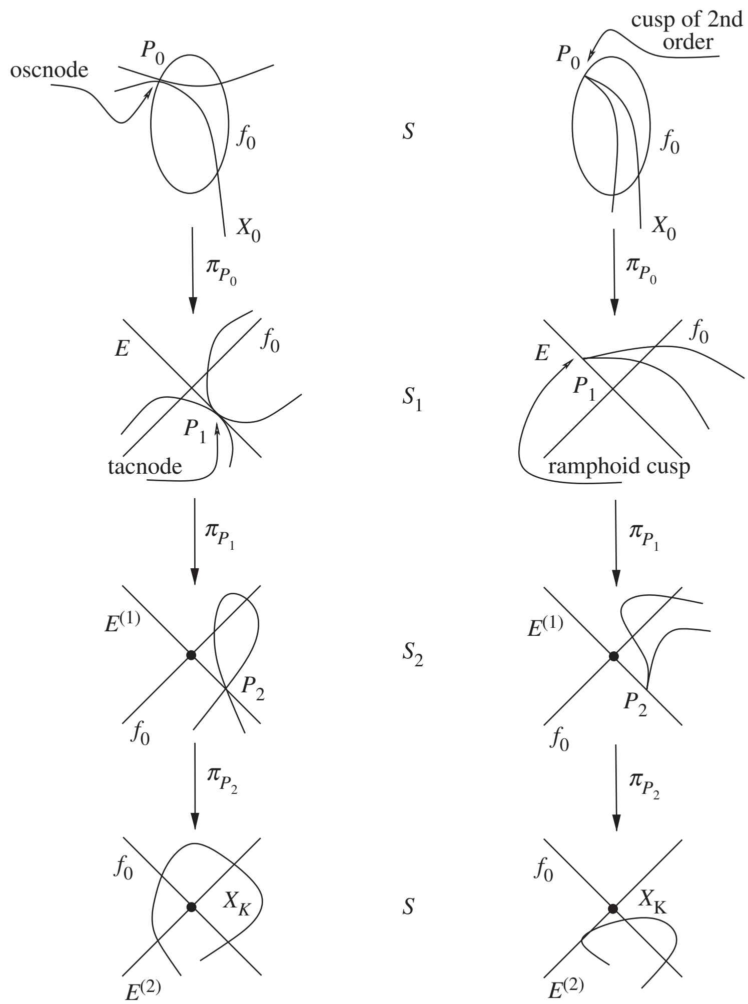

Figure 3. Type (3A): possible singularities of X_0.

Example 5.6. In the same way, we get the possible singularities in the case $F_3(A)$, as Figure 3 shows.

The above study can be easily generalized, obtaining the following result.

Proposition 5.7. *The possible singularities of $X_0 \subset S_0$ arising from a fibre of S of type $F_n(A)$, where $n \geqslant 2$, are the following points on the same fibre $f_0 \subset S_0$.*

- *If $n = 2$, there is either one double point of second kind (either a transversal tacnode or a transversal ramphoid cusp) or two double points of first kind (either node or cusp).*
- *If $n \geqslant 3$, there is either one double point of the nth kind (transversal) or two double points of lower kind.*

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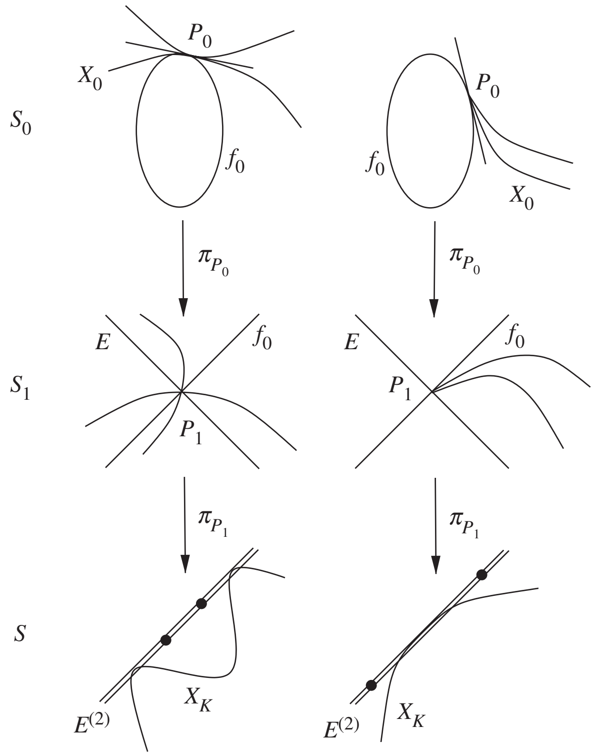

Figure 4. Type (2D): possible singularities of X_0.

Note that in the case of two double points on f_0, these two points are of kind h and k, where $h + k = n$.

Example 5.8. Assume now that S is a surface ruled by conics having a fibre f of type $(2D)$. Clearly (see [**6**, § 3]), this fibre arises from a fibre $f_0 \subset S_0$ by projecting it from two infinitely near points. More precisely, if $\pi\colon S_0 \to S$ is the considered projection, then $\pi = \pi_{P_1} \circ \pi_{P_0}$, where $P_0 \in f_0$, and if $f_1 := f_0 + E \subset \pi_{P_0}(S_0)$, then $P_1 := f_0 \cdot E$. As noted in [**6**], the fibre of S corresponding to f_0 is given by $f = 2E^{(2)}$: it is a totally degenerate conic containing two singular points of S, which correspond to the lines f_0 and E. Since f consists of a double line, the four-gonal divisor can be either $2A + 2B$ (where $A, B \in E^{(2)}$ are distinct points, non-singular for S) or $4A$, as Figure 4 shows.

It is clear that the first configuration leads to a tangential tacnode, and the second one gives a tangential ramphoid cusp of first order. Using the same argument as before, we easily get the following result.

Proposition 5.9. *The possible singularities of $X_0 \subset S_0$ arising from a fibre of S of type $F_n(D)$, where $n \geqslant 2$, consist of a unique singular point of the corresponding fibre $f_0 \subset S_0$ as follows.*

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- *If $n = 2$, then there exists either a tangential tacnode or a tangential ramphoid cusp.*
- *If $n \geqslant 3$, then there exists a tangential double point of the nth kind.*

Combining Example 5.2, Propositions 5.7 and 5.9, we obtain the following complete description of the possible singularities of X_0.

Theorem 5.10. *Let S be a surface ruled by conics containing X_K, and let $X_0 \subset S_0$ be birational models of X_K and S, respectively, where S_0 is a g.r.c. surface. Let $\pi\colon S_0 \to S$ be the usual projection. Assume that f is the unique singular fibre of S and set f_0 to be the corresponding fibre of S_0.*

The singular points of X_0 then belong to f_0 and are, as long as f is of type F_1, $F_n(A)$, $F_n(D)$ for $n \geqslant 2$, of one of the following types.

- *(F_1) One singular point: either a node or a cusp, both of them either tangential or transversal.*
- *($F_n(A)$) Only transversal singular points. More precisely:*
  - (a) *one double point of the nth kind,*
  - (b) *two double points of orders $h, k < n$, where $h + k = n$.*
- *($F_n(D)$) Only one tangential double point of the nth kind.*

In particular, all the singular points of X_0 are double points.

6. 'Standard' birational models of $X_K \subset S$

In § 4 we studied the set $\mathrm{GRC}_L(S)$ consisting of the g.r.c. surfaces S_0 such that S can be obtained from S_0 by a sequence of L monoidal transformations (L is the level of S). We now determine one such surface in a sort of 'canonical' way: this will be called the 'standard' birational model of S.

Proposition 6.1. *Let $X_0 \subset S_0 \in \mathrm{GRC}_L(S)$ be as usual. Then, $\mathrm{GRC}_L(S) = \{\mathrm{elm}_\Sigma(S_0) \mid \Sigma \subseteq \mathrm{Sing}(X_0)\}$, i.e. each $S_0' \in \mathrm{GRC}_L(S)$ can be obtained from S_0 by a sequence of elementary transformations centred in singular points of X_0 (or infinitely near to them) and conversely.*

Proof. Consider a surface $S_0' \in \mathrm{GRC}_L(S)$ and the corresponding model of X_K, say $X_0' \subset S_0'$. As in Proposition 4.2, denote by $\pi\colon S_0 \to S$ and $\pi'\colon S_0' \to S$ the projections centred in the singular points (possibly infinitely near) of X_0 and X_0', respectively. We can then define a sequence of elementary transformations (centred in some of the singular points of X_0) from S_0 to S_0'.

Conversely, note that each elementary transformation of S_0 can be obtained by considering an embedded model of S_0 that is ruled by lines and projecting it from a finite number of points. In this way, we get a birational model S_0' of S that is a geometrically ruled surface. If $X_0' \subset S_0'$ is the corresponding curve, it is clear that $\delta(X_0') = \delta(X_0)$ if and only if the above projection is centred in singular points of X_0 (this is due to the fact that the singular points of X_0 are double points for Theorem 5.10). Therefore, if $S_0' = \mathrm{elm}_{\Sigma}(S_0)$, where $\Sigma \subseteq \mathrm{Sing}(X_0)$, using Remark 4.4, the level of S_0' coincides with $\delta(X_0') = \delta(X_0) = L$; hence, $S_0' \in \mathrm{GRC}_L(S)$. □

Definition 6.2. A surface $\bar{S}_0 \in \mathrm{GRC}_L(S)$ is a *standard model* of a surface S ruled by conics if its invariant is

$$t := \min\{\tau_0 = t(S_0) \mid S_0 \in \mathrm{GRC}_L(S)\}.$$

If, moreover, $\bar{X}_0 := \rho(X_K) \subset \bar{S}_0$ is the corresponding birational model of $X_K \subset S$, we also say that $\bar{X}_0$ is a *standard model* of X_K. Finally, we denote the corresponding invariant λ_0 of $\bar{S}_0$ by λ.

Theorem 6.3. *Let S be as before, let L be its level, let $S_0 \in \mathrm{GRC}_L(S)$ be a birational model of S of invariant τ_0, and let X_0 be the model of X_K on S_0. If we assume that $t > 0$, then the following facts hold.*

(i) *If S_0 is a standard model, then the singular points of X_0 belong to the minimum unisecant C_0 of S_0.*

(ii) *There is exactly one standard model $\bar{S}_0$ of S.*

(iii) *If the singular points of X_0 belong to the minimum unisecant C_0 of S_0, then $S_0 = \bar{S}_0$.*

Proof. First, consider the model $X' \subset R_{1,\tau_0+1} \cong S_0$. We know that $X' \sim 4C_0 + (\lambda_0 + \tau_0)f$ and $\delta(X') = 3(\lambda_0 - \tau_0 - 1) - g$ by Proposition 4.2. In particular, the level of S is $L = 3(\lambda_0 - \tau_0 - 1) - g$.

Consider a singular point T of X' and the projection π_T from T. From Proposition 6.1, $\pi_T(R_{1,\tau_0+1})$ belongs to $\mathrm{GRC}_L(S)$.

(i) If S_0 is a standard model, then $\tau_0 = t$. Assume that the point T does not belong to C_0. The invariant of $\pi_T(R_{1,t+1})$ is then $t - 1$, while t is the minimum invariant of the surfaces belonging to $\mathrm{GRC}_L(S)$.

(ii) Let $\bar{S}_0 \cong R_{1,t+1}$ be a standard model and let S_0' be another surface in $\mathrm{GRC}_L(S)$. From Proposition 6.1, we know that $S_0' = \mathrm{elm}_{\Sigma}(\bar{S}_0)$, where $\Sigma \subseteq \mathrm{Sing}(\bar{X}_0)$. For simplicity, assume that $\Sigma = \{T\}$, where T is a singular point of $\bar{X}_0$. From (i), we have that $T \in C_0$ and, from Theorem 5.10, we know that T is a double point of $\bar{X}_0$, so $T = A_1 + A_2$, where $\Phi := A_1 + A_2 + A_3 + A_4$ is the four-gonal divisor on the fibre $\bar{f}_0$ containing T.

Clearly, $S_0' = \pi_T(R_{1,t+1})$, so the curve X_0' has a double point $(A_3 + A_4)$ on the fibre $\bar{f}_0'$, and such a point does not belong to the unisecant curve C_0' of S_0'. Therefore, we get from (i) that S_0' is not a standard model of S.

(iii) This follows by an analogous argument. □

Proposition 6.4. *With the above notation, if $t > 0$, then the singular points of $\bar{X}_0$ belong to distinct fibres.*

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Proof. Also in this case consider the model $X' \subset R_{1,t+1} \cong \bar{S}_0$ and assume that there exists a fibre containing two distinct singular points of X', P_1 and P_2, say. Clearly, one of them, P_1 say, does not belong to C_0. So, by projecting $R_{1,t+1}$ from P_1 we get a contradiction with the argument used in Theorem 6.3. □

Theorem 6.5. *With the notation above, the surface S has degree*

$$\deg(S) = 4(\lambda - t - 2) - \delta_t = g + \lambda - t - 5.$$

Proof. Since $\bar{S}_0 = \varphi_{2C_0+(\lambda-2)f}(\mathbb{F}_t)$ and $C_0^2 = -t$, we have

$$\deg(\bar{S}_0) = (2C_0 + (\lambda-2)f)^2 = 4(\lambda - t - 2).$$

Moreover, from Remark 4.4 we have that $\deg(S) = \deg(\bar{S}_0) - \delta_t$, so the first equality holds. The second equality follows immediately from $\delta_t = 3(\lambda - t - 1) - g$ (see Proposition 4.2 (iii)). □

7. Bounds on the invariants λ and t

We return to the global description of the four-gonal curve X; as usual, $X_K \subset S \subset V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)) \subset \mathbb{P}^{g-1}$ (where S is as in Theorem 3.1) and $\bar{X}_0 \subset \bar{S}_0 \cong \mathbb{F}_t$ are standard models of them. Since the model $X_t \subset \mathbb{F}_t$ is again a four-secant curve, it is of type $X_t \sim 4C_0 + (\lambda + t)f$.

So far we have defined the integers, a, b, c, t, δ, λ (here, $\delta := \delta_t$), which are *invariants* of the curve X. All of them are useful to describe its geometry. We start with the dependence of a, b, c on t, δ, λ.

Remark 7.1. Consider the isomorphism $\varphi_{2C_0+(\lambda-2)f} \colon \mathbb{F}_t \to \bar{S}_0 \subset \mathbb{P}^{g-1+\delta}$ and the volume $V_{\bar{S}_0} \subset \mathbb{P}^{g-1+\delta}$ generated by $\bar{S}_0$. From [**4**, Proposition 1.8], we have that

$$V_{\bar{S}_0} = \mathbb{P}(\mathcal{O}(\lambda - 2 - 2t) \oplus \mathcal{O}(\lambda - 2 - t) \oplus \mathcal{O}(\lambda - 2)).$$

If we consider the projection $\pi \colon \mathbb{P}^{g-1+\delta} \to \mathbb{P}^{g-1}$ centred at the singular locus of $\bar{X}_0$, it is clear that $\pi(V_{\bar{S}_0}) = V_S$. Using Theorem 6.3 (i), if $t > 0$, then the singular points of $\bar{X}_0$ are contained in the unisecant of minimum degree of $\bar{S}_0$ and, hence, of $V_{\bar{S}_0}$. Moreover, if these points are all distinct, then V_S has the form

$$V_S = \mathbb{P}(\mathcal{O}(\lambda - 2 - 2t - \delta) \oplus \mathcal{O}(\lambda - 2 - t) \oplus \mathcal{O}(\lambda - 2)).$$

On the other hand, taking into account that $c = g - 3 - a - b$, the scroll above is

$$V_S = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(g - 3 - a - b)).$$

Hence, comparing the two expressions of V_S and using $\delta = 3(\lambda - t - 1) - g$ (see Proposition 4.2 (iii)), we obtain that

$$a = g + t - 2\lambda + 1 \quad \text{and} \quad b = \lambda - t - 2.$$

Note that, if $t > 0$ but the δ double points of $\bar{X}_0$ are not all distinct, $a \geqslant g + t - 2\lambda + 1$.

Proposition 7.2. *With the above notation, if* $V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$, *then*

$$a + b \geqslant \frac{g-5}{2}.$$

Proof. We consider the curve $X_K \subset V$ and the ruled surface

$$R_{a,b} = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b)) \subset V.$$

In order to apply the intersection formula (IF) in § 2, we first observe that $R_{a,b}$ and X_K meet properly on V, i.e. $\dim(R_{a,b} \cap X_K) = \dim(R_{a,b}) + \dim(X_K) - \dim(V) = 0$.

To see this, note that X_K cannot be contained in $R_{a,b}$, otherwise the general four-gonal divisor on X_K would span a line instead of a plane, in contradiction to the geometric Riemann–Roch theorem.

Hence, $\dim(R_{a,b} \cap X_K) = 0$ and we can apply (IF), which gives the (non-negative) degree of the intersection: $0 \leqslant \deg_V(R_{a,b} \cdot X_K) = 4(a+b)+2g-2-4(g-3) = 2(a+b)-g+5$, and this is the required inequality. □

The lower bound of λ in terms of t given in Proposition 4.2 ($\lambda \geqslant \max\{3t, t+5\}$) can be improved.

Remark 7.3. Assume that $t \geqslant 1$ and that the δ singular points of X_t are distinct. Clearly,

$$2\delta \leqslant \int C_0 \cdot X_t = \int C_0 \cdot (4C_0 + (\lambda + t)f) = \lambda - 3t;$$

hence, $\lambda \geqslant 2\delta + 3t$. Since $\delta = 3(\lambda - t - 1) - g$ (see Proposition 4.2 (iii)), we easily obtain that

$$\lambda \leqslant \frac{2g + 3t + 6}{5}. \tag{7.1}$$

Proposition 7.4. *The following properties hold.*

(i) *For any* t,
$$\lambda \geqslant \frac{g}{3} + t + 1.$$

(ii) *If* $t = 0$, *then*
$$\lambda \leqslant \frac{g+3}{2}.$$

(iii) *If* $t \geqslant 1$, *then*
$$\lambda \leqslant t + \frac{g+3}{2} \quad \text{and} \quad t \leqslant \frac{g+3}{4}.$$

(iv) *If* $t \geqslant 1$ *and the double points of* X *are all distinct, then*
$$\lambda \leqslant \frac{g+3}{2} \quad \text{and} \quad t \leqslant \frac{g+3}{6}.$$

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Proof. (i) This follows from Proposition 4.2 (i), since $p_a(\bar{X}_0) = 3(\lambda - t - 1) \geqslant g$.

(ii)–(iii) Using Theorems 3.1 and 6.5, we have that

$$g + \lambda - t - 5 = \deg(S) \leqslant \left\lceil \frac{3g-8}{2} \right\rceil \quad \Rightarrow \quad \lambda - t \leqslant \left\lceil \frac{3g-8}{2} \right\rceil - g + 5 = \left\lceil \frac{g+2}{2} \right\rceil;$$

hence, we obtain the required bounds either if $t = 0$ or if $t \geqslant 1$. Moreover, from Proposition 4.2 we have $\lambda \geqslant 3t$, so, using the previous bound of λ in (iii), we finally get $t \leqslant \lambda/3 \leqslant t/3 + (g+3)/6$, and this concludes the proof.

(iv) In this case, we can apply Remark 7.3. Using $3(\lambda - t - 1) - g = \delta \geqslant 0$ followed by (7.1), we get that

$$t \leqslant \lambda - \frac{g+3}{3} \leqslant \frac{2g+3t+6}{5} - \frac{g+3}{3} \quad \Rightarrow \quad t \leqslant \frac{g+3}{6}.$$

Using this bound and (7.1) we finally get $\lambda \leqslant (g+3)/2$. □

8. Geometric meaning of the invariant $\boldsymbol{\lambda}$

We keep the notation of the previous section: S is a surface ruled by conics such that $X_K \subset S \subset V$, and L denotes its level. Take a standard model $\bar{S}_0 \in \mathrm{GRC}_L(S)$ and consider its embedded model $R_{1,t+1} \subset \mathbb{P}^{t+3}$.

As usual, we denote by $X' \subset R_{1,t+1}$ the corresponding model of X_K, where $X' \sim 4C_0 + (\lambda + t)f$.

Remark 8.1. Note that such an X' has only double points as singularities (see Theorem 5.10).

Remark 8.2. Denote by $H_{X'}$ the hyperplane section of $X' \subset R := R_{1,t+1} \subset \mathbb{P}^{t+3}$. Since $H_R \sim C_0 + (t+1)f$,

$$H_{X'} = H_R \cdot X' \sim \Phi + \Delta, \quad \text{where } \Phi \in g^1_4 \text{ and } \Delta \in g^{1+t}_{\lambda+t}.$$

In particular, $\deg(H_{X'}) = \lambda + t + 4$ and one can verify that X' is the embedding of minimum degree of X_K.

Definition 8.3. A linear system $|D|$ on a curve X is called *primitive* if, for each point $P \in X$, the linear system $|D + P|$ has P as a base point. Equivalently, $\dim |D + P| = \dim |D|$.

It is not difficult to see that the following property of $X' \subset \mathbb{P}^{t+3}$, here stated for a standard model $\bar{S}_0$, also holds for any birational model $S_0 \in \mathrm{GRC}_L(S)$.

Proposition 8.4. *Let $\bar{S}_0 \cong R_{1,t+1} \subset \mathbb{P}^{t+3}$ be a standard model of S. Let Φ and Δ be as before and let $X' = X_{\Phi+\Delta} \subset R_{1,t+1}$ be as usual. If $g > 13$, then the following facts hold.*

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(i) *The divisor* $\Phi+\Delta$ *is a special divisor on* X*; in particular,* $K-\Phi-\Delta$ *is an effective divisor.*

(ii) *The curve* $X' \subset \mathbb{P}^{t+3}$ *is linearly normal.*

Proof. (i) It is enough to show that $h^0(\mathcal{O}(K-\Phi-\Delta)) > 0$ or, equivalently, by the Riemann–Roch theorem, that $\lambda < g-1$. If $t=0$, it follows immediately from Proposition 7.4 (ii).

If $t \geqslant 1$, still from Proposition 7.4 (iii), we have that $\lambda \leqslant t+(g+3)/2$ and $t \leqslant (g+3)/4$, so $\lambda \leqslant (3g+9)/4 < g-1$, where the last inequality is true since $g > 13$ by assumption. Finally, since $\Phi+\Delta$ is a special divisor, $K-\Phi-\Delta$ is an effective divisor.

(ii) We recall that (as in Remark 7.1) the surface $\bar{S}_0$ is naturally embedded, via the isomorphism $\varphi_{2C_0+(\lambda-2)f}$, in a projective space, namely, $\bar{S}_0 \subset V_{\bar{S}_0} \subset \mathbb{P}^{g-1+\delta}$, where

$$V_{\bar{S}_0} = \mathbb{P}(\mathcal{O}(\lambda-2-2t) \oplus \mathcal{O}(\lambda-2-t) \oplus \mathcal{O}(\lambda-2))$$

and $t \geqslant 0$. If $t > 0$, defining $M := \langle \varphi_{2C_0+(\lambda-2)f}((\lambda-3-t)\Phi) \rangle$, it is clear that

$$\pi_M \colon V_{\bar{S}_0} \to \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(t+1)) = R_{1,t+1}.$$

This map can be factorized as follows: setting Σ as the divisor of the singular points of $\bar{X}_0$ and taking into account that $K-\Phi-\Delta$ is an effective divisor on X from (i), set

$$L := \langle \varphi_{2C_0+(\lambda-2)f}(\Sigma) \rangle, \qquad N := \langle \varphi_K(K-\Phi-\Delta) \rangle.$$

We then have the following diagram:

$$\begin{array}{ccccccccc}
 & & \bar{X}_0 & \subset & \bar{S}_0 & \subset & V_{\bar{S}_0} & \subset & \mathbb{P}^{g-1+\delta} \\
 & \overset{\bar{\varphi}}{\nearrow} & \downarrow & & \downarrow & & \downarrow & & \downarrow \pi_L \\
\mathbb{F}_t \supset X_t & \xrightarrow{\varphi_K} & X_K & \subset & S & \subset & V & \subset & \mathbb{P}^{g-1} \\
 & \underset{\varphi'}{\searrow} & \downarrow & & & \searrow & \downarrow & & \downarrow \pi_N \\
 & & X' & & \subset & & R_{1,t+1} & \subset & \mathbb{P}^{t+3}
\end{array} \qquad (8.1)$$

where $\bar{\varphi} := \varphi_{2C_0+(\lambda-2)f}$, $\varphi' = \varphi_{\Phi+\Delta}$ and $\pi_N \circ \pi_L = \pi_M$.

Note that $\bar{X}_0$ is not linearly normal. Namely, $\bar{X}_0$ is not special; if it was linearly normal, then $\dim\langle\Phi\rangle = 3$ in $\mathbb{P}^{g-1+\delta}$, while $\bar{X}_0$ is contained in the scroll $V_{\bar{S}_0}$ which is ruled by planes.

Hence, we have to consider its normalization $\tilde{X} \subset \mathbb{P}^{g-1+2\delta}$, and the corresponding scroll

$$W := \bigcup_{\Phi \in g^1_4} \langle \Phi \rangle \subset \mathbb{P}^{g-1+2\delta}.$$

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It is easy to see that W is ruled by planes. Setting $\tilde{L} := \langle \Sigma \rangle \subset \mathbb{P}^{g-1+2\delta}$, the projection $\pi_{\tilde{L}}$ factorizes through the normalization map, say Π, as follows:

$$\begin{array}{ccccc} \tilde{X} & \subset & W & \subset & \mathbb{P}^{g-1+2\delta} \\ \downarrow & & \downarrow & & \downarrow \Pi \\ \bar{X}_0 & \subset & V_{\bar{S}_0} & \subset & \mathbb{P}^{g-1+\delta} \\ \downarrow & & \downarrow & & \downarrow \pi_L \\ X_K & \subset & V & \subset & \mathbb{P}^{g-1} \end{array} \tag{8.2}$$

and $\pi_L \circ \Pi = \pi_{\tilde{L}}$. Setting $\tilde{M} := \langle (\lambda - 3 - t)\Phi \rangle \subset \mathbb{P}^{g-1+2\delta}$ and keeping in mind (8.1) and (8.2) we obtain:

$$\begin{array}{ccccc} \tilde{X} & \subset & W & \subset & \mathbb{P}^{g-1+2\delta} \\ \downarrow & & \downarrow & & \downarrow \pi_{\tilde{L}} \\ X_K & \subset & V & \subset & \mathbb{P}^{g-1} \\ \downarrow & & \downarrow & & \downarrow \pi_N \\ X' & \subset & R_{1,t+1} & \subset & \mathbb{P}^{t+3} \end{array}$$

where $\pi_N \circ \pi_{\tilde{L}} = \pi_{\tilde{M}}$. Since $\pi_{\tilde{M}} \colon \tilde{X} \to X'$ and $\tilde{X}$ is linearly normal, X' is also linearly normal.

If $t = 0$, the proof follows in a similar way. □

Proposition 8.5. *Let $\bar{S}_0 \cong R_{1,t+1} \subset \mathbb{P}^{t+3}$, and let Φ, Δ and $X' = X_{\Phi+\Delta}$ be as usual. If $g > 13$, then the following hold.*

(i) *The linear system $|\Delta|$ defined above is primitive.*

(ii) *If $B \subset \Delta$ is a divisor on X' such that $B \in g^1_\beta \neq g^1_4$, then $B \sim \Delta - A_1 - \cdots - A_t$, for suitable $A_i \in X' \setminus C_0$ for all i. In particular, $\beta = \lambda$.*

Proof. (i) Assume that there exists $P \in X'$ such that $\Delta + P \in g^{2+t}_{\lambda+t+1}$, and consider the model of X_K given by $X_{\Delta+P} \subset \mathbb{P}^{t+2}$. Keeping in mind Proposition 8.4, we have that $X' = X_{\Phi+\Delta}$ is linearly normal in $\mathbb{P}^{t+3}$. Hence, we can consider the following diagram:

$$\begin{array}{ccccccc} & & X_{\Phi+\Delta} & \subset R_{1,t+1} & \subset & \mathbb{P}^{t+3} \\ & \nearrow & \downarrow & & & \downarrow \pi_{\langle \Phi - P \rangle} \\ X & \longrightarrow & X_{\Delta+P} & & \subset & \mathbb{P}^{t+2} \\ & \searrow & \downarrow & & & \downarrow \pi_P \\ & & X_\Delta & & \subset & \mathbb{P}^{t+1} \end{array}$$

therefore, $\Phi - P$ is a triple point of $X' = X_{\Phi+\Delta}$, in contrast with Remark 8.1.

(ii) The result is obvious for $t = 0$, so we can assume that $t > 0$. Since $\langle \Phi \rangle$ is a fibre of $R_{1,t+1}$, the projection $\pi_{\langle \Phi \rangle}\colon \mathbb{P}^{t+3} \to \mathbb{P}^{t+1}$, centred in the line $\langle \Phi \rangle$, maps $R_{1,t+1}$ onto the cone $R_{0,t}$.

Moreover, $H_{X'} \sim \Phi + \Delta$, so $\pi_{\langle \Phi \rangle}(X') = X_\Delta = \varphi_\Delta(X) \subset R_{0,t}$. Since all the singularities of X' belong to C_0 (see Theorem 6.3), X_Δ has only one singular point in $C := \pi_{\langle \Phi \rangle}(C_0)$, which is the vertex of the cone $R_{0,t}$.

In order to obtain a linear series of dimension 1 on $X_\Delta \subset \mathbb{P}^{t+1}$, it is necessary to project it from t points, say $A_1, \dots, A_t$, of X_Δ. If each of these points is different from C, then we get the required $B \in g^1_\beta$, where $\beta = \deg(\Delta) - t = \lambda$. If, for some i, it occurs that $A_i = C$, then $\pi_C(R_{0,t}) = \mathcal{C} \subset \mathbb{P}^t$, where $\mathcal{C}$ is a rational normal curve of degree t: in this case $B \in g^1_4$, in contrast with the assumption that $g^1_\beta \neq g^1_4$. □

Definition 8.6. A linear system $|\Delta|$ on the curve X is called *minimal* if it satisfies Proposition 8.5 (i) and (ii).

Remark 8.7. Note that, if we perform the previous construction with respect to a birational model $S_0 \in \mathrm{GRC}_L(S)$, which is not a standard model, the corresponding series $|\Delta|$ is primitive but not minimal.

Remark 8.8. If $t = 0$, i.e. $|\Delta| = g^1_\lambda$, then $|\Delta|$ is minimal if and only if it is primitive.

We have seen in Proposition 8.5 that, if $R_{1,t+1}$ is isomorphic to a standard model, the associated series $|\Delta|$ on X' is minimal. The converse is also true, as the following result shows.

Proposition 8.9. *Let X be as usual and consider two divisors $\Phi \in g^1_4$ and $\Delta \in g^{1+t}_{\lambda+t}$. If the linear series $|\Delta|$ is minimal on X, then $X_{\Phi+\Delta} \subset R_{1,t+1}$ is isomorphic to a standard model of $X_K \subset S$.*

Proof. We have to consider two cases; either

$$\dim\langle \varphi_{\Phi+\Delta}(\Phi) \rangle = 1 \quad \text{or} \quad \dim\langle \varphi_{\Phi+\Delta}(\Phi) \rangle = 2.$$

(1) In this case, since $\deg(\Phi) = 4$, $X_{\Phi+\Delta}$ is contained in a geometrically ruled surface as a four-secant curve. Moreover, since $\dim|\Delta| = t + 1$, the invariant of such a ruled surface is t. Therefore, $X_{\Phi+\Delta} \subset R_{h,t+h}$ for a suitable $h \geqslant 1$.

Assume first that $h \geqslant 2$. Consider, as in the proof of Proposition 8.5 (ii), the projection $\pi_{\langle \Phi \rangle}\colon R_{h,t+h} \to R_{h-1,t+h-1}$, where $\pi_{\langle \Phi \rangle}(X_{\Phi+\Delta}) = X_\Delta$. Clearly, $H_R \sim U + hf$, where U is a unisecant of degree $t + h$. Therefore,

$$\Phi + \Delta = H_R \cdot X_{\Phi+\Delta} \sim h\Phi + U \cdot X_{\Phi+\Delta}$$

(see Remark 8.2). Since $h \geqslant 2$, it follows that $\Delta \sim (h-1)\Phi + U \cdot X_{\Phi+\Delta}$, so $\Phi \subset \Delta$. Hence, $\Delta - \Phi \in g^{t-1}_{\lambda+t-4}$. Therefore, there exist $t - 2$ points, say $A_1, \dots, A_{t-2}$, such that $\Delta - \Phi - A_1 - \cdots - A_{t-2} \in g^1_{\lambda-2}$. But this is impossible since $|\Delta|$ is minimal; hence, it satisfies Proposition 8.5 (ii). This proves that $h = 1$, so $X_{\Phi+\Delta} \subset R_{1,t+1}$.

If $X_{\Phi+\Delta}$ has a multiple point P not belonging to C_0, then we can project it from P and $t - 1$ general points of the curve, obtaining a divisor $B \subset \Delta$ such that $B \in g^1_\lambda$ and

$\bar{\lambda} < \lambda$. Therefore, all the singular points of $X_{\Phi+\Delta} \subset R_{1,t+1}$ belong to C_0, and this implies (from Theorem 6.3) that $R_{1,t+1}$ is a standard model.

(2) In this case the curve is contained in the scroll V, ruled by planes, whose fibres are $\langle \varphi_{\Phi+\Delta}(\Phi) \rangle$, $\Phi \in g^1_4$. So we set, for suitable $a \leqslant b \leqslant c$, $X_{\Phi+\Delta} \subset V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$. Clearly, among the unisecant curves U^b of degree b such that $U^b \subset R_{a,b} \subset V$, we can choose one, say U, that does not meet $X_{\Phi+\Delta}$ (otherwise, $X_{\Phi+\Delta}$ would be contained in the ruled surface $R_{a,b} \subset V$, in contradiction to the assumption). Therefore, if we consider the projection $\pi_{\langle U \rangle} \colon V \to R_{a,c}$, it is clear that $\pi_{\langle U \rangle}(X_{\Phi+\Delta})$ is again a curve, say $\bar{X}_{\Phi+\Delta}$, whose hyperplane divisor is still $\Phi + \Delta$, but $\bar{X}_{\Phi+\Delta} \subset R_{a,c}$, also contrary to the assumption. $\square$

The remaining part of this section is devoted to the case $t = 0$. Here, the linear series $|\Delta|$ is denoted by $|\Lambda|$, since its degree is λ, as noted in Remark 8.8.

We show that this linear series is, in general, not unique. In order to determine all such series g^1_λ, let $X_K \subset S \subset V$ be as usual and assume that $t(S) = 0$. Let $\Phi \in g^1_4$, let $\Lambda' \in g^1_{\lambda'}$ (where $\lambda' > 4$) and let $X_{\Phi+\Lambda'} := \varphi_{\Phi+\Lambda'}(X) \subset R_{1,1}$. Denote by $|l|$ and $|l'|$ the two rulings of $R_{1,1}$.

Notation. If $P \in R_{1,1}$, denote by l_P and l'_P the lines of the two rulings passing through P. Moreover, if A is a double point of $X_{\Phi+\Lambda'}$, denote by A_1 and A_2 the corresponding points on the canonical model of the curve, i.e. $A_1, A_2 \in X_K$ are such that $\varphi_{\Phi+\Lambda'}(A_1) = \varphi_{\Phi+\Lambda'}(A_2) = A$.

Proposition 8.10. *In the above situation, each pair of double points, A and B say, of $X_{\Phi+\Lambda'}$ such that $l_A \neq l_B$ and $l'_A \neq l'_B$ determines a linear series $|\bar{\Lambda}'| \neq |\Lambda'|$ of degree λ'.*

Proof. Take the four-gonal and the λ'-gonal divisors of $|\Lambda'|$ containing, respectively, the two double points:

$$A_1 + A_2 + A'_1 + A'_2 \in g^1_4, \qquad A_1 + A_2 + P_1 + \cdots + P_{\lambda'-2} \in |\Lambda'|,$$
$$B_1 + B_2 + B'_1 + B'_2 \in g^1_4, \qquad B_1 + B_2 + Q_1 + \cdots + Q_{\lambda'-2} \in |\Lambda'|.$$

Set $\bar{\Lambda}' = \Phi + \Lambda' - (A_1 + A_2 + B_1 + B_2)$; clearly, $|\bar{\Lambda}'|$ is a linear series of degree λ' and distinct from $|\Lambda'|$. $\square$

Remark 8.11. Let $X_K \subset S$ be as usual and assume that $t = 0$ and λ are the invariants of S. Let $\Phi \in g^1_4$, $\Lambda \in g^1_\lambda$ be two divisors on X. In the general case, the δ double points of $X' = X_{\Phi+\Lambda} \subset R_{1,1}$ belong to different lines of the two rulings $|l|$ and $|l'|$. Therefore, from the above result it is clear that there are $\binom{\delta}{2}$ linear series $|\Lambda|$ of degree λ; with each of them we can associate a model of X lying on $R_{1,1}$. In particular, if $|\bar{\Lambda}|$ is one of these series, the corresponding model $X_{\Phi+\bar{\Lambda}}$ still has δ double points, since the pair (A, B) has been replaced by (A', B'), where $A' := \varphi_{\Phi+\bar{\Lambda}}(A'_1) = \varphi_{\Phi+\bar{\Lambda}}(A'_2)$ and $B' := \varphi_{\Phi+\bar{\Lambda}}(B'_1) = \varphi_{\Phi+\bar{\Lambda}}(B'_2)$, following the notation in Proposition 8.10.

Theorem 8.12. *Let $X_K \subset S \subset V$ and let S be a surface ruled by conics of minimum degree. Let t and λ be the invariants of S defined before. If $t = 0$, then the invariant λ is the minimum degree of a linear series distinct from the g^1_4, i.e.*

$$\lambda = \min\{r \mid X \text{ has a complete and base-point-free linear series } g^1_r \text{ and } r > 4\}.$$

Moreover, assume that $|\Lambda|$ and $|\Lambda'|$ are two distinct linear series of degree λ, and let S and S' be the associated surfaces. The following facts then hold.

(i) *If $\lambda \neq (g+3)/2$, then $S = S'$.*

(ii) *If $\lambda = (g+3)/2$, then S and S' are not necessarily coincident, but belong to a pencil of surfaces. Moreover, each element of this pencil is a surface ruled by conics, associated with a linear series of degree λ, and having degree $(3g-7)/2$.*

Proof. Recall that λ is defined at the beginning of this section as the invariant of X such that a standard model of X is a divisor of type $(4, \lambda)$ on $R_{1,1}$. Consider a linear series $g^1_{\lambda'} \neq g^1_\lambda$; we need to show that $\lambda' \geqslant \lambda$. Suppose that $\lambda' < \lambda$. If $g^1_{\lambda'}$ is minimal, consider $\Lambda' \in g^1_{\lambda'}$. Clearly, $X_{\Phi+\Lambda'} \subset R_{1,1}$ is a standard model.

If $g^1_{\lambda'}$ is not minimal, then it is not primitive (from Remark 8.8), so there exist t' points, say $A_1, \dots, A_{t'}$, such that $\Delta := \Lambda' + A_1 + \cdots + A_{t'}$ is both primitive and minimal. Therefore, $X_{\Phi+\Delta} \subset R_{1,t'+1}$ is a standard model. Hence, the corresponding surface S' ruled by conics is such that $X_K \subset S' \subset V$ and $\deg(S') = g + \lambda' - t' - 5$. Assume that $S' \neq S$; since $X_K \subseteq S \cap S'$, by (IF) we have that

$$\deg(X_K) \leqslant \int_V S \cdot S' = 2\deg(S) + 2\deg(S') - 4\deg(V).$$

Hence, $2g - 2 \leqslant 2(2g + \lambda + \lambda' - t - t' - 10) - 4(g-3)$, so $\lambda + \lambda' \geqslant t + t' + g + 3$. Since $\lambda' < \lambda$, we obtain that

$$\lambda > \frac{g+3}{2} + \frac{t+t'}{2} = \frac{g+3}{2} + \frac{t'}{2},$$

where the last equality comes from the assumption that $t = 0$. On the other hand, $\lambda \leqslant (g+3)/2$ from Proposition 7.4. Hence, $t' < 0$, and this is impossible. Therefore, we have proved that if $S' \neq S$, then $\lambda' \geqslant \lambda$.

Assume now that $S' = S$. Clearly, $t' = t = 0$ and $\deg(S) = \deg(S')$. Hence, from Theorem 6.5, it follows that $\lambda = \lambda'$.

In this way, we have proved the first part of the statement.

(i) Assume now that $\lambda \neq (g+3)/2$ and $S \neq S'$. We can then use (IF) as before and, from the assumption that $\lambda = \lambda'$, we obtain that $\lambda \geqslant (g+3)/2 + t'/2$. We again apply Proposition 7.4 to S, so $\lambda \leqslant (g+3)/2$. Comparing these inequalities, we obtain that $t' = 0$; hence, $\lambda = (g+3)/2$, contrary to the assumption.

(ii) Suppose now that $\lambda = (g+3)/2$. In this case, from Theorem 6.5, $\deg(S) = g+\lambda-5 = (3g-7)/2$. Therefore, $\deg(S') = g - \lambda - t' - 5 \leqslant \deg(S)$, and this implies that $t' = 0$ and $\deg(S') = \deg(S) = (3g-7)/2$. So, by Theorem 3.1, the result follows.

□

9. Bounds for the invariants a and b

In this section we determine the range of the invariants a and b of the four-gonal curve X.

We keep the notation of § 7, where $\bar{X}_0 \subset \bar{S}_0 \subset \bar{V}$ are standard models of $X_K \subset S \subset V$ and $\pi\colon \mathbb{P}^{g-1+\delta} \to \mathbb{P}^{g-1}$ is the projection centred on the singular locus of $\bar{X}_0$.

Recall also that $V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$ and

$$\bar{V} = V_{\bar{S}} = \mathbb{P}(\mathcal{O}(\lambda - 2 - 2t) \oplus \mathcal{O}(\lambda - 2 - t) \oplus \mathcal{O}(\lambda - 2)).$$

Moreover, from Proposition 4.2 (iii) we have $\delta = 3(\lambda - t - 1) - g$, and from Proposition 7.4 we obtain the following range of the invariant λ:

$$\frac{g+3}{3} \leqslant \lambda - t \leqslant \frac{g+3}{2}. \tag{9.1}$$

Remark 9.1. Note that, from the above expression of $\bar{V}$, it follows that $a \leqslant \lambda - 2 - 2t$, $b \leqslant \lambda - 2 - t$, $c \leqslant \lambda - 2$. Moreover, since $a + b + c = g - 3$, there are only two independent invariants, a and b say.

Notation. Clearly, if $a < b$, there exists a unique directrix on V with degree a. In this case, we denote by A such a directrix of V, by $\bar{A} \subset \bar{V}$ the preimage of A via π, by δ_A the number of the double points (possibly infinitely near) of $\bar{X}_0$ lying on $\bar{A}$, and by $\bar{a}$ the degree of $\bar{A}$. Then,

$$a = \bar{a} - \delta_A. \tag{9.2}$$

Proposition 9.2. *Let $t > 0$ and let U be a directrix on $\bar{S}_0$. If $\deg(U) < \lambda - 2$, then $U = C_0$.*

Proof. It is enough to consider the isomorphism $\varphi_{2C_0+(\lambda-2)f}\colon \mathbb{F}_t \to \bar{S}_0$ and the unisecant irreducible curves C_0 and $U = C_0 + \alpha f$ on $\mathbb{F}_t$. If $U \neq C_0$, then $\alpha \geqslant t$ from Lemma 2.1. So,

$$\deg_{\bar{S}_0}(U) = \int_{\bar{S}_0} (C_0 + \alpha f) \cdot (2C_0 + (\lambda - 2)f) = \lambda - 2 + 2\alpha - 2t \geqslant \lambda - 2,$$

and the result follows. □

Proposition 9.3. *Let $t \geqslant 0$. The directrix $\bar{A}$ of $\bar{V}$ is then contained in $\bar{S}_0$.*

Proof. Assume that $\bar{A} \not\subset \bar{S}_0$. Taking into account that $\deg(\bar{S}_0) = 4(\lambda - t - 2)$, as computed in Theorem 6.5, and $\deg(\bar{V}) = 3(\lambda - t - 2)$, using the intersection formula we then have that

$$\int_{\bar{V}} \bar{X}_0 \cdot \bar{A} \leqslant \int_{\bar{V}} \bar{S}_0 \cdot \bar{A} = \deg(\bar{S}_0) + 2\deg(\bar{A}) - 2\deg(\bar{V}) = 2\bar{a} - 2\lambda + 2t + 4.$$

Therefore, if the δ_A singular points are distinct, it follows that

$$\delta_A \leqslant \frac{1}{2} \int_{\bar{V}} \bar{X}_0 \cdot \bar{A} = \bar{a} - \lambda + t + 2.$$

In the case of infinitely near points, it is not so difficult to show that the same relation holds.

In this way, from (9.2), we have the following bound of a: $a = \bar{a} - \delta_A \geqslant \lambda - t - 2$, which is the minimum degree of a directrix of V.

Consider the directrix $\pi(C_0) \subset V$. Since $\deg_{\bar{V}}(C_0) = \lambda - 2t - 2$ and the centre of π contains at least one point of C_0, $\deg_V(\pi(C_0)) \leqslant \lambda - 2t - 3 < \lambda - t - 2$; this concludes the proof. □

Remark 9.4. Consider the unisecant $\bar{A} \subset \bar{S}_0 \cong \mathbb{F}_t$. Clearly, from Lemma 2.1, we have $\bar{A} \sim C_0 + \alpha f$ for some $\alpha \geqslant t$ or $\alpha = 0$. Therefore, as computed in the proof of Proposition 9.2, we have

$$\bar{a} = \deg_{\bar{S}_0}(\bar{A}) = \lambda - 2t + 2\alpha - 2, \tag{9.3}$$

$$\bar{A} \cdot \bar{X}_0 = \int_{\bar{S}_0} (C_0 + \alpha f)(4C_0 + (\lambda + t)f) = \lambda - 3t + 4\alpha,$$

$$\delta_A \leqslant \frac{\bar{A} \cdot \bar{X}_0}{2} = \frac{\lambda - 3t + 4\alpha}{2}. \tag{9.4}$$

It is immediate to see, from (9.2), (9.3) and (9.4), that

$$a = \bar{a} - \delta_A \geqslant \frac{\lambda - t - 4}{2}. \tag{9.5}$$

Note that this bound of a does not depend on α.

Remark 9.5. Note that, since $\delta_A \leqslant \delta$, from (9.2) we have that $a = \bar{a} - \delta_A \geqslant \bar{a} - \delta$; so, taking into account that $\delta = 3(\lambda - t - 1) - g$, from (9.3) we immediately obtain that

$$a \geqslant \lambda - 2t + 2\alpha - 2 - 3(\lambda - 1 - t) + g = g - 2\lambda + t + 2\alpha + 1 \geqslant g - 2\lambda + t + 1. \tag{9.6}$$

Remark 9.6. In order to compare the two bounds of a given by (9.5) and (9.6), just note that

$$\frac{\lambda - t - 4}{2} < g - 2\lambda + t + 1 \quad \Longleftrightarrow \quad \lambda < \frac{2g + 3t + 6}{5}.$$

This leads us to consider the best lower bound of a in each of the two ranges of λ.

Keeping in mind the previous remarks, we immediately have the following.

Proposition 9.7. *The invariant a has the following lower bound:*

$$a_{\min} := a_{\min}(g, \lambda, t) = \begin{cases} \left\lceil \dfrac{\lambda - t - 4}{2} \right\rceil & \text{if } \lambda \geqslant \dfrac{2g + 3t + 6}{5}, \\ g - 2\lambda + t + 1 & \text{if } \lambda \leqslant \dfrac{2g + 3t + 6}{5}, \end{cases}$$

and these bounds are attained if and only if $\bar{A} = C_0$.

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Remark 9.8. We can also obtain an 'absolute' lower bound of a, just observing that $a_{\min}$ can be realized when $\delta_A = \delta$; hence, when $(\lambda - t - 4)/2 = g - 2\lambda + t + 1$ or, equivalently (from Remark 9.6), when $\lambda = (2g + 3t + 6)/5$. It is immediate to see that, on this line of the plane (t, λ), the two values of $a_{\min}(g, \lambda, t)$ coincide and are equal to

$$a_{\min}(g, t) = \frac{g - t - 7}{5}. \tag{9.7}$$

Clearly, the minimum value of a is obtained for the maximum value of t (if $t > 0$). Therefore, keeping in mind that $\lambda \geqslant 3t$ (by Proposition 4.2), it is clear that the minimum value of a corresponds to the common point of the lines $\lambda = (2g + 3t + 6)/5$ and $\lambda = 3t$. We finish the argument by observing that $(2g+3t+6)/5 = 3t$ if and only if $t = (g+3)/6$, and substituting this value into (9.7) we obtain that $a_{\min}(g) = (g - 9)/6$. Note that, in this case, $\lambda = 3t = (g + 3)/2$. Summing up, we have proved that if $t > 0$, then $a_{\min}(g) = (g - 9)/6$ for $t = (g + 3)/6$ and $\lambda = (g + 3)/2$.

On the other hand, if $t = 0$, the value of (9.7) occurs for $\lambda = (2g + 6)/5$, and we have that $a_{\min}(g) = (g - 7)/5$ for $\lambda = (2g + 6)/5$.

Corollary 9.9. *With the above notation we have that,*

$$\text{for all } t \geqslant 0, \quad a \geqslant \frac{g - 9}{6}, \qquad \text{while for } t = 0, \quad a \geqslant \frac{g - 7}{5}.$$

In particular, V_S is not a cone for $t \geqslant 0$ and $g \geqslant 10$ or $t = 0$ and $g \geqslant 8$.

Proposition 9.10. *Keeping the above notation, the invariants a and b can vary in the following two ranges:*

$$a_{\min} \leqslant a \leqslant \frac{g - 3}{3}, \tag{R$_2$}$$

$$g - \lambda - 1 \leqslant a + b \leqslant \frac{2(g - 3)}{3}. \tag{R$_3$}$$

Proof. The two inequalities on the right-hand side in (R$_2$) and (R$_3$) follow from $a \leqslant b \leqslant c$ and $a + b + c = g - 3$. For the left-hand inequality of (R$_3$), note that $c \leqslant \lambda - 2$ by Remark 9.1; hence, $a + b = g - 3 - c \geqslant g - 3 - (\lambda - 2)$. $\square$

Remark 9.11. If $a < (g - \lambda - 1)/2$, then $a < b$; hence, A is unique.

10. Existence of curves of given invariants λ, a, b when $t = 0$

Remark 10.1. If $t = 0$, a standard model $\bar{S}_0$ of S is isomorphic to $\mathbb{F}_0$ via $\varphi_{2l+(\lambda-2)l'} \colon \mathbb{F}_0 \to \bar{S}_0 \subset \mathbb{P}^{3\lambda-4}$ and $\bar{X}_0 \sim 4l + \lambda l'$ on $\bar{S}_0$. Moreover, the projection from $\bar{V}$ to V is $\pi \colon \mathbb{P}^{3\lambda-4} \to \mathbb{P}^{g-1}$, $\bar{V} = \mathbb{P}(\mathcal{O}(\lambda - 2)^{\oplus 3})$ and Proposition 4.2 (iii), (9.1), (R$_2$), (R$_3$)

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become, respectively,

$$\delta = 3(\lambda - 1) - g, \tag{10.1}$$

$$\frac{g+3}{3} \leqslant \lambda \leqslant \frac{g+3}{2}, \tag{R$_1$}$$

$$a_{\min} \leqslant a \leqslant \frac{g-3}{3}, \tag{R$_2$}$$

$$g - \lambda - 1 \leqslant a + b \leqslant \frac{2(g-3)}{3}, \tag{R$_3$}$$

where

$$a_{\min} = \begin{cases} \left\lceil \dfrac{\lambda - 4}{2} \right\rceil & \text{if } \lambda \geqslant \dfrac{2g+6}{5}, \\ g - 2\lambda + 1 & \text{if } \lambda \leqslant \dfrac{2g+6}{5}. \end{cases}$$

Note that $(2g+6)/5$ belongs to the range of λ given in (R_1). Moreover, $\lambda = (2g+6)/5$ if and only if $\delta = \lambda/2$.

At this point, beside the map $\varphi := \varphi_{2l+(\lambda-2)l'}$, it is useful to consider a further model of S given by the isomorphism

$$\psi := \varphi_{4l+\lambda l'} \colon \mathbb{F}_0 \to S' \subset \mathbb{P}^{5\lambda+4}.$$

Notation. From now on, we denote a geometrically ruled surface $\varphi_{nl+ml'}(\mathbb{F}_0) \subset \mathbb{P}^{(n+1)(m+1)-1}$ by $S_{n,m}$.

In this way, $S' = S_{4,\lambda}$ and we set $f \colon S' \to \bar{S}_0$, the isomorphism being given by $\varphi = f \circ \psi$.

Remark 10.2. A hyperplane section $H \cdot S'$ of $S' \subset \mathbb{P}^{5\lambda+4}$ corresponds, via the morphism ψ, to a curve $X_H \subset \mathbb{F}_0$ of type $(4, \lambda)$. It is not difficult to show, using Theorem 5.10, that $P \in \mathbb{F}_0$ is a double point of X_H if and only if H contains the tangent plane $T_P(S')$ (here, P means $\psi(P) \in S'$).

Remark 10.3. Let $S := S_{n,m} \subset \mathbb{P}^{(n+1)(m+1)-1}$ and let $Y \subset S$ be a divisor whose decomposition into irreducible and reduced components is $Y = Y_1 \cup \cdots \cup Y_s$. Let $P_1, \ldots, P_\delta$ be points of Y, and denote by δ_i the number of these points belonging to the component Y_i. Let

$$L := \langle T_{P_1}(S), \ldots, T_{P_\delta}(S) \rangle$$

be the linear space spanned by the δ tangent planes. Clearly, if H is any hyperplane containing L, H intersects Y_i in at least $2\delta_i$ points. Therefore, if $2\delta_i > \deg(Y_i)$, then H contains Y_i.

The above observation leads to the following.

Definition 10.4. We say that $P_1, \ldots, P_\delta$ *trivially degenerate* the component Y_i if $2\delta_i > \deg(Y_i)$. Moreover, we say that $P_1, \ldots, P_\delta$ *trivially degenerate* the curve Y if this occurs for at least one component of Y.

Remark 10.5. Let $S' = S_{4,\lambda}$ be as before. Assume that $a \leqslant b \leqslant c$ fulfil the relations (R_1), (R_2) and (R_3).

(a) Let $M \sim l$ be a divisor of S'. Clearly, $\deg(M) = H \cdot M = \lambda$. We consider $\lambda - 2 - a$ distinct points of M, say $P_1, \dots, P_{\lambda-2-a}$. Clearly, $P_1, \dots, P_{\lambda-2-a}$ do not trivially degenerate M if and only if $2(\lambda - 2 - a) \leqslant \deg(M) = \lambda$ if and only if $a \geqslant (\lambda-4)/2$, and this is true by ($\mathrm{R}_2$).

(b) In the same way, if $N \sim l$ is a divisor of S', and $P_1, \dots, P_{\lambda-2-b}$ are distinct points of N, then $2(\lambda-2-b) \leqslant 2(\lambda-2-a) \leqslant \deg(N) = \lambda$, again by ($\mathrm{R}_2$). So $P_1, \dots, P_{\lambda-2-b}$ do not trivially degenerate N.

(c) Consider now a divisor $Q \sim (\lambda - 2 - c)l'$ consisting of $\lambda - 2 - c$ distinct components and a set of distinct points $P_1, \dots, P_{\lambda-2-c}$, one on each component of Q. Obviously, $P_1, \dots, P_{\lambda-2-c}$ do not trivially degenerate Q.

Theorem 10.6. *Let g, a, b, λ be positive integers, with $g \geqslant 10$, and consider the following inequalities:*

$$\frac{g+3}{3} \leqslant \lambda \leqslant \frac{g+3}{2}, \tag{R_1}$$

$$a_{\min} \leqslant a \leqslant \frac{g-3}{3}, \tag{R_2}$$

$$g - \lambda - 1 \leqslant a + b \leqslant \frac{2(g-3)}{3}, \tag{R_3}$$

where

$$a_{\min} = \begin{cases} \left\lceil \dfrac{\lambda-4}{2} \right\rceil & \text{if } \lambda \geqslant \dfrac{2g+6}{5}, \\ g - 2\lambda + 1 & \text{if } \lambda \leqslant \dfrac{2g+6}{5}. \end{cases}$$

There then exists a four-gonal curve of genus g and invariants a, b, λ if and only if (R_1), (R_2), (R_3) *are verified.*

Proof. If there exists a four-gonal curve of genus g and invariants a, b, λ, then (R_1), (R_2), (R_3) come from Remark 10.1.

Conversely, we choose g, λ, a, b satisfying the inequalities (R_1), (R_2), (R_3). Using Remark 10.2, it is enough to show that there exists an irreducible hyperplane section $H \cdot S'$ of $S' = S_{4,\lambda}$, i.e. a curve $X_H \sim 4l + \lambda l'$ on $\mathbb{F}_0$, of genus g and invariants a, b.

Take the following three divisors of S': M, N, Q, where $M \sim l \sim N$ ($M \neq N$) and $Q \sim (\lambda - 2 - c)l'$ consists of distinct lines. Moreover, consider $\lambda - 2 - a$ distinct points of M, $\lambda - 2 - b$ distinct points of N, and $\lambda - 2 - c$ distinct points of Q, one on each line and none belonging to M or N.

Note that $M + N + Q \in |2l + (\lambda - 2 - c)l'|$ and (10.1) implies that $(\lambda - 2 - a) + (\lambda - 2 - b) + (\lambda - 2 - c) = \delta$.

Therefore, also taking into account Remark 10.5, it is immediate to see that the hypotheses of Lemma 11.4 are verified; we can then deduce that the linear space L

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spanned by the tangent planes to S' at the above δ points does not contain any further point of S'. In particular, a general hyperplane $H \supset L$ corresponds to an irreducible curve $X_H \sim 4l + \lambda l'$ having exactly δ nodes, so its genus is $g(X_H) = 3(\lambda - 1) - \delta = g$.

Consider the isomorphism $f\colon S' \to \bar{S}_0$, defined previously, and set $\bar{A} := f(M)$, $\bar{B} := f(N)$. Clearly,

$$\deg(\bar{A}) = \deg(\bar{B}) = \lambda - 2.$$

Set $\bar{X}_0 := \varphi(X_H) \subset \bar{S}_0$ and denote by δ_A and δ_B the number of the double points of $\bar{X}_0$ lying on $\bar{A}$ and on $\bar{B}$, respectively. From the construction, it is clear that

$$\delta_A = \lambda - 2 - a \quad \text{and} \quad \delta_B = \lambda - 2 - b.$$

Setting $A, B \subset S \subset V$ the projections of $\bar{A}$ and $\bar{B}$, respectively, via $\pi_{\langle \bar{\Delta} \rangle}\colon \bar{S}_0 \to S$, from (9.2) we have that $\deg(A) = \deg(\bar{A}) - \delta_A = \lambda - 2 - \delta_A = a$ and $\deg(B) = \deg(\bar{B}) - \delta_B = \lambda - 2 - \delta_B = b$. In this way, one can easily deduce that $V = V_S = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$, so a and b are the other two invariants of X. □

In order to complete the proof of Theorem 10.6, we need to prove the 'key-lemma' stated in Lemma 11.4. The next section is devoted to this purpose.

11. Proof of the key-lemma

In order to prove the key-lemma (Lemma 11.4), we need some preliminary technical results.

Lemma 11.1. *Let $S := S_{n,m}$ and let $D \sim hl + kl' \subset S$ be a divisor, where $h \leqslant n + 1$ and $k \leqslant m + 1$. Then,*

(i) $\dim\langle D \rangle = h(m + 1) + k(n + 1) - hk - 1$.

Moreover, if D is irreducible, then

(ii) *D is a non-special curve,*

(iii) *D is a linearly normal curve in $\langle D \rangle$.*

Proof. (i) Assume first that $h \leqslant n$ and $k \leqslant m$. It is clear that, setting $S' := S_{n-h,m-k}$, we have $\dim\langle D \rangle = h^0(\mathcal{O}_S(1)) - h^0(\mathcal{O}_{S'}(1)) - 1$, and this proves the above relation.

The remaining cases are $h = n + 1$ and $k \leqslant m + 1$ or $h \leqslant n + 1$ and $k = m + 1$. In both cases, $D \sim hl + kl'$ cannot be contained in any hyperplane section $H \cdot S \sim nl + ml'$ of S. Hence, $\langle D \rangle = \langle S \rangle$, so $\dim\langle D \rangle = \dim\langle S \rangle = (n + 1)(m + 1) - 1$, and this gives the formula in the statement when $h = n + 1$ or $k = m + 1$.

(ii) It is enough to show that $\deg(D) > 2p_a(D) - 2$. Taking into account that $\deg(D) = hm + kn$ and $p_a(D) = hk - h - k + 1$, and using the assumption that $n \geqslant h - 1$ and $m \geqslant k - 1$, we obtain that $\deg(D) = hm + kn \geqslant h(k - 1) + (h - 1)k > 2hk - 2h - 2k = 2p_a(D) - 2$.

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(iii) It is enough to prove that $h^0(D, \mathcal{O}_D(1)) = \dim\langle D\rangle + 1$. Since D is non-special, as proved before, applying the Riemann–Roch theorem, we obtain that $h^0(\mathcal{O}_D(1)) = \deg(D) - p_a(D) + 1$, and this coincides with $\dim\langle D\rangle + 1$, as one can easily verify. Hence, D is linearly normal in $\langle D\rangle$. □

Lemma 11.2. *Let $S := S_{2,k}$, where $k \geqslant 2$, and consider d distinct points $P_1, \dots, P_d \in S$, where $d \leqslant 2k + 1$. Setting $J := \langle P_1, \dots, P_d\rangle$, if $\dim(J) < d - 1$, then there exists a unisecant curve U on S such that $\#(U \cap \{P_1, \dots, P_d\}) \geqslant \deg(U) + 1$. In particular, $U \subset S \cap J$.*

Proof. Assume, for simplicity, that the considered points belong to distinct fibres of S'. Since $\dim|l + kl'| = 2k + 1 \geqslant d$, there exists a unisecant curve linearly equivalent to $l + kl'$ containing $P_1, \dots, P_d$. Therefore, we can find a unisecant, U' say, of minimum degree containing $P_1, \dots, P_d$. Clearly, $U' \sim l + \epsilon l'$, where $\epsilon \leqslant k$; moreover, $U' = U + l'_1 + \cdots + l'_\alpha$, where U is irreducible, $P_1, \dots, P_{d-\alpha} \in U$ and $P_{d-\alpha+i} \in l'_i \setminus U$ for $i = 1, \dots, \alpha$. We show that U is the required unisecant curve. Were this not the case, setting $\beta := \deg(U) + 1 - (d - \alpha)$, it would follow that $\beta > 0$. Let $T := \langle J, A_1, \dots, A_\beta\rangle$, where $A_j \in U$. Clearly, $U \subset T$; hence, T meets each fibre l'_i in two points: $P_{d-\alpha+i}$ and $U \cap l'_i$. Since the fibres are conics, choosing $B_i \in l'_i$, the linear space

$$\Sigma := \langle J, A_1, \dots, A_\beta, B_1, \dots, B_\alpha\rangle$$

contains $\langle U'\rangle$. Therefore, $\dim\langle U'\rangle \leqslant \dim(\Sigma) \leqslant \dim(J) + \alpha + \beta = \dim(J) + \deg(U) + 1 - d + 2\alpha$. On the other hand, using Lemma 11.1, $\dim\langle U'\rangle = \deg(U') = \deg(U) + 2\alpha$, so $\dim(J) \geqslant d - 1$, against the assumption.

It is easy to extend this proof to the case where at most two of the d points belong to the same fibre. □

Lemma 11.3. *Let $S := S_{4,\lambda}$, where $\lambda \geqslant 4$, and let $\tilde{D} \in |2l + \epsilon l'|$ be a bisecant curve on S such that $\tilde{D}$ does not contain any fibre of S. Consider $d + 1$ points $P, P_1, \dots, P_d$ as follows: $P \in S$, $P_1, \dots, P_d \in \tilde{D}$ such that they do not trivially degenerate $\tilde{D}$ and at most two of them belong to the same fibre. Assume that $P_1, \dots, P_m$ are double points of $\tilde{D}$ (for $0 \leqslant m \leqslant d$) and that $P_{m+1}, \dots, P_d$ are simple points of $\tilde{D}$. Let*

$$T := \langle P, T_{P_1}(S), \dots, T_{P_m}(S), t_{P_{m+1}}(\tilde{D}), \dots, t_{P_d}(\tilde{D})\rangle,$$

where $T_{P_i}(S)$ and $t_{P_i}(\tilde{D})$ denote the tangent plane to S and the tangent line to $\tilde{D}$, respectively, at P_i.

If $\epsilon \leqslant \lambda$ and $d \leqslant \lambda$, then $\dim(T) = 2d + m$.

Proof. For simplicity, assume that $P \in \tilde{D}$ and $P_1, \dots, P_d$ belong to distinct fibres of S. In this situation, $T \subseteq \langle\tilde{D}\rangle$ and $m \leqslant d \leqslant \epsilon$.

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Claim. T is a proper subspace of $\langle\tilde{D}\rangle$.

In order to prove this, observe that, by Lemma 11.1 and the assumption $d \leqslant \lambda$, we have that

$$\dim\langle\tilde{D}\rangle = 2\lambda + 3\epsilon + 1 \geqslant 2d + 3\epsilon + 1.$$

As noted before, $m \leqslant \epsilon$; hence, $\dim\langle\tilde{D}\rangle \geqslant 2d + 3m + 1 > 2d + m \geqslant \dim(T)$, and this proves the claim.

Let $N := \dim\langle\tilde{D}\rangle$ and consider the projection $\pi_T \colon \mathbb{P}^N \to \mathbb{P}^n$ with centre T for a suitable n. Clearly, by the claim above, $n > 0$. Let $R := R(\tilde{D})$ be the ruled surface generated by $\tilde{D}$ via the ruling on S. Since T is a multisecant space of R and $P_1, \dots, P_d$ belong to distinct fibres, $T \cap R$ contains a unisecant curve (see [**4**, Lemma 1.5]), Y say. Therefore, $\pi_T(R) = \pi_T(\tilde{D})$ is a rational normal curve of degree n in $\mathbb{P}^n$. In particular,

$$N - n = \dim\langle\tilde{D}\rangle - \dim\langle\pi_T(\tilde{D})\rangle = \dim(T) + 1. \tag{11.1}$$

In order to prove the statement, observe that it holds that $\dim(T) \leqslant 2d + m$.

First case ($\tilde{D}$ is irreducible.) Since $\pi_{T|\tilde{D}}$ is a map of degree 2,

$$n = \deg(\pi_T(\tilde{D})) = \frac{\deg(\tilde{D}) - \int T \cdot \tilde{D}}{2}. \tag{11.2}$$

Moreover, from Lemma 11.1 (iii) we get $N = \dim\langle\tilde{D}\rangle = h^0(\mathcal{O}_{\tilde{D}}(1)) - 1 = \deg(\tilde{D}) - p_a(\tilde{D})$, so, using (11.1), we obtain that

$$\begin{aligned}\dim(T) &= N - n - 1\\ &= \deg(\tilde{D}) - p_a(\tilde{D}) - \frac{\deg(\tilde{D}) - \int T \cdot \tilde{D}}{2} - 1\\ &= \frac{\deg(\tilde{D}) + \int T \cdot \tilde{D}}{2} - p_a(\tilde{D}) - 1.\end{aligned}$$

Note that $\deg(\tilde{D}) = 4\epsilon + 2\lambda$ and $p_a(\tilde{D}) = \epsilon - 1$; moreover, by the definition of T, $\int T \cdot \tilde{D} \geqslant 2d + 2m + 1$. Hence, we obtain that $\dim(T) \geqslant \epsilon + \lambda + d + m + 1/2$. Thus, if we assume that $\dim(T) < 2d + m$, we get $\epsilon + \lambda + d + m + 1/2 < 2d + m$, so $d > \lambda + \epsilon + 1/2$, contrary to the assumption that $d \leqslant \lambda$.

Second case ($\tilde{D}$ is reducible.) Let $\tilde{D} = U_1 + U_2$, where U_i are irreducible unisecant curves. Let d_i be the number of points among $P_1, \dots, P_d$ belonging to U_i. Clearly, $P_1, \dots, P_m \in U_1 \cap U_2$, so $d = d_1 + d_2 - m$. Moreover, we have that

$$\dim\langle\tilde{D}\rangle = \dim\langle U_1\rangle + \dim\langle U_2\rangle - \int U_1 \cdot U_2 + 1. \tag{11.3}$$

Since T is a proper subspace of $\langle\tilde{D}\rangle$, as proved in the previous claim, $\tilde{D} \not\subset T$. Therefore, only two cases can occur: either $U_i \not\subset T$ for $i = 1, 2$ or (for instance) $U_1 \subset T$ and $U_2 \not\subset T$.

If $U_i \not\subset T$ for $i = 1, 2$, then $\pi_T(\tilde{D}) = \pi_T(U_1) = \pi_T(U_2)$, so

$$n = \dim\langle \pi_T(\tilde{D})\rangle = \dim\langle \pi_T(U_i)\rangle = \deg(\pi_T(U_i)) = \deg(U_i) - \int T \cdot U_i \quad \text{for } i = 1, 2. \tag{11.4}$$

Adding the previous relations (11.4) for $i = 1$ and $i = 2$, we obtain that $2n = \deg(U_1 + U_2) - \int T \cdot (U_1 + U_2)$, so this equality coincides with (11.2) and we conclude the proof as in the first case.

We are left to study the case $U_1 \subset T$, i.e. $U_1 = Y$. Since T contains the tangent lines to U_2 at all the d_2 points defined before, and since $U_1 \subset T$ and the m double points of $\tilde{D}$ belong to $U_1 \cap U_2$,

$$\int T \cdot U_2 = 2d_2 + \int U_1 \cdot U_2 - m.$$

In this case, (11.4) holds only for U_2, so it becomes

$$\dim\langle \pi_T(\tilde{D})\rangle = \deg(U_2) - \left(2d_2 + \int U_1 \cdot U_2 - m\right).$$

Therefore, using the relation above and (11.3), and taking into account that $\dim\langle U_i\rangle = \deg(U_i)$, we obtain that $\dim\langle \tilde{D}\rangle - \dim\langle \pi_T(\tilde{D})\rangle = \deg(U_1) + 2d_2 - m + 1$. We now substitute $d_2 = d + m - d_1$ and use (11.1), obtaining that $\dim(T) + 1 = \deg(U_1) + 2d + 2m - 2d_1 - m + 1$. Finally, recall that the P_i do not trivially degenerate $\tilde{D}$; hence, $2d_1 \leqslant \deg(U_1)$. So, we obtain that

$$\dim(T) + 1 \geqslant 2d + m + 1,$$

as required. In the general case, the proof follows in a similar way. □

Notation. Since we consider, in the following result, both $S' := S_{4,\lambda}$ and $S_{2,c+2}$, we denote the divisors on these surfaces by $D_4, \tilde{D}_4, \ldots$ and $D_2, \tilde{D}_2, \ldots$, respectively.

Lemma 11.4 (key-lemma). *Let g, a, b, c, λ be positive integers satisfying (2.2), (R_1), (R_2), (R_3).*

Let $S' := S_{4,\lambda} \subset \mathbb{P}^{5\lambda+4}$ and let $D_4 \in |2l + (\lambda - 2 - c)l'|$ be a curve on S' of type $D_4 = \tilde{D}_4 + \sum_{i=1}^{\alpha} l'_i$, where α is an integer such that $0 \leqslant \alpha \leqslant \lambda - 2 - c$ and $\tilde{D}_4$ is a suitable bisecant divisor not containing any irreducible component linearly equivalent to l'.

We take $\delta = 3(\lambda - 1) - g$ distinct points on D_4 that do not trivially degenerate D_4: set $P_1, \ldots, P_{\delta-\alpha} \in \tilde{D}_4$ and $P'_1, \ldots, P'_\alpha \in \sum_{i=1}^{\alpha} l'_i$ such that $P'_i \in l'_i \setminus \tilde{D}_4$ for $i = 1, \ldots, \alpha$. Setting

$$L := \langle T_{P_1}(S'), \ldots, T_{P_{\delta-\alpha}}(S'), T_{P'_1}(S'), \ldots, T_{P'_\alpha}(S')\rangle,$$

if $P \in S'$ is any further point such that $P \notin L$ and $L' := \langle P, L\rangle$, then we have

$$\dim(L') = 3\delta.$$

In particular, $\dim(L) = 3\delta - 1$, i.e. L is of maximum dimension and the intersection of L and S' consists only of the points $P_1, \ldots, P_{\delta-\alpha}, P'_1, \ldots, P'_\alpha$.

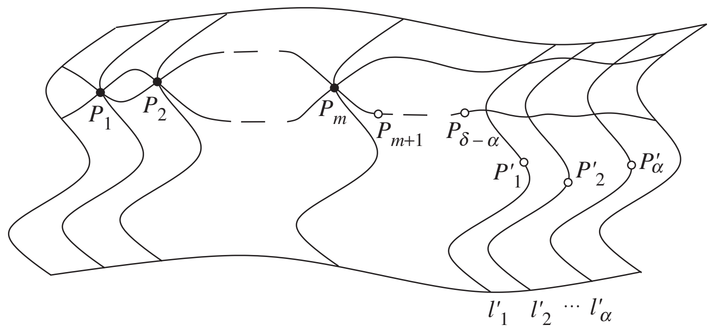

Figure 5. The divisor D_4 on the surface $S_{4,\lambda}$.

Proof. First note that $\dim(L') \leqslant 3\delta$ and $\dim(L) \leqslant 3\delta - 1$. It is, thus, enough to show that $\dim(L') \geqslant 3\delta$.

Assume first that $P \notin \tilde{D}_4$.

Step 1 (computation of the dimension of $\boldsymbol{\Sigma := \langle L', D_4\rangle}$). Among the chosen points $P_1, \dots, P_{\delta-\alpha} \in \tilde{D}_4$, consider those that are singular points of $\tilde{D}_4$, say $P_1, \dots, P_m$, for some $0 \leqslant m \leqslant \delta - \alpha$.

Clearly, since they are double points of $\tilde{D}_4$, the tangent plane at each of them is contained in $\langle \tilde{D}_4 \rangle$. On the other hand, the tangent plane at the remaining $\delta - m$ points intersects $\langle D_4 \rangle$ in a line (either tangent to $\tilde{D}_4$ for $P_{m+1}, \dots, P_{\delta-\alpha}$, or tangent to l'_i for the points of type P'_i). Briefly,

$$\left.\begin{array}{ll} T_{P_i}(S') \subset \langle \tilde{D}_4 \rangle & \text{for } i = 1, \dots, m, \\ T_{P_i}(S') \cap \langle D_4 \rangle = t_{P_i}(D_4) = t_{P_i}(\tilde{D}_4) & \text{for } i = m+1, \dots, \delta - \alpha, \\ T_{P'_j}(S') \cap \langle D_4 \rangle = t_{P'_j}(D_4) = t_{P'_j}(l'_j) & \text{for } j = 1, \dots, \alpha. \end{array}\right\} \tag{11.5}$$

Now consider the projection $\pi := \pi_{\langle D_4 \rangle} \colon S' = S_{4,\lambda} \to S_{2,c+2}$, and set

$$J := \pi(\Sigma) = \langle \bar{P}, \bar{P}_{m+1}, \dots, \bar{P}_{\delta-\alpha}, \bar{P}'_1, \dots, \bar{P}'_\alpha \rangle,$$

where $\bar{P} := \pi(P)$, $\bar{P}_i := \pi(T_{P_i}(S'))$ for $i = m+1, \dots, \delta - \alpha$, and $\bar{P}'_j := \pi(T_{P'_j}(S'))$ for $j = 1, \dots, \alpha$.

By the definition of J, we clearly have that

$$\dim(\Sigma) = \dim(J) + \dim\langle D_4 \rangle + 1. \tag{11.6}$$

Step 2 (computation of the dimension of $\boldsymbol{J}$). Observe that the isomorphisms $\varphi_{4l+\lambda l'}$ and $\varphi_{2l+(c+2)l'}$ induce a canonical isomorphism, say χ, as follows:

$$\begin{array}{ccc} & \mathbb{F}_0 & \\ \swarrow^{\varphi_{4l+\lambda l'}} & & \searrow^{\varphi_{2l+(c+2)l'}} \\ S_{4,\lambda} & \xrightarrow{\quad\chi\quad} & S_{2,c+2} \end{array}$$

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and χ coincides with π on $S_{4,\lambda} \setminus D_4$. Therefore, setting $D_2 := \chi(D_4) \subset S_{2,c+2}$, the points $\bar{P}_{m+1}, \ldots, \bar{P}_{\delta-\alpha}$, $\bar{P}'_1, \ldots, \bar{P}'_\alpha$ belong to D_2. Clearly, $\dim(J) \leqslant \delta - m$. We want to show that $\dim(J) = \delta - m$.

Assume that $\dim(J) < \delta - m$. In order to apply Lemma 11.2, we need to compare the number of points spanning J with the integer c. On the one hand, from (10.1) and (R_1) we have that $\delta = 3(\lambda - 1) - g \leqslant (g+3)/2$. On the other hand, from (R_3), we get $c \geqslant (g-3)/3$, i.e. $g \leqslant 3c + 3$. Therefore, we obtain that

$$\delta - m \leqslant \delta \leqslant \frac{g+3}{2} \leqslant \frac{3c+6}{2} < 2c + 5 \quad \Rightarrow \quad \delta - m + 1 \leqslant 2(c+2) + 1.$$

So, we can apply Lemma 11.2 to J (which is spanned by $\delta - m + 1$ points and has dimension smaller than $\delta - m$) and $S_{2,c+2}$. In this way we obtain that there exists a unisecant curve $\bar{U} \subset J \cap S_{2,c+2}$ such that, setting r the number of the points among $\bar{P}, \bar{P}_{m+1}, \ldots, \bar{P}_{\delta-\alpha}, \bar{P}'_1, \ldots, \bar{P}'_\alpha$ belonging to $\bar{U}$, $\deg(\bar{U}) \leqslant r - 1$. Let $\bar{U} \sim l + \epsilon l'$; then, $\deg(\bar{U}) = c + 2 + 2\epsilon$.

Claim. The unisecant $\bar{U}$ is not contained in D_2.

If not, let $U := \chi^{-1}(\bar{U})$ and let h be the number of the points among P, the P_i and P'_j belonging to U. On the one hand, since these points do not trivially degenerate D_4 (by assumption) and $U \subset D_4$ (since $\bar{U} \subset D_2$ by the assumption of the claim), $2h \leqslant \deg(U)$.

On the other hand, $h \geqslant r$ by the definitions of h and r and from $\chi(U) = \bar{U}$. From all these observations, it follows that $\deg(U) \geqslant 2h \geqslant 2r \geqslant 2(\deg(\bar{U}) + 1) = 2(c + 3 + 2\epsilon)$. Since $\deg(U) = \lambda + 4\epsilon$, we obtain that $2c + 6 \leqslant \lambda$. Using the bound $c \geqslant (g-3)/3$, we finally get $\lambda \geqslant \frac{2}{3}g + 4$, against (R_1). In this way the claim is proved.

Since $\bar{U}$ is not contained in D_2, their intersection surely contains the r points introduced before. So

$$r \leqslant \int_{S_{2,c+2}} \bar{U} \cdot D_2 = (l + \epsilon l') \cdot (2l + (\lambda - 2 - c)l') = \lambda - 2 - c + 2\epsilon.$$

The above relation and $\deg(\bar{U}) \leqslant r-1$ give that $c+2+2\epsilon = \deg(\bar{U}) \leqslant r-1 \leqslant \lambda-3-c+2\epsilon$, so $\lambda \geqslant 2c+5$, and this leads to a contradiction, as in the proof of the claim above.

Hence, such a unisecant curve $\bar{U}$ does not exist, and this implies that

$$\dim(J) = \delta - m. \tag{11.7}$$

Step 3 (computation of the dimension of L'). Putting (11.6) and (11.7) together, we finally obtain that

$$\dim(\Sigma) = \dim\langle D_4 \rangle + \delta - m + 1. \tag{11.8}$$

We now compare $\dim(\Sigma)$ with $\dim(L')$. Consider the linear space

$$T := \langle P, T_{P_1}(S'), \ldots, T_{P_m}(S'), t_{P_{m+1}}(\tilde{D}_4), \ldots, t_{P_{\delta-\alpha}}(\tilde{D}_4) \rangle \subseteq L'.$$

Note that, from (R_1), we have $g \geqslant 2\lambda - 3$; hence, $\delta - \alpha \leqslant \delta = 3(\lambda - 1) - g \leqslant \lambda$. Therefore, the assumptions in Lemma 11.3 are satisfied by $S_{4,\lambda}$, $\tilde{D}_4$ and T, with respect to the points $P, P_1, \ldots, P_{\delta-\alpha}$. We then obtain that

$$\dim(T) = 2(\delta - \alpha) + m. \tag{11.9}$$

Since $T \subseteq \langle \tilde{D}_4, P\rangle$ by (11.5), there exist β points, say $R_1, \ldots, R_\beta \in \tilde{D}_4$ such that $\langle T, R_1, \ldots, R_\beta\rangle$ coincides with $\langle \tilde{D}_4, P\rangle$, where

$$\beta = \dim\langle \tilde{D}_4, P\rangle - \dim(T) \leqslant \dim\langle \tilde{D}_4\rangle - \dim(T) + 1. \qquad (11.10)$$

Therefore, the linear space $\langle L', R_1, \ldots, R_\beta\rangle$ contains $\langle \tilde{D}_4, P\rangle$, so it meets each fibre $l'_{P'_j}$ (for $j = 1, \ldots, \alpha$) in four points: two of them are $l'_{P'_j} \cap \tilde{D}_4$ and the remaining ones are $l'_{P'_j} \cap T_{P'_j}(S')$. Hence, if we add to this space a further point, say A_j, on each fibre, the obtained linear space also contains the quartic curves $l'_{P'_1}, \ldots, l'_{P'_\alpha}$, and, hence, the whole divisor D_4. In this way we have proved that $\langle L', R_1, \ldots, R_\beta, A_1, \ldots, A_\alpha\rangle \supset \langle L', D_4\rangle = \Sigma$, so

$$\dim(\Sigma) \leqslant \dim(L') + \alpha + \beta. \qquad (11.11)$$

Using (11.8) and (11.11) we obtain that $\dim\langle D_4\rangle + \delta - m + 1 = \dim(\Sigma) \leqslant \dim(L') + \alpha + \beta$, and, from this, using (11.10) we get that $\dim\langle D_4\rangle + \delta - m + 1 \leqslant \dim(L') + \alpha + \dim\langle \tilde{D}_4\rangle - \dim(T) + 1$. Finally, using (11.9) we obtain that

$$\begin{aligned} \dim(L') &\geqslant \delta - m + \dim\langle D_4\rangle - \dim\langle \tilde{D}_4\rangle - \alpha + 2(\delta - \alpha) + m \\ &= 3\delta - 3\alpha + \dim\langle D_4\rangle - \dim\langle \tilde{D}_4\rangle \\ &= 3\delta, \end{aligned}$$

where the last equality easily comes from Lemma 11.1.

Note that the statement has been proved in the case $P \notin \tilde{D}_4$, but the case $P \in \tilde{D}_4$ follows in a similar way, with some warnings. Namely, in Step 1, the main difference concerns the linear space $J := \pi(\Sigma) = \langle \bar{P}_{m+1}, \ldots, \bar{P}_{\delta-\alpha}, \bar{P}'_1, \ldots, \bar{P}'_\alpha\rangle$ obtained from Σ by projecting from $\langle D_4\rangle$, and (11.6) still holds. In Step 2, since $\delta - m + 1 \leqslant 2(c + 2) + 1$, *a fortiori*, it holds that $\delta - m \leqslant 2(c + 2) + 1$. So, also in this case, Lemma 11.2 can be applied to J, which is spanned by $\delta - m$ points and is assumed to have dimension smaller than $\delta - m - 1$. Using the same argument we can prove the analogous statement of (11.7), i.e. $\dim(J) = \delta - m - 1$. Finally, in Step 3 we obtain the analogous statement of (11.8) and, precisely, that $\dim(\Sigma) = \dim\langle D_4\rangle + \delta - m$. In the following argument, the result of Lemma 11.3 is used; since it holds for any P, in this case (11.9) is also verified. It is now immediate to see that (11.10) becomes $\beta = \dim\langle \tilde{D}_4\rangle - \dim(T)$, and we again obtain that $\dim\langle D_4\rangle + \delta - m = \dim(\Sigma) \leqslant \dim(L') + \alpha + \beta$.

Using the new form of (11.10) we finally obtain that $\dim\langle D_4\rangle + \delta - m \leqslant \dim(L') + \alpha + \dim\langle \tilde{D}_4\rangle - \dim(T)$, which completes the proof as in the general case. □

Remark 11.5. The above result also holds if at most two of the points $P_1, \ldots, P_d$ belong to the same fibre.

Corollary 11.6. *For every curve* $\bar{D} \sim 2l + (\lambda - 2 - c)l' \subset \bar{S}_0 \cong \mathbb{F}_0$ *and for every choice of* $P_1, \ldots, P_\delta \in \bar{D}$ *that do not trivially degenerate* $\bar{D}$*, there exists a curve* $\bar{X}_0 \subset \bar{S}_0$ *whose double points are exactly* $P_1, \ldots, P_\delta$ *and whose characters are* a*,* b*,* λ*, where* $a + b = g - 3 - c$*.*

We conclude this section with some remarks about the construction of the bisecant curves D_4 and $\tilde{D}_4$.

We consider a geometrically ruled surface contained in V and having minimum degree; each such surface corresponds to a quotient of type

$$\mathcal{F} := \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c) \to \mathcal{O}(a) \oplus \mathcal{O}(b) \to 0, \tag{11.12}$$

i.e. it is of the type $R := R_{a,b} = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b))$.

Remark 11.7. Since these quotients correspond to $\mathcal{F}(-c)$, tensorizing (11.12) by $\mathcal{O}(-c)$ we obtain that

$$0 \to \mathcal{O} \to \mathcal{O}(a-c) \oplus \mathcal{O}(b-c) \oplus \mathcal{O} \to \mathcal{O}(a-c) \oplus \mathcal{O}(b-c) \to 0,$$

so

$$h^0(\mathcal{F}(-c)) = \begin{cases} 3 & \text{if } a = b = c, \\ 2 & \text{if } a < b = c, \\ 1 & \text{if } b < c \end{cases} \quad \text{or, equivalently,} \quad \dim|R_{a,b}| = \begin{cases} 2 & \text{if } a = b = c, \\ 1 & \text{if } a < b = c, \\ 0 & \text{if } b < c. \end{cases}$$

Remark 11.8. Set $\bar{V} := V_{\bar{S}_0}$ and, as usual, let Σ be the set of the double points of $\bar{X}_0$. We have the diagram

$$\begin{array}{ccccc} \bar{S}_0 & \subset & \bar{V} & \supset & \bar{R} \\ \downarrow & & \downarrow{\scriptstyle \pi_\Sigma} & & \downarrow \\ S & \subset & V & \supset & R \end{array}$$

where $\bar{R} := \pi_\Sigma^{-1}(R)$. Setting $\delta_R := \sharp(\Sigma \cap \bar{R})$, i.e. the number of the double points (possibly infinitely near) of $\bar{X}_0$ lying on $\bar{R}$, it is clear that $\deg(\bar{R}) = \deg(R) + \delta_R = a + b + \delta_R$.

Lemma 11.9. *Let $R \in |R_{a,b}|$ be a ruled surface on $V = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)$ and let $\bar{S}_0 = S_{2,\lambda-2}$ as usual. Then,*

$$\tilde{D} := \bar{R} \cdot \bar{S}_0 \sim 2l + (\lambda - 2 - c - \delta + \delta_R)l',$$

and there exists a unique bisecant curve $\bar{D} \sim 2l + (\lambda - 2 - c)l' \subset \bar{S}_0$ such that $\Sigma \subset \bar{D}$ and $\bar{D} \supseteq \tilde{D}$. Moreover, as soon as R varies in $|R_{a,b}|$, $\bar{D}$ varies in a linear system of dimension 0, 1, 2 *if $b < c$, $a < b = c$, $a = b = c$, respectively.*

Proof. Let $H_{\bar{V}}$ be a hyperplane section of $\bar{V}$ containing $\bar{R}$. Since each hyperplane section cannot contain any other unisecant component out of $\bar{R}$, $H_{\bar{V}} \sim \bar{R} + \tau F_{\bar{V}}$, where $F_{\bar{V}}$ is the generic fibre of $\bar{V}$ and τ is a non-negative integer. Clearly, since $\deg(H_{\bar{V}}) = \deg(\bar{V}) = \deg(V) + \delta = a + b + c + \delta$ and $\deg(\bar{R}) = a + b + \delta_R$, we obtain that

$$\bar{R} \sim H_{\bar{V}} - (c + \delta - \delta_R)F_{\bar{V}}.$$

Taking into account that $H_{\bar{V}} \cdot \bar{S}_0 = 2l + (\lambda - 2)l'$ and $F_{\bar{V}} \cdot \bar{S}_0 = l'$, we obtain that

$$\bar{R} \cdot \bar{S}_0 \sim 2l + (\lambda - 2)l' - (c + \delta - \delta_R)l' = 2l + (\lambda - 2 - c - \delta + \delta_R)l',$$

as required. Note that only δ_R points of Σ lie on $\tilde{D}$, and the remaining $\delta - \delta_R$ lie on $\delta - \delta_R$ fibres (possibly coincident) of $\bar{S}_0$, say $l'_1, \dots, l'_{\delta-\delta_R}$. Hence,

$$\Sigma \subset \tilde{D} \cup l'_1 \cup \cdots \cup l'_{\delta-\delta_R} \sim 2l + (\lambda - 2 - c)l',$$

so, setting $\bar{D} := \tilde{D} \cup l'_1 \cup \cdots \cup l'_{\delta-\delta_R}$, we obtain that $\bar{D}$ is linearly equivalent to $2l + (\lambda - 2 - c)l'$ and contains both Σ and $\tilde{D}$, as required. Finally, from the above construction, the divisor $\bar{D}$ is unique for each $\bar{R}$. The last statement follows from Remark 11.7. □

Keeping the above notation, one can immediately compute the degree of $\bar{D}$:

$$\deg(\bar{D}) = \int (2l + (\lambda - 2 - c)l') \cdot (2l + (\lambda - 2)l') = 4(\lambda - 2) - 2c. \tag{11.13}$$

Observe that $\bar{R}$ is the ruled surface generated by the ruling of $\bar{V}$ on $\tilde{D}$, i.e.

$$\bar{R} = \bigcup_{P,Q \in \tilde{D} \cap F_{\bar{V}}} l_{P,Q},$$

where $l_{P,Q}$ denotes the line passing through the points P and Q. In particular, $\bar{R}$ is determined by $\tilde{D}$; to stress this fact, we write $\bar{R} = \bar{R}(\tilde{D})$.

12. Moduli spaces of four-gonal curves with $t = 0$

In this section we study the moduli spaces of four-gonal curves with given invariants; in particular, we determine whether they are irreducible and find their dimension. Moreover, we give a stratification of these spaces using the invariants introduced in the previous sections.

Let X be a four-gonal curve of genus g and consider its canonical model $X_K \subset S \subset V \subset \mathbb{P}^{g-1}$, where (from Theorem 3.1) S is a surface ruled by conics, of minimum degree and unique, unless g is odd and $\deg(S) = (3g - 7)/2$. In this case, there exists a pencil of such surfaces.

Assume that S has invariant $t = 0$, i.e. its (embedded) standard model is the quadric surface $R_{1,1} \subset \mathbb{P}^3$, on which X can be realized as a curve $X' \sim 4l + \lambda l'$ having only double points as singularities: we write $X = X(g, \lambda)$. Moreover, if $V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$, then a and b are further invariants of X, and we write $X = X(g, \lambda, a, b)$.

Remark 12.1. If X is as before, then, by Remark 8.11, it is clear that it has a finite number of models X', at most $\binom{\delta}{2}$, on $R_{1,1}$ unless g is odd and $\deg(S) = (3g - 7)/2$. In this case, there is a one-dimensional family of such models of X. More precisely, one model comes from another via an elementary transformation of type $\mathrm{elm}_{A,B}$, where A and B are two double points of X' as in Remark 8.11. In this way, denoting by X'' another model of X on $R_{1,1}$ and by ξ an elementary transformation as before, the set

$$\Xi_{X'} := \{\xi \colon X' \to X''\}$$

consists of at most $\binom{\delta}{2}$ elements if $\deg(S) \leqslant \lceil (3g-8)/2 \rceil$, while $\dim(\Xi_{X'}) = 1$ if $\deg(S) = (3g - 7)/2$.

Note that $\Xi_{X'}$ has exactly $\binom{\delta}{2}$ elements in the general case.

We denote by $\mathcal{A}_\lambda$ the open subset of the linear system $|4l+\lambda l'|$ on $R_{1,1}$ parametrizing the irreducible curves of such a linear system, and set

$$\mathcal{W}_g^\lambda := \{X' \in \mathcal{A}_\lambda \mid X = X(g,\lambda) \text{ and } X' \text{ has } \delta \text{ double points on distinct fibres}\},$$
$$\mathcal{W}_g^\lambda(a,b) := \{X' \in \mathcal{W}_g^\lambda \mid X = X(g,\lambda,a,b)\}.$$

We denote by $\mathcal{M}_{g,4}$ the moduli space of four-gonal curves of genus g and let

$$\theta\colon \mathcal{W}_g^\lambda \to \mathcal{M}_{g,4}$$

be the usual projection defined by $\theta(X') = [X]$, where $[X]$ is the isomorphism class of the four-gonal curve X in $\mathcal{M}_{g,4}$. Finally, set

$$\mathcal{M}_g^\lambda := \theta(\mathcal{W}_g^\lambda), \qquad \mathcal{M}_g^\lambda(a,b) := \theta(\mathcal{W}_g^\lambda(a,b)).$$

It is clear that, in order to compute the dimension of these moduli spaces, we need to find both the dimensions of $\mathcal{W}_g^\lambda$ (respectively, $\mathcal{W}_g^\lambda(a,b)$) and of the general fibre of θ.

Remark 12.2. From Theorem 10.6, the locally closed subsets $\mathcal{W}_g^\lambda(a,b)$ and, hence, $\mathcal{W}_g^\lambda$ are not empty, as long as a, b, λ fulfil (R_1), (R_2), (R_3).

Lemma 12.3. *Let $X', Y' \in \mathcal{W}_g^\lambda$ be two curves on $R_{1,1}$. If $[X] = [Y]$ in $\mathcal{M}_g^\lambda$, there exists an automorphism β of the quadric surface $R_{1,1}$ and a morphism $\xi \in \Xi_{Y'}$ such that $Y' = \xi(\beta(X'))$. Therefore, the dimension of the general fibre of θ is*

$$\dim(\theta^{-1}([X])) = \begin{cases} 7 & \text{if } g \text{ is odd and } \lambda = \left\lceil \dfrac{g+2}{2} \right\rceil, \\ 6 & \text{otherwise.} \end{cases}$$

Proof. Since $X \cong Y$, $X_K \cong Y_K$ and there exists a linear automorphism, α say, of $\mathbb{P}^{g-1}$ such that $\alpha(X_K) = Y_K$. Let S_X and S_Y be the surfaces, ruled by conics and of minimum degree such that $X_K \subset S_X \subset \mathbb{P}^{g-1}$ and $Y_K \subset S_Y \subset \mathbb{P}^{g-1}$. Assume that these surfaces are unique; therefore, $\alpha(S_X) = S_Y$.

Consider diagram (8.1) for both X and Y: setting $N_X := \langle \varphi_X(K_X - \Phi_X - \Lambda_X)\rangle$ and N_Y analogously, we have the following:

$$\begin{array}{ccccccccc}
\mathbb{P}^{g-1} & \supset & S_X & \supset & X_K & \xrightarrow{\pi_{N_X}} & X_{\Phi_X+\Lambda_X} = X' & \subset & R_{1,1}(X) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle\beta} \\
\mathbb{P}^{g-1} & \supset & S_Y & \supset & Y_K & \xrightarrow{\pi_{N_Y}} & Y_{\Phi_Y+\Lambda_Y} = Y' & \subset & R_{1,1}(Y)
\end{array}$$

where β is the isomorphism between the quadrics $R_{1,1}(X)$ and $R_{1,1}(Y)$ induced by α. Up to a linear change of coordinates in $\mathbb{P}^3$, we can assume that $R_{1,1}(X) = R_{1,1}(Y)$, so $\beta \in \mathrm{Aut}(R_{1,1})$.

Consider then the curves Y' and $\beta(X')$ lying on $R_{1,1}$: from the construction above, we obtain that they are both models of Y on a quadric. Therefore, applying Remark 12.1, we get that there exists $\xi \in \Xi_{Y'}$ such that $Y' = \xi(\beta(X'))$, as requested.

When S_X and S_Y are not unique they vary in a pencil (see Theorem 3.1), and the proof follows in a similar way.

The second part of the statement follows from the first part, namely, it is clear that $\dim(\theta^{-1}([X])) = \dim(\mathrm{Aut}(R_{1,1}))+\dim(\Xi_X)$. On the one hand, observe that $\mathrm{Aut}(R_{1,1}) \cong \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathrm{PGL}(2) \times \mathrm{PGL}(2)$ has dimension 6. On the other hand, by Remark 12.1,

$$\dim(\Xi_X) = \begin{cases} 1 & \text{if } g \text{ is odd and } \deg(S) = \dfrac{3g-7}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, note (using Theorem 6.5) that $g + \lambda - 5 = \deg(S) = (3g - 7)/2$ or, equivalently, $\lambda = (g + 3)/2 = \lceil (g + 2)/2 \rceil$, where the last equality holds since g is odd. □

We recall (see § 10) that if $X' \in \mathcal{W}_g^\lambda$, then $X' \subset R_{1,1} \cong \mathbb{F}_0$ and $\varphi_{4l+\lambda l'} \colon \mathbb{F}_0 \to S' \subset \mathbb{P}^{5\lambda+4}$. In particular, we can associate with X' a hyperplane H_X of $\mathbb{P}^{5\lambda+4}$. By Remark 10.2 we have that X' has $P_1, \ldots, P_\delta$ as double points if and only if H_X contains the linear space

$$L_{P_1,\ldots,P_\delta} := \langle T_{P_1}(S'), \ldots, T_{P_\delta}(S')\rangle.$$

We can then identify $\mathcal{W}_g^\lambda$ with its image via the injective morphism $i \colon \mathcal{W}_g^\lambda \to \check{\mathbb{P}}^{5\lambda+4}$ defined by $X' \mapsto H_X$.

In order to compute the dimension of $\mathcal{W}_g^\lambda$ and of $\mathcal{W}_g^\lambda(a, b)$, and to prove their irreducibility, we need further preliminary observations.

Remark 12.4. The key-lemma (Lemma 11.4) has been proved under the assumption that $(P_1, \ldots, P_\delta)$ are distinct points. For instance, if $\delta = 2$, this result states that

$$\dim L_{P_1,P_2} = \dim\langle T_{P_1}(S'), T_{P_2}(S')\rangle = 5.$$

If P_2 is infinitely near to P_1, given a local system of coordinates of S' in a neighbourhood of P_1, the tangent plane to S' at P_1 is generated by P_1 and the first derived vectors both along the bisecant $\tilde{D}$ and along the fibre l'_1. Hence, it is easy to see that the linear space L_{P_1,P_2} is generated by the above generators of $T_{P_1}(S')$ and by two further second derived vectors and a third derived vector. One can show that all of them are linearly independent, so, also in this case, $\dim L_{P_1,P_2} = 5$.

It is not difficult to prove, if k is any integer $(1 \leqslant k \leqslant \delta - 1)$ and the considered points are $P_1, P_2, \ldots, P_{k+1}, \ldots, P_\delta$, where $P_2, \ldots, P_{k+1}$ are infinitely near to P_1, that $\dim L_{P_1,\ldots P_\delta} \geqslant 3\delta - k$.

Lemma 12.5. *We consider the morphism*

$$\Psi \colon \mathcal{W}_g^\lambda \to \mathrm{Sym}^\delta(R_{1,1}) \quad \textit{defined by } X' \mapsto (P_1, \ldots, P_\delta),$$

where $\Sigma = P_1 + \cdots + P_\delta$ is the singular locus of $X' \subset R_{1,1}$. The general fibre of Ψ then has dimension

(i) *$\dim(\Psi^{-1}(P_1, \ldots, P_\delta)) = 5\lambda + 4 - 3\delta$ if $P_1, \ldots, P_\delta$ are distinct points,*

(ii) *$\dim(\Psi^{-1}(P_1, \ldots, P_\delta)) \leqslant 5\lambda + 3 - 3\delta + k$ if $P_2, \ldots, P_{k+1}$ are infinitely near to P_1 for some $k \geqslant 1$.*

Proof. By definition, $\mathcal{W}_g^\lambda$ consists of the irreducible curves of type $(4,\lambda)$ on $R_{1,1}$ having δ double points on distinct fibres. So, taking into account the above injective morphism $i\colon \mathcal{W}_g^\lambda \to \check{\mathbb{P}}^{5\lambda+4}$ and the fact that $X' \in \mathcal{W}_g^\lambda$ has $P_1,\dots,P_\delta$ as double points if and only if the hyperplane $H_X := i(X')$ contains the linear space $L_{P_1,\dots,P_\delta}$, it is clear that the general fibre $\Psi^{-1}(P_1,\dots,P_\delta)$ is isomorphic to an open subset of $\{H \in \check{\mathbb{P}}^{5\lambda+4} \mid H \supset L_{P_1,\dots,P_\delta}\}$, since the general hyperplane containing $L_{P_1,\dots,P_\delta}$ contains the tangent planes to S' only at the chosen points. This means, exactly, that

$$\dim(\Psi^{-1}(P_1,\dots,P_\delta)) = 5\lambda + 4 - (\dim L_{P_1,\dots P_\delta} + 1).$$

(i) If $P_1,\dots,P_\delta$ are distinct, then in the key-lemma (Lemma 11.4) we have shown that the dimension of $L_{P_1,\dots,P_\delta}$ is $3\delta - 1$ independently of the position of the considered points. So, in this case, $\Psi^{-1}(P_1,\dots,P_\delta)$ is irreducible of dimension $5\lambda + 4 - 3\delta$.

(ii) If $P_1,\dots,P_\delta$ are not distinct, as in the assumption, then the fibre of Ψ could have bigger dimension. Nevertheless, we can get an upper bound on this dimension by taking into account Remark 12.4, obtaining that $\dim(\Psi^{-1}(P_1,\dots,P_\delta))$ is at most $5\lambda + 4 - (3\delta - k + 1)$, and this proves the second part.

□

Proposition 12.6. *For each λ satisfying*

$$\frac{g+3}{3} \leqslant \lambda \leqslant \left\lceil \frac{g+2}{2} \right\rceil, \tag{R$_1$}$$

the locally closed subset $\mathcal{W}_g^\lambda$ is irreducible of dimension $g + 2\lambda + 7$.

Proof. Setting $\mathrm{Sym} := \mathrm{Sym}^\delta(R_{1,1})$, consider the map $\Psi\colon \mathcal{W}_g^\lambda \to \mathrm{Sym}$ defined in Lemma 12.5. Note that Ψ is dominant and $\dim(\mathrm{Sym}) = 2\delta$. Recall also that the δ singular points of the general curve $X' \in \mathcal{W}_g^\lambda$ are in general position on $R_{1,1}$ by Lemma 11.4.

If $P_1,\dots,P_\delta$ are distinct points, by Lemma 12.5 we get that $\dim(\Psi^{-1}(P_1,\dots,P_\delta)) = 5\lambda + 4 - 3\delta$. Therefore,

$$\dim(\mathcal{W}_g^\lambda) = \dim(\Psi^{-1}(P_1,\dots,P_\delta)) + \dim(\mathrm{Sym}) = 5\lambda + 4 - \delta = g + 2\lambda + 7,$$

where the last equality follows from $\delta = 3(\lambda - 1) - g$.

Assume now that $P_2,\dots,P_{k+1}$ are infinitely near to P_1 for some $k \geqslant 1$. The fibre of Ψ at the point $(P_1,\dots,P_\delta) \in \mathrm{Sym}$ then has dimension at most $5\lambda + 3 - 3\delta + k$ by Lemma 12.5. The difference between such an integer and $5\lambda + 4 - 3\delta$ is at most $k - 1 < 2k = \mathrm{codim}_{\mathrm{Sym}}(\Delta)$, where $\Delta := \{(Q_1,\dots,Q_\delta) \in \mathrm{Sym} \mid Q_1 = \cdots = Q_{k+1}\}$. Clearly, Δ is a closed subset of Sym and contains the considered element $(P_1,\dots,P_\delta)$. Therefore, the variety consisting of the fibres on the points of Δ is a proper closed subset of $\mathcal{W}_g^\lambda$. □

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Remark 12.7. Recall that $\mathcal{M}_{g,4}$ is a closed irreducible subset of the moduli space $\mathcal{M}_g$, and has dimension $2g+3$. We set the maximum value of λ (see (R_1)) to be

$$\lambda_{\max} := \left\lceil \frac{g+2}{2} \right\rceil.$$

Then, from Proposition 12.6,

$$\dim(\mathcal{W}_g^{\lambda_{\max}}) = g + 2\lambda_{\max} + 7.$$

We recall that the fibre of $\theta\colon \mathcal{W}_g^{\lambda_{\max}} \to \mathcal{M}_g^{\lambda_{\max}}$ has dimension either 6 or 7, according to whether g is even or odd, respectively (from Lemma 12.3). Hence,

$$\dim(\mathcal{M}_g^{\lambda_{\max}}) = \begin{cases} g + 2\dfrac{g+2}{2} + 1 = 2g+3 & \text{if } g \text{ is even,} \\ g + 2\dfrac{g+3}{2} = 2g+3 & \text{if } g \text{ is odd.} \end{cases}$$

Therefore, in both cases, we have that $\dim(\mathcal{M}_g^{\lambda_{\max}}) = \dim(\mathcal{M}_{g,4})$; in other words, the general four-gonal curve has invariant $\lambda_{\max}$.

Remark 12.8. We know that, if $t > 0$, X admits a standard model $X' \subset R_{1,t+1}$. Nevertheless, also in this case, it is possible to define another model of X, X'' say, on a quadric surface $R_{1,1}$. Clearly, in this situation, X'' not only has double points as singularities, but also triple points.

Namely, let $Q_1, \dots, Q_t$ be simple points of X', belonging to t distinct fibres of $R_{1,t+1}$, and consider the projection from these points:

$$\begin{array}{ccc} X' & \subset & R_{1,t+1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_{Q_1,\dots,Q_t}} \\ X'' & \subset & R_{1,1} \end{array}$$

Since X' meets each fibre of $R_{1,t+1}$ in the four points of the gonal divisor, the singularities of X'' are the δ double points of X' and, in addition, t triple points, all of them belonging to the same line l.

It is clear that the closure $\bar{\mathcal{W}}_g^{\lambda}$ of $\mathcal{W}_g^{\lambda}$ in $\mathcal{A}_\lambda$ also contains the curves of the invariants g, λ and $t > 0$, and it is not difficult to see that the closed subset consisting of such curves has dimension smaller than $\dim(\mathcal{W}_g^{\lambda})$.

Using Remark 12.2, Lemma 12.3, Proposition 12.6 and Remarks 12.7 and 12.8, we immediately obtain the following result, which is the first part of the Main Theorem stated in § 1 (here, $\bar{\mathcal{M}}_g^{\lambda}$ denotes the closure of $\mathcal{M}_g^{\lambda}$ in the moduli space $\mathcal{M}_{g,4}$ of four-gonal curves).

Theorem 12.9. *There exists a stratification of the moduli space $\mathcal{M}_{g,4}$ of four-gonal curves given by*

$$\mathcal{M}_{g,4} = \bar{\mathcal{M}}_g^{\lceil (g+2)/2 \rceil} \supset \bar{\mathcal{M}}_g^{\lceil g/2 \rceil} \supset \cdots \supset \bar{\mathcal{M}}_g^{\lambda} \supset \cdots \supset \bar{\mathcal{M}}_g^{\lceil (g+3)/3 \rceil},$$

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and $\bar{\mathcal{M}}_g^\lambda$ are irreducible locally closed subsets of dimension $g+2\lambda+1$ for $(g+3)/3 \leqslant \lambda < \lceil (g+2)/2 \rceil$.

In order to show the second part of the Main Theorem, we start with some preliminary facts.

We keep the notation of Lemma 11.9, where $\tilde{D}$ denotes a divisor of $\bar{S}_0 = S_{2,\lambda-2} \subset \mathbb{P}^{g-1+\delta}$ linearly equivalent to $2l + (\lambda - 2 - c - \delta + \delta_R)l'$ and containing δ_R points among $P_1, \dots, P_\delta$.

Also recall (see § 9) that the unisecant $\bar{A} \subset \bar{V}$ is the preimage, via π, of the (unique if $a < b$) unisecant of degree a of V. Moreover, $\bar{R} := \pi^{-1}(R)$, where $R := R_{a,b}$, so $\bar{A} \subset \bar{R} = \bar{R}(\tilde{D})$ as described in Lemma 11.9.

In the forthcoming computations we use the following relations a few times (coming from $a+b+c = g-3$ and from (10.1)):

$$c = g - 3 - a - b, \qquad 3\lambda = \delta + g + 3. \tag{12.1}$$

Lemma 12.10. *Let $\tilde{D} \subset \bar{S}_0$ and $\bar{R} := \bar{R}(\tilde{D})$ be as before. Let $\bar{A} \in \mathrm{Un}^{a+\delta_R}(\bar{R})$ and $\Gamma := \tilde{D} \cdot \bar{A}$. Assume that $a \geqslant (g - \lambda - 1)/2$. Then,*

(i) $\deg(\Gamma) = 4(\lambda - 2) - 2b - 2c - 2(\delta - \delta_R)$,

(ii) $h^0(\mathcal{O}_{\bar{R}}(\bar{A})) = h^0(\mathcal{O}_{\tilde{D}}(\Gamma))$,

(iii) *assuming also that $\delta_R = \delta$ and either $a > (g - \lambda - 1)/2$ or $a = (g - \lambda - 1)/2$ and $a < b$,*
$$H^0(\mathcal{O}_{\bar{R}}(\bar{A})) \cong H^0(\mathcal{O}_{\tilde{D}}(\Gamma)).$$

Proof. (i) Recall that, keeping the notation in Lemma 11.9, $\bar{D} = \tilde{D} + (\delta - \delta_R)l'$. So $\deg(\tilde{D}) = \deg(\bar{D}) - 2(\delta - \delta_R)$, since $\bar{S}_0$ is ruled by conics. Hence, using (11.13), we obtain that $\deg(\tilde{D}) = 4(\lambda - 2) - 2(c + \delta - \delta_R)$. Therefore, applying (IF) and Remark 11.8, we have that

$$\begin{aligned} \deg(\Gamma) &= 2\deg(\bar{A}) + \deg(\tilde{D}) - 2\deg(\bar{R}) \\ &= 2(a + \delta_R) + 4(\lambda - 2) - 2(c + \delta - \delta_R) - 2(a + b + \delta_R) \\ &= 4(\lambda - 2) - 2b - 2c - 2(\delta - \delta_R). \end{aligned}$$

(ii) We first show that Γ is a non-special divisor on $\tilde{D}$. Since $\tilde{D}$ is of type $(2, \lambda - 2 - c - (\delta - \delta_R))$ on the quadric, $p_a(\tilde{D}) = \lambda - 3 - c - (\delta - \delta_R)$. A sufficient condition in order to have non-special Γ is that $\deg(\Gamma) > 2p_a(\tilde{D}) - 2$, or, equivalently,

$$4(\lambda - 2) - 2b - 2c - 2(\delta - \delta_R) > 2(\lambda - 3 - c - (\delta - \delta_R)) - 2,$$

i.e. $\lambda - b > 0$, and this is true since $b \leqslant \lambda - 2$. Therefore, $h^1(\mathcal{O}_{\tilde{D}}(\Gamma)) = 0$ and, by the Riemann–Roch theorem, also using (12.1), we obtain that $h^0(\mathcal{O}_{\tilde{D}}(\Gamma)) - 1 = \deg(\Gamma) - p_a(\tilde{D}) = a - b + \delta_R + 1$. Moreover, $h^0(\mathcal{O}_{\bar{R}}(\bar{A})) - 1 = \dim_{\bar{R}}(|\bar{A}|) = \dim(\mathrm{Un}^{a+\delta_R}(\bar{R})) = a - b + \delta_R + 1$ by (UF). Hence, we obtain that

$$h^0(\mathcal{O}_{\tilde{D}}(\Gamma)) = a - b + \delta_R + 2 = h^0(\mathcal{O}_{\bar{R}}(\bar{A})).$$

(iii) In order to prove the claim, consider the exact sequence

$$0 \to \mathcal{I}_{\tilde{D}/\bar{R}}(\bar{A}) \to \mathcal{O}_{\bar{R}}(\bar{A}) \to \mathcal{O}_{\tilde{D}}(\Gamma) \to 0. \tag{12.2}$$

By (ii), it suffices to show that the map $f\colon H^0(\mathcal{O}_{\bar{R}}(\bar{A})) \to H^0(\mathcal{O}_{\tilde{D}}(\Gamma))$ induced by (12.2) is injective.

Clearly, this holds if and only if there exists a unique $\bar{A} \in \mathrm{Un}^{a+\delta_R}(\bar{R})$ passing through Γ, and this holds if $\int \bar{A}^2 < \deg(\Gamma)$. From (IF) and Remark 11.8 we obtain that

$$\int \bar{A}^2 = 2\deg(\bar{A}) - \deg(\bar{R}) = 2(a+\delta_R) - (a+b+\delta_R) = a - b + \delta_R.$$

Therefore, the condition $\int \bar{A}^2 < \deg(\Gamma)$ becomes $a-b+\delta_R < 4(\lambda-2)-2b-2c-2(\delta-\delta_R)$. Again using (12.1), the above inequality is equivalent to $\lambda - g + a + b + 1 - (\delta - \delta_R) > 0$. By assumption, $\delta - \delta_R = 0$, so $a + b > g - \lambda - 1$ and, using the further assumptions on a and b, the claim is proved. □

Before stating the second part of the Main Theorem, we set

$$\epsilon := \begin{cases} 0 & \text{if } b < c, \\ 1 & \text{if } a < b = c, \\ 2 & \text{if } a = b = c, \end{cases} \qquad \tau := \begin{cases} 0 & \text{if } a < b, \\ 1 & \text{if } a = b \end{cases} \qquad \text{and} \quad \xi := \begin{cases} 1 & \text{if } \lambda = \dfrac{g+3}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 12.11. *Let g, λ, a, b be positive integers satisfying* (R_1), (R_2), (R_3) *and $c = g - 3 - a - b$. If $a \geqslant (g - \lambda - 1)/2$, then $\mathcal{M}_g^\lambda(a,b)$ is an irreducible variety of dimension $2(2a + b + \lambda) + 10 - g - \epsilon - \tau - \xi$.*

Proof. From Remark 12.2 and Lemma 12.3, it is enough to show that $\mathcal{W}_g^\lambda(a,b)$ is irreducible of the right dimension.

Keeping the notation in Lemma 12.5, set $Y_g^\lambda(a,b) := \Psi(\mathcal{W}_g^\lambda(a,b))$.

Claim. $\Psi^{-1}(Y_g^\lambda(a,b)) \subset \mathcal{W}_g^\lambda(a,b)$.

This is equivalent to the following property: let $X'' \in \mathcal{W}_g^\lambda$ be such that $\Psi(X'') = (P_1,\ldots,P_\delta) = \Psi(X')$, where $X' \in \mathcal{W}_g^\lambda(a,b)$; then, $X'' \in \mathcal{W}_g^\lambda(a,b)$. This is true, since $\pi_{\langle P_1,\ldots,P_\delta\rangle}(\bar{V})$ is the scroll $V = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$ associated with both X' and X'', and this proves the claim.

Step 1 (irreducibility and dimension of $\boldsymbol{\mathcal{W}_g^\lambda(a,b)}$). From the claim above we can consider the restriction of Ψ,

$$\psi\colon \mathcal{W}_g^\lambda(a,b) \to Y_g^\lambda(a,b).$$

From Lemma 12.5, $\dim(\psi^{-1}(P_1,\ldots,P_\delta)) = 5\lambda + 4 - 3\delta$ if $P_1,\ldots,P_\delta$ are distinct points.

Using the same argument as that in the proof of Proposition 12.6, in the case of infinitely near points one easily shows that the variety consisting of the fibres on the points of Δ is a proper closed subset of $\mathcal{W}_g^\lambda(a,b)$. For this reason, $\mathcal{W}_g^\lambda(a,b)$ is irreducible if $Y_g^\lambda(a,b)$ is irreducible and

$$\dim(\mathcal{W}_g^\lambda(a,b)) = \dim(Y_g^\lambda(a,b)) + 5\lambda + 4 - 3\delta. \tag{12.3}$$

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Step 2 (irreducibility and dimension of $Y_g^\lambda(a,b)$). Recall that the singular locus $\Sigma = P_1 + \cdots + P_\delta$ of $X' \subset R_{1,1}$ is contained in a suitable bisecant curve $\bar{D} \sim 2l + (\lambda - 2 - c)l' \subset R_{1,1}$ by Lemma 11.9 (there, the result concerns $\bar{S}_0$; here, it concerns $R_{1,1}$).

It is not hard to show that there exists an open subset, Y^0 say, of $Y_g^\lambda(a,b)$ whose elements $(P_1,\dots,P_\delta)$ fulfil the following property: there exists $\bar{D} \in |2l+(\lambda-2-c)l'|$ not containing fibres and such that $P_1,\dots,P_\delta \in \bar{D} \cap \bar{A}$, for a suitable $\bar{A} \in \mathrm{Un}^{a+\delta}(\bar{R})$, where $\bar{R} := \bar{R}(\bar{D})$. In particular, on this subset $\delta_R = \delta$.

We check that the above condition is compatible with the degrees of the involved divisors, i.e. setting $\Gamma := \bar{D} \cap \bar{A}$, we must have that $\delta \leqslant \deg(\Gamma)$. From Lemma 12.10 (ii), taking into account that here $\delta = \delta_R$ and using (12.1) as usual, it is easy to see that $\deg(\Gamma) = 2a + \lambda - g + 1 + \delta \geqslant \delta$, since $2a + \lambda - g + 1 \geqslant 0$. Namely, this is equivalent to $a \geqslant (g - \lambda - 1)/2$, which holds by assumption.

Consider then the correspondence $Z_{a,b}^\lambda \subset |2l + (\lambda - 2 - c)l'| \times \mathrm{Sym}^\delta(R_{1,1})$ defined by

$$Z_{a,b}^\lambda := \{(\bar{D}, P_1, \dots, P_\delta) \mid \text{there exists } \bar{A} \in \mathrm{Un}^{a+\delta}(\bar{R}(\bar{D})) \text{ such that } P_1,\dots,P_\delta \in \bar{D} \cap \bar{A}\}.$$

Consider now the two canonical projections, where Ω is the open subset of $|2l+(\lambda-2-c)l'|$ consisting of curves not containing fibres:

$$\begin{array}{ccc} & Z_{a,b}^\lambda & \\ {}^{p}\swarrow & & \searrow^{q} \\ |2l+(\lambda-2-c)l'| \supset \Omega & & Y^0 \subset Y_g^\lambda(a,b) \subset \mathrm{Sym}^\delta(R_{1,1}) \end{array}$$

By Lemma 11.9, every element $(P_1,\dots,P_\delta)$ of Y^0 determines either a unique $\bar{D} \sim 2l + (\lambda - 2 - c)l'$ (if $b < c$) or a pencil (if $a < b = c$) or a two-dimensional linear system (if $a = b = c$) of such curves. This implies that the general fibre of q is irreducible of dimension ϵ, where $\epsilon = 0, 1, 2$ as long as $b < c$, $a < b = c$, $a = b = c$, respectively. Furthermore, p is surjective by Corollary 11.6.

Defining by $Z_{\bar{D}} := p^{-1}(\bar{D})$ any fibre of p, we have that if $Z_{\bar{D}}$ is irreducible, then $Y_g^\lambda(a,b)$ is irreducible and

$$\begin{aligned} \dim(Y_g^\lambda(a,b)) &= \dim(Z_{a,b}^\lambda) - \epsilon \\ &= \dim(Z_{\bar{D}}) + \dim(|\bar{D}|) - \epsilon \\ &= \dim(Z_{\bar{D}}) + 3(\lambda - 1 - c) - 1 - \epsilon. \end{aligned} \tag{12.4}$$

Step 3 (irreducibility and dimension of $Z_{\bar{D}}$). It is clear that

$$Z_{\bar{D}} \cong \{(P_1,\dots,P_\delta) \in \mathrm{Sym}^\delta(\bar{D}) \mid \text{there exists } \bar{A} \in \mathrm{Un}^{a+\delta}(\bar{R}) \\ \text{such that} P_1,\dots,P_\delta \in \bar{D} \cap \bar{A}\}.$$

In order to compute the dimension and to prove the irreducibility of $Z_{\bar{D}}$, consider the correspondence (where $\Gamma = \bar{D} \cap \bar{A}$ is as before)

$$T_{\bar{D}} := \{(P_1',\dots,P_\delta',\bar{A}) \mid P_1',\dots,P_\delta' \in \Gamma\} \subset \mathrm{Sym}^\delta(\bar{D}) \times \mathrm{Un}^{a+\delta}(\bar{R}),$$

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and the two projections:

$$\begin{array}{ccc} & T_{\bar{D}} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \operatorname{Sym}^{\delta}(\bar{D}) & & \operatorname{Un}^{a+\delta}(\bar{R}) \end{array}$$

Obviously, $\operatorname{Im}(\pi_1) = Z_{\bar{D}}$, and π_2 is a finite surjective morphism; hence, denoting by τ the dimension of the fibres of π_1, we obtain that

$$\dim(Z_{\bar{D}}) = \dim(T_{\bar{D}}) - \tau = \dim(\operatorname{Un}^{a+\delta}(\bar{R})) - \tau = a - b + \delta + 1 - \tau. \tag{12.5}$$

We find the possible values of τ. In the proof of Lemma 12.10 (iii) we show that $\int \bar{A}^2 = a - b + \delta$; with the same argument used there to prove the uniqueness of the unisecant $\bar{A}$ passing through a certain divisor, it is immediate to see that

$$\tau = 0 \quad\Longleftrightarrow\quad \int \bar{A}^2 < \delta \quad\Longleftrightarrow\quad a - b + \delta < \delta \quad\Longleftrightarrow\quad a < b.$$

Using the same argument we obtain that

$$\tau \geqslant 1 \quad\Longleftrightarrow\quad \int \bar{A}^2 \geqslant \delta \quad\Longleftrightarrow\quad a - b + \delta \geqslant \delta \quad\Longleftrightarrow\quad a = b \text{ and } \int \bar{A}^2 = \delta.$$

Hence, necessarily, $\tau = 1$ and $a = b$. It remains to show that $Z_{\bar{D}}$ is irreducible. Since $Z_{\bar{D}} = \pi_1(T_{\bar{D}})$, it is enough to show that $T_{\bar{D}}$ itself is irreducible. Assume first that

$$a > \frac{g - \lambda - 1}{2} \quad \text{or} \quad a = \frac{g - \lambda - 1}{2} < b.$$

It follows from Lemma 12.10 (iii) that $H^0(\mathcal{O}_{\bar{R}}(\bar{A})) \cong H^0(\mathcal{O}_{\bar{D}}(\Gamma))$; hence,

$$T_{\bar{D}} \cong \{(P'_1, \ldots, P'_\delta, \Gamma') \mid P'_1, \ldots, P'_\delta \in \Gamma'\} \subset \operatorname{Sym}^{\delta}(\bar{D}) \times |\Gamma|.$$

Consider the morphism $\varphi_\Gamma \colon \bar{D} \to \mathbb{P}^r$, where $r = \dim|\Gamma| = a - b + \delta + 1$ (as computed in the proof of Lemma 12.10 (ii)); if $\bar{D}'$ denotes the image of $\bar{D}$ in $\mathbb{P}^r$, it is clear that

$$T_{\bar{D}} \cong \{(P'_1, \ldots, P'_\delta, H) \mid P'_1, \ldots, P'_\delta \in H \cap \bar{D}'\} \subset \operatorname{Sym}^{\delta}(\bar{D}') \times \check{\mathbb{P}}^r.$$

The irreducibility of $T_{\bar{D}}$ is a consequence of Lemma 12.12.

Finally, we have to consider the last case:

$$a = \frac{g - \lambda - 1}{2} = b.$$

Since $c = g - 3 - (a + b) = \lambda - 2$, from Lemma 12.10 (i) we have $\deg(\Gamma) = 4(\lambda - 2) - 2b - 2c = 3\lambda - 3 - g = \delta$. Therefore, $\pi_2 \colon T_{\bar{D}} \to \operatorname{Un}^{a+\delta}(\bar{R})$ is an isomorphism; hence, $T_{\bar{D}}$ is irreducible of dimension $\delta + 1$ (since $a = b$).

Finally, observe that, if $\bar{\bar{D}} \notin \Omega$ in Step 2, one can easily prove that $\dim(Z_{\bar{\bar{D}}}) = a - b + \delta_R + 1 - \tau$. In particular, $\dim(Z_{\bar{\bar{D}}}) < \dim(Z_{\bar{D}})$; hence, $p^{-1}(|2l + (\lambda - 2 - c)l'| \setminus \Omega)$ is a Zariski locally closed subset of $Z^{\lambda}_{a,b}$.

Step 4 (final computation). We can now compute the dimension of the moduli space using (12.3), (12.4), (12.5) and (12.1):

$$\begin{aligned}\dim(\mathcal{W}_g^\lambda(a,b)) &= \dim(Y_g^\lambda(a,b)) + 5\lambda + 4 - 3\delta \\ &= \dim(Z_{\bar{D}}) + 3(\lambda - 1 - c) + 3 - \epsilon + 5\lambda - 3\delta \\ &= 2(2a + b + \lambda) + 16 - g - \epsilon - \tau.\end{aligned}$$

Hence, from Lemma 12.3, we obtain that

$$\dim(\mathcal{M}_g^\lambda(a,b)) = \dim(\mathcal{W}_g^\lambda(a,b)) - 6 - \xi = 2(2a + b + \lambda) + 10 - g - \epsilon - \tau - \xi,$$

and this proves the claim. □

It remains to show the following fact.

Lemma 12.12. *Let $X \subset \mathbb{P}^r$ be a (smooth) irreducible curve, let k be an integer such that $k \leqslant \deg(X)$, and let*

$$V_X := \{(P_1, \dots, P_k; H) \mid P_1, \dots, P_k \in H \cap X\} \subset \mathrm{Sym}^k(X) \times \check{\mathbb{P}}^r.$$

The variety V_X is then irreducible.

Proof. This is a straightforward generalization of the proof of the uniform position lemma [**9**]. □

We now prove the last part of the Main Theorem. We first need some preliminary results; we recall that, if $a < (g - \lambda - 1)/2$, $\bar{A} \subset \bar{S}_0 \subset \bar{V}$ (from Proposition 9.3).

Lemma 12.13. *Let $a < (g - \lambda - 1)/2$ and $[X] \in \mathcal{M}_g^\lambda(a,b)$. In $\theta^{-1}([X])$ there then exists a curve $X' \subset R_{1,1}$ such that $\bar{A} \sim l$. In particular, $\deg(\bar{A}) = \lambda - 2$ and $\delta_A = \lambda - 2 - a$.*

Proof. Let $\bar{A} \sim l + \alpha l' \subset \bar{S}_0 = \varphi_{2l+(\lambda-2)l'}(\mathbb{F}_0) \subset \mathbb{P}^{3\lambda-4}$ and assume that $\alpha \geqslant 1$. Since

$$\deg_{\bar{S}_0}(\bar{A}) = \int (l + \alpha l') \cdot (2l + (\lambda - 2)l') = \lambda - 2 + 2\alpha \tag{12.6}$$

and $\deg(A) = a \leqslant \lambda - 2$ (from Remark 9.1), the number of double points of $\bar{X}_0$ lying on $\bar{A}$ is, from (9.2), $\delta_A = \deg(\bar{A}) - \deg(A) = \lambda - 2 + 2\alpha - a \geqslant 2\alpha$. Therefore, since $\bar{A}$ meets each line of the ruling l of $\bar{S}_0$ in α points, there are at least two double points of $\bar{X}_0$, N_1 and N_2 say, belonging to $\bar{A}$ and not belonging to a same line l. Consider now the isomorphism $\varphi_{l+2l'} \colon R_{1,1} \cong \bar{S}_0 \to \tilde{S} \cong R_{2,2}$ and set $\tilde{A} := \varphi(\bar{A}) \sim \tilde{l} + \alpha\tilde{l}'$; for simplicity, we still denote by N_1 and N_2 the images of these points in $\tilde{S}$.

Clearly, $\deg(\tilde{A}) = \alpha + 2$ and the projection $\pi_{\langle N_1, N_2\rangle} \colon \tilde{S} \to R_{1,1}$ maps $\tilde{A}$ to a unisecant curve $\bar{A}^*$ of degree α (since $N_1, N_2 \in \tilde{A}$) lying on $R_{1,1}$. Hence, $\bar{A}^* \sim l + (\alpha - 1)l'$; in particular, from (12.6), $\deg_{\bar{S}_0}(\bar{A}^*) = \lambda - 2 + 2(\alpha - 1)$.

Set $X' := (\pi_{\langle N_1, N_2\rangle} \circ \varphi_{l+2l'})(X) \subset R_{1,1}$ and let $A^* \subset S$ be the curve corresponding to $\bar{A}^* \subset R_{1,1}$. Since the number of the double points of X' lying on $\bar{A}^*$ is $\delta_A - 2$, we get that

$$\deg(A^*) = \deg_{\bar{S}_0}(\bar{A}^*) - (\delta_A - 2) = \lambda - 2 + 2\alpha - \delta_A = a = \deg(A);$$

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this implies that $A^* = A$. Iterating this procedure we obtain a model of X such that $\alpha = 0$; hence, $\bar{A} \sim l$ and the other requirements are fulfilled. □

Corollary 12.14. *Let $a < (g-\lambda-1)/2$ and let $\tilde{\mathcal{W}}_g^\lambda(a,b) \subset \mathcal{W}_g^\lambda(a,b)$ be the following set:*

$$\tilde{\mathcal{W}}_g^\lambda(a,b) := \{X' \in \mathcal{W}_g^\lambda(a,b) \mid X' \subset R_{1,1},\ \bar{A} \sim l\}.$$

The restriction

$$\theta\colon \tilde{\mathcal{W}}_g^\lambda(a,b) \to \mathcal{M}_g^\lambda(a,b)$$

is then surjective and the fibres have dimension 6 unless g is odd and $\lambda = (g+3)/2$, in which case they have dimension 7.

Proof. The surjectivity is immediate by Lemma 12.13, and the dimension of the fibres can be computed using the same argument as in Lemma 12.3. □

We set

$$\epsilon := \begin{cases} 0 & \text{if } b < c, \\ 1 & \text{if } a < b = c \end{cases} \qquad \text{and} \quad \xi := \begin{cases} 1 & \text{if } \lambda = \dfrac{g+3}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the case $a = b = c$ (which corresponds to $\epsilon = 2$ in Theorem 12.11) does not occur here. Namely, we now consider the range $a < (g-\lambda-1)/2$: the relation $a = b = c$ would contradict (R_1).

Theorem 12.15. *Let g, λ, a, b be positive integers satisfying* (R_1), (R_2), (R_3) *and $c = g-3-a-b$. If $a < (g-\lambda-1)/2$, then $\mathcal{M}_g^\lambda(a,b)$ is an irreducible variety of dimension $2(a+b)+\lambda+8-\epsilon-\xi$.*

Proof. Using Corollary 12.14, we can slightly modify the construction in Theorem 12.11; essentially, we use $\tilde{\mathcal{W}}_g^\lambda(a,b)$ instead of $\mathcal{W}_g^\lambda(a,b)$. In particular, we consider models $X' \subset R_{1,1}$ of X such that $\bar{A} \sim l$ and $\bar{A} \subset \bar{D} \sim 2l + (\lambda-2-c)l'$. Namely, if $\bar{A} \not\subset \bar{D}$, then $\delta_A \leqslant \bar{A}\cdot\bar{D}$; but $\delta_A = \lambda-2-a$ (from Lemma 12.13) while $\bar{A}\cdot\bar{D} = \lambda-2-c$, and this is impossible since $a < c$.

Setting $\tilde{Y}_g^\lambda(a,b)$ the image of $\tilde{\mathcal{W}}_g^\lambda(a,b)$ via the map $\Psi\colon \mathcal{W}_g^\lambda \to \mathrm{Sym}^\delta(R_{1,1})$, we have that

$$\tilde{Y}_g^\lambda(a,b) = \{(P_1,\dots,P_\delta) \mid \text{there exist } \bar{A} \in |l|,\ \bar{B} \in |l+(\lambda-2-c)l'|\colon \\ P_1,\dots,P_{\lambda-2-a} \in \bar{A},\ P_{\lambda-1-a},\dots,P_\delta \in \bar{B}\},$$

and the analogous statement of (12.3) holds:

$$\dim(\tilde{\mathcal{W}}_g^\lambda(a,b)) = \dim(\tilde{Y}_g^\lambda(a,b)) + 5\lambda + 4 - 3\delta. \tag{12.7}$$

Consider the correspondence $Z_{a,b}^\lambda \subset |l| \times |l+(\lambda-2-c)l'| \times \mathrm{Sym}^\delta(R_{1,1})$ defined by

$$Z_{a,b}^\lambda := \{(\bar{A},\bar{B},(P_1,\dots,P_\delta)) \mid P_1,\dots,P_{\lambda-2-a} \in \bar{A},\ P_{\lambda-1-a},\dots,P_\delta \in \bar{B}\}.$$

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Note that b is determined from a and c. Consider now the two canonical projections:

$$\begin{array}{ccc} & Z^{\lambda}_{a,b} & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ |l| \times |l + (\lambda - 2 - c)l'| & & \tilde{Y}^{\lambda}_{g}(a,b) \subset \mathrm{Sym}^{\delta}(R_{1,1}) \end{array}$$

Using the same argument as in Theorem 12.11, one can see that the fibres of q are irreducible of dimension ϵ. Note that, in this case, ϵ can assume only the values 0 and 1, since the assumption that $a < (g-\lambda-1)/2$ implies that $a < b$; otherwise, $a+b < g-\lambda-1$, against (R_3) (see Theorem 10.6). Note that p is surjective from Corollary 11.6. Moreover, the general fibre $p^{-1}(\bar{A}, \bar{B})$ of p is isomorphic to $\mathrm{Sym}^{\lambda-2-a}(\bar{A}) \times \mathrm{Sym}^{\delta-\lambda+2+a}(\bar{B})$, so it is irreducible of dimension δ. Therefore, $Z^{\lambda}_{a,b}$ and, hence, $\tilde{Y}^{\lambda}_{g}(a,b)$ are irreducible and

$$\dim(\tilde{Y}^{\lambda}_{g}(a,b)) = \dim(Z^{\lambda}_{a,b}) - \epsilon = \dim|l| + \dim|l + (\lambda-2-c)l'| + \delta - \epsilon = 2(\lambda-1-c) + \delta - \epsilon,$$

so, using (12.7) we obtain that

$$\dim(\tilde{\mathcal{W}}^{\lambda}_{g}(a,b)) = 2(\lambda-1-c) + \delta - \epsilon + 5\lambda + 4 - 3\delta = 2(3\lambda + 1 - c - \delta) + \lambda - \epsilon.$$

Using (12.1), we get that $3\lambda + 1 - c - \delta = 3\lambda + 1 - (g-3-a-b) - 3(\lambda-1) + g = a+b+7$, so

$$\dim(\tilde{\mathcal{W}}^{\lambda}_{g}(a,b)) = 2(a+b) + 14 + \lambda - \epsilon.$$

Applying Corollary 12.14, we obtain that

$$\dim(\mathcal{M}^{\lambda}_{g}(a,b)) = \dim(\tilde{\mathcal{W}}^{\lambda}_{g}(a,b)) - 6 - \xi = 2(a+b) + 8 + \lambda - \epsilon - \xi,$$

as required. □

Remark 12.16. If $a < (g-\lambda-1)/2$, then $\delta = 3(\lambda-1) - g > 0$; in particular, $\lambda > (g+3)/3$. To show this, just remark that $g \leqslant 3\lambda - 3$ by (R_1); hence, $a < (g-\lambda-1)/2 \leqslant (3\lambda-3-\lambda-1)/2 = \lambda-2$, so, from Lemma 12.13, $\delta \geqslant \delta_A = \lambda-2-a > 0$.

Corollary 12.17. *Set, as usual, $a \leqslant b \leqslant c$ and $a+b+c = g-3$. The following facts hold.*

(1) *The general curve $X(g,\lambda,a,b)$ of $\mathcal{M}^{\lambda}_{g}$ satisfies $a+b \geqslant (2g-8)/3$.*

(2) *For the general curve $X(g,\lambda,a,b)$ of $\mathcal{M}^{\lambda}_{g}$, the values of a, b, $c = g-3-(a+b)$ are determined by the class of g (mod 3). In particular,*

   (i) *if $g = 3p$, then $(a,b,c) = (p-1,p-1,p-1)$;*

   (ii) *if $g = 3p+1$, then $(a,b,c) = (p-1,p-1,p)$;*

   (iii) *if $g = 3p+2$, then $(a,b,c) = (p-1,p,p)$.*

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(3) *Conversely, for the above values of a and b we obtain a stratum of maximal dimension, i.e.*

$$\dim(\mathcal{M}_g^\lambda(a,b)) = \dim(\mathcal{M}_g^\lambda).$$

Consequently,

$$\text{a curve } X(g,\lambda,a,b) \in \mathcal{M}_g^\lambda \text{ is general} \quad \Longleftrightarrow \quad a,b,c \in \left\{ \left[\frac{g-3}{3}\right], \left[\frac{g-1}{3}\right] \right\}.$$

Proof. (1) We have to show that, if $a+b < (2g-8)/3$, $\dim(\mathcal{M}_g^\lambda(a,b)) < \dim(\mathcal{M}_g^\lambda)$. We rewrite the above condition correspondingly for the possible values of $g \pmod 3$ as follows.

- For $g = 3p$,

$$a+b \leqslant 2p-3 \quad \Rightarrow \quad a \leqslant p-2 \quad \Rightarrow \quad 2a+b \leqslant 3p-5.$$

- For $g = 3p+1$,

$$a+b \leqslant 2p-3 \quad \Rightarrow \quad a \leqslant p-2 \quad \Rightarrow \quad 2a+b \leqslant 3p-5.$$

- For $g = 3p+2$,

$$a+b \leqslant 2p-2 \quad \Rightarrow \quad a \leqslant p-1 \quad \Rightarrow \quad 2a+b \leqslant 3p-3.$$

Clearly, in all these cases,

$$a+b \leqslant \frac{2g-9}{3} \quad \text{and} \quad 2a+b \leqslant g-5. \tag{12.8}$$

From Theorem 12.11 (respectively, Theorem 12.15) and using (12.8) we immediately obtain that

$$\begin{aligned} a \geqslant \frac{g-\lambda-1}{2} \quad &\Rightarrow \quad \dim(\mathcal{M}_g^\lambda(a,b)) \leqslant 2(2a+b+\lambda)+10-g-\xi \\ &\qquad\qquad\qquad\qquad \leqslant 2(g-5+\lambda)+10-g-\xi \\ &\qquad\qquad\qquad\qquad = g+2\lambda-\xi, \\ a < \frac{g-\lambda-1}{2} \quad &\Rightarrow \quad \dim(\mathcal{M}_g^\lambda(a,b)) \leqslant 2(a+b)+\lambda+8-\xi \\ &\qquad\qquad\qquad\qquad \leqslant \frac{4g-18}{3}+\lambda+8-\xi \\ &\qquad\qquad\qquad\qquad = g+\lambda+1+\frac{g+3}{3}-\xi, \end{aligned}$$

where, in both cases,

$$\xi := \begin{cases} 1 & \text{if } \lambda = \dfrac{g+3}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

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Note that, in the second case, from Remark 12.16 we have that $(g+3)/3 < \lambda$. Therefore, for every value of a it holds that

$$\dim(\mathcal{M}_g^\lambda(a,b)) < g + 2\lambda + 1 - \xi. \tag{12.9}$$

Finally, recall that, from Theorem 12.9, $\dim(\mathcal{M}_g^\lambda) = g+2\lambda+1$ for all $(g+3)/3 < \lambda < \lambda_{\max}$, where $\lambda_{\max} = \lceil (g+2)/2 \rceil$.

On the other hand, from Lemma 12.3, Proposition 12.6 and Remark 12.7 it turns out that

$$\dim(\mathcal{M}_g^{\lambda_{\max}}) = \begin{cases} g + 2\lambda_{\max} & \text{if } g \text{ is odd,} \\ g + 2\lambda_{\max} + 1 & \text{if } g \text{ is even.} \end{cases}$$

Therefore, if $\lambda < \lambda_{\max}$ or g is even, then $\xi = 0$, so (12.9) gives that

$$\dim(\mathcal{M}_g^\lambda(a,b)) < g + 2\lambda + 1 = \dim(\mathcal{M}_g^\lambda).$$

Otherwise, $\lambda = \lambda_{\max}$ and g is odd; then $\xi = 1$, so (12.9) gives that

$$\dim(\mathcal{M}_g^\lambda(a,b)) < g + 2\lambda = \dim(\mathcal{M}_g^\lambda),$$

and this proves the first part of the statement.

(2) We consider a general curve $X(g,\lambda,a,b) \in \mathcal{M}_g^\lambda$. We have just proved that $a + b \geqslant (2g-8)/3$. From the condition (R_3) we get that

$$\frac{2g-8}{3} \leqslant a + b \leqslant \frac{2g-6}{3} \quad \Rightarrow \quad a + b = \left[\frac{2g-6}{3}\right];$$

hence, $a+b$ is uniquely determined. Therefore, since $c = g - 3 - (a+b)$ and $a \leqslant b \leqslant c$, we obtain the following.

- For $g = 3p$,

$$a + b = 2p - 2 \quad \Rightarrow \quad c = p - 1 \quad \Rightarrow \quad (a,b,c) = (p-1, p-1, p-1).$$

- For $g = 3p+1$,

$$a + b = 2p - 2 \quad \Rightarrow \quad c = p \quad \Rightarrow \quad (a,b,c) = \begin{cases} (p-1, p-1, p), \\ (p-2, p, p). \end{cases}$$

- For $g = 3p+2$,

$$a + b = 2p - 1 \quad \Rightarrow \quad c = p \quad \Rightarrow \quad (a,b,c) = (p-1, p, p).$$

Note that the case where $g = 3p+1$ and $(a,b,c) = (p-2,p,p)$ does not correspond to a general curve since, in this case, $X(g,\lambda,a,b)$ belongs to a proper closed subset of $\mathcal{M}_g^\lambda$.

To show this, we consider the two ranges of a and the corresponding dimensions of the moduli spaces found in Theorems 12.11 and 12.15, respectively.

(I) $a \geqslant (g-\lambda-1)/2$. We have that

$$\dim(\mathcal{M}_g^\lambda(a,b)) \leqslant 2(2a+b+\lambda)+10-g = 2(3p-4+\lambda)+10-(3p+1) = 3p+2\lambda+1,$$

while $\dim(\mathcal{M}_g^\lambda) = g+2\lambda+1 = 3p+2\lambda+2$.

(II) $a < (g-\lambda-1)/2$. Substituting $g = 3p+1$ into (R_1) and into the bound of a in the assumption, we obtain, respectively, that

$$\lambda \geqslant \frac{g+3}{3} = p + \frac{4}{3} \quad \Rightarrow \quad \lambda \geqslant p+2,$$
$$p-2 = a < \frac{g-\lambda-1}{2} \quad \Rightarrow \quad \lambda \leqslant p+3.$$

Using Theorem 12.15, under the assumption that $(a,b,c) = (p-2,p,p)$, we obtain that $\epsilon = 1$ and $\xi = 0$; hence,

$$\dim(\mathcal{M}_g^\lambda(a,b)) = 2(a+b)+\lambda+8-\epsilon-\xi = 2(2p-2)+\lambda+8-1 = 4p+\lambda+3.$$

On the other hand, $\dim(\mathcal{M}_g^\lambda) = g+2\lambda+1 = 3p+2\lambda+2$. Examining the two possible cases of λ, we immediately get that

$$\dim(\mathcal{M}_g^\lambda(a,b)) = \begin{cases} 5p+5 & \text{if } \lambda = p+2, \\ 5p+6 & \text{if } \lambda = p+3, \end{cases}$$

while

$$\dim(\mathcal{M}_g^\lambda) = \begin{cases} 5p+6 & \text{if } \lambda = p+2, \\ 5p+8 & \text{if } \lambda = p+3, \end{cases}$$

and this proves the second part.

(3) It remains to show that the strata corresponding to the values (i), (ii), (iii) of (a,b,c) are maximal.

First note that the inequalities $a < (g-\lambda-1)/2$ and $\lambda \geqslant (g+3)/3$ (the latter coming from (R_1)) become, respectively,

(i)
$$p-1 < \frac{3p-\lambda-1}{2} \quad \text{and} \quad \lambda \geqslant \frac{3p+3}{3},$$

(ii)
$$p-1 < \frac{3p-\lambda}{2} \quad \text{and} \quad \lambda \geqslant \frac{3p+4}{3},$$

(iii)
$$p-1 < \frac{3p-\lambda+1}{2} \quad \text{and} \quad \lambda \geqslant \frac{3p+5}{3}.$$

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In cases (i) and (ii) we get a contradiction, while in (iii) we get that $\lambda = p+2$. So, in cases (i) and (ii), $a \geqslant (g-\lambda-1)/2$ necessarily.

Second, observe that if $a \geqslant (g-\lambda-1)/2$, then Theorem 12.11 can be applied, and we have that

$$\dim(\mathcal{M}_g^\lambda(a,b)) = 2(2a+b+\lambda) + 10 - g - \epsilon - \tau - \xi, \qquad (*)$$

where $\xi = 1$ if and only if $\lambda = (g+3)/2$. This happens if g is odd, so $\lambda = (g+3)/2 = \lceil (g+2)/2 \rceil$. Keeping the notation and the result in Remark 12.7, where $\lambda_{\max} := \lceil (g+2)/2 \rceil$, we have that $\dim(\mathcal{M}_g^{\lambda_{\max}}) = 2g+3 = \dim(\mathcal{M}_{g,4})$. Otherwise, $\xi = 0$ and $\lambda < \lceil (g+2)/2 \rceil$; in this case, from Theorem 12.9, $\dim(\mathcal{M}_g^\lambda) = g + 2\lambda + 1$. Now consider each possibility.

Case (i) $g = 3p$, $(a,b,c) = (p-1,p-1,p-1)$.

Since $\epsilon = 2$ and $\tau = 1$, from $(*)$ we obtain that

$$\dim(\mathcal{M}_g^\lambda(a,b)) = 2(3p-3+\lambda) + 10 - 3p - 2 - 1 - \xi = 3p + 2\lambda + 1 - \xi = g + 2\lambda + 1 - \xi.$$

Therefore,

$$\lambda = \left\lceil \frac{g+2}{2} \right\rceil \quad \Rightarrow \quad \xi = 1 \text{ and } \dim(\mathcal{M}_g^\lambda(a,b)) = g + 2\lambda = g + 2\frac{g+3}{2}$$
$$= 2g+3 = \dim(\mathcal{M}_g^\lambda),$$
$$\lambda < \left\lceil \frac{g+2}{2} \right\rceil \quad \Rightarrow \quad \xi = 0 \text{ and } \dim(\mathcal{M}_g^\lambda(a,b)) = g + 2\lambda + 1 = \dim(\mathcal{M}_g^\lambda).$$

Case (ii) $g = 3p+1$, $(a,b,c) = (p-1,p-1,p)$.

Since $\epsilon = 0$ and $\tau = 1$, from $(*)$ we again obtain that $\dim(\mathcal{M}_g^\lambda(a,b)) = 3p+2\lambda+2-\xi = g+2\lambda+1-\xi$. Using the same argument as before, we prove the claim.

Case (iii) $g = 3p+2$, $(a,b,c) = (p-1,p,p)$.

(I) If $a \geqslant (g-\lambda-1)/2$, the proof follows as above, using $(*)$ where $\epsilon = 1$ and $\tau = 0$.

(II) If $a < (g-\lambda-1)/2$, the dimension of the strata is computed in Theorem 12.15, where one can find that

$$\dim(\mathcal{M}_g^\lambda(a,b)) = 2(a+b) + \lambda + 8 - \epsilon - \xi. \qquad (**)$$

In our situation, $\epsilon = 1$ and $\xi = 0$, since $\lambda \neq (g+3)/2$, with $g = 3p+2$ and $\lambda = p+2$, as remarked before. So $(**)$ gives that $\dim(\mathcal{M}_g^\lambda(a,b)) = 5p+7$. On the other hand, $\dim(\mathcal{M}_g^\lambda) = g + 2\lambda + 1 = 5p+7$.

The final claim comes from (2.2) and [**11**, Chapter V, Corollary 2.19], together with a straightforward computation on the values in (i), (ii), (iii), taking into account that $a \leqslant b \leqslant c$. □

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13. Moduli spaces of four-gonal curves with $t \geqslant 1$

We recall that if $t \geqslant 1$ and the double points of the standard model $\bar{X}_0$ are distinct, then the bounds of the invariants λ and t are described in Proposition 7.4 (i)–(iv), while the invariants a and b are determined by λ and t (see Remark 7.1). More precisely,

$$\frac{g+3}{3} + t \leqslant \lambda \leqslant \frac{g+3}{2} + t, \qquad 1 \leqslant t \leqslant \frac{g+3}{6},$$
$$a = g - 2\lambda + t + 1, \qquad b = \lambda - t - 2, \qquad c = \lambda - 2.$$

As a consequence, the subvariety of $\mathcal{W}_g^\lambda$ parametrizing the curves of invariants g, λ, t, a, b can be simply denoted by $\mathcal{W}_g^\lambda(t)$. In order to describe such a variety, we perform a construction similar to that in Theorem 12.11.

We denote by $\mathcal{A}_\lambda^t$ the open subset of the linear system $|4C_0 + (\lambda + t)f|$ on $R_{1,t+1}$ parametrizing the irreducible curves of such a linear system, and set

$$\mathcal{W}_g^\lambda(t) := \{X' \in \mathcal{A}_\lambda^t \mid X = X(g, \lambda, t) \text{ and it has } \delta \text{ distinct double points on } C_0\}.$$

Consider the morphism $\varphi\colon R_{1,t+1} \to S' \subset \mathbb{P}^N$, where $\varphi := \varphi_{4C_0+(\lambda+t)f}$.

It is clear that $N = h^0(R_{1,t+1}, \mathcal{O}_{R_{1,t+1}}(4C_0 + (\lambda + t)f)) - 1 = 5(\lambda - t) + 4$ (from [**4**, Proposition 1.8]), and we can identify $\mathcal{W}_g^\lambda(t)$ with the set $\{H \in \check{\mathbb{P}}^N \mid H \supset \langle T_{P_1}(S'), \dots T_{P_\delta}(S')\rangle,\ P_i \in C_0\}$. Therefore, consider the correspondence

$$\tilde{W} = \{(H; P_1, \dots, P_\delta) \mid H \supset \langle T_{P_1}(S'), \dots, T_{P_\delta}(S')\rangle\} \subset \check{\mathbb{P}}^N \times \operatorname{Sym}^\delta(\mathbb{P}^1)$$

and the projections

$$\begin{array}{ccc} & \tilde{W} & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ \check{\mathbb{P}}^N & & \operatorname{Sym}^\delta(\mathbb{P}^1) \end{array}$$

Obviously, $\pi_1(\tilde{W}) = \overline{\mathcal{W}_g^\lambda(t)}$ and π_1 is an isomorphism on an open subset of $\mathcal{W}_g^\lambda(t)$. Moreover, π_2 is surjective and the fibres have dimension $N - \dim\langle T_{P_1}(S'), \dots, T_{P_\delta}(S')\rangle$.

One can show (as in Lemma 11.4) that it also holds in the case $t \geqslant 1$ that the space $\langle T_{P_1}(S'), \dots, T_{P_\delta}(S')\rangle$ has maximum dimension, i.e. $3\delta - 1$. Hence, $\dim(\mathcal{W}_g^\lambda(t)) = \dim \tilde{W} = N - (3\delta - 1) + \delta = 5(\lambda - t + 1) - 2\delta$, so, using Proposition 4.2 (iii), we obtain that

$$\dim(\mathcal{W}_g^\lambda(t)) = 2g + t - \lambda + 1.$$

As well as in the case $t = 0$, one can show that these varieties are not empty. Furthermore, we recall that the automorphism group of a rational ruled surface $R_{1,t+1} \subset \mathbb{P}^{t+2}$ has dimension $t + 5$ if $t \geqslant 1$, and dimension 6 if $t = 0$ (as we already noted in Lemma 12.3). These two facts, together with the previous computation of $\dim(\mathcal{W}_g^\lambda(t))$, immediately give the following result.

Theorem 13.1. *Let g, λ, t be positive integers satisfying $g \geqslant 10$, $(g+3)/3 + t \leqslant \lambda \leqslant (g+3)/2 + t$, $1 \leqslant t \leqslant (g+3)/6$. Then, $\mathcal{M}_g^\lambda(t)$ is an irreducible variety of dimension $2g - \lambda - 4$.*

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