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GROUPS IN WHICH EVERY SUBGROUP IS MODULAR-BY-FINITE

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A group G is called a *BCF-group* if there is a positive integer k such that $|X : X_G| \leqslant k$ for each subgroup X of G. The structure of BCF-groups has been studied by Buckley, Lennox, Neumann, Smith and Wiegold; they proved in particular that locally finite groups with the property BCF are Abelian-by-finite. As a group lattice version of this concept, we say that a group G is a *BMF-group* if there is a positive integer k such that every subgroup X of G contains a modular subgroup Y of G for which the index $|X : Y|$ is finite and the number of its prime divisors with multiplicity is bounded by k (it is known that that such number can be characterised by purely lattice-theoretic considerations, and so it is invariant under lattice isomorphisms of groups). It is proved here that any locally finite BMF-group contains a subgroup of finite index with modular subgroup lattice.

1. INTRODUCTION

A subgroup of a group G is called *modular* if it is a modular element of the lattice $\mathfrak{L}(G)$ of all subgroups of G. Clearly every normal subgroup of a group is modular, but arbitrary modular subgroups need not be normal; thus modularity may be considered as a lattice generalisation of normality. Lattices in which all elements are modular are also called *modular*. Obviously, the subgroup lattice of any Abelian group is modular, and hence groups with modular subgroup lattice naturally arise in the study of lattice isomorphisms of Abelian groups. The structure of groups with modular subgroup lattice has been completely described by Iwasawa [**10, 11**] and Schmidt [**14**]. We refer to [**15**] for a detailed account of results concerning modular subgroups of groups.

In a relevant article of 1955, Neumann [**12**] investigated the structure of groups whose subgroups have finitely many conjugates and that of groups in which every subgroup has finite index in its normal closure, and proved that such properties characterise central-by-finite groups and finite-by-Abelian groups, respectively. It has recently been shown that both Neumann's theorems can be considered from a lattice-theoretic point of view (see [**5, 8, 9**]).

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A subgroup X of a group G is said to be *normal-by-finite* if the core X_G of X in G has finite index in X, and a group G is a *CF-group* if all its subgroups are normal-by-finite. Moreover, G is called a *BCF-group* if there exists a positive integer k such that $|X : X_G| \leqslant k$ for each subgroup X of G. The structure of CF-groups has been studied by Buckley, Lennox, Neumann, Smith and Wiegold (see [**2, 16**]); in particular, it turns out that any locally finite BCF-group is Abelian-by-finite. If φ is a *projectivity* from a group G onto a group $\overline{G}$ (that is, an isomorphism from the lattice $\mathfrak{L}(G)$ onto the subgroup lattice $\mathfrak{L}(\overline{G})$ of $\overline{G}$), and N is a normal subgroup of G, then the image N^φ of N is a modular element of the lattice $\mathfrak{L}(\overline{G})$. Furthermore, if H and K are subgroups of G such that $H \leqslant K$ and the index $|K : H|$ is finite, then H^φ has finite index in K^φ (see [**15**, Theorem 6.1.7]). Thus the image of any normal-by-finite subgroup of G contains a subgroup of finite index which is modular in $\overline{G}$.

We shall say that a subgroup X of a group G is *modular-by-finite* if it contains a modular subgroup Y of G such that the index $|X : Y|$ is finite. The group G is called an *MF-group* if all its subgroups are modular-by-finite. Thus every projective image of a CF-group is an MF-group. Although the finiteness of the index of a subgroup can be characterised by means of lattice-theoretic concepts, it turns out immediately that the index itself is not preserved under projectivities. On the other hand, if H is a subgroup of finite index of a group G, a lattice characterisation of the number of prime factors (with multiplicity) of the index $|G : H|$ has recently been given (see [**4**]), so that in particular such number, that will be denoted by $\|G : H\|$, is invariant under projectivities. This property can be used to introduce a lattice analogue for BCF-groups: a group G is called a *BMF-group* if every subgroup X of G contains a modular subgroup Y of G such that the index $|X : Y|$ is finite and the number of prime divisors (with multiplicity) of $|X : Y|$ is at most k, where k is a positive integer independent of X. If X is any subgroup of a group G, the join of all modular subgroups of G contained in X is likewise a modular subgroup, which is called the *modular core* of X in $\mathfrak{L}(G)$ and is denoted by $\mathrm{core}_{\mathfrak{L}(G)} X$; thus a group G is a BMF-group if and only if there exists a positive integer k such that $\|X : \mathrm{core}_{\mathfrak{L}(G)} X\| \leqslant k$ for each subgroup X of G. Clearly, the class of BMF-groups is projectively invariant, and projective images of BCF-groups have the property BMF. The aim of this article is to provide lattice correspondings of the above quoted theorems on groups with normal-by-finite subgroups. Our main result is the following.

THEOREM. *Let G be a locally finite BMF-group. Then G contains a subgroup of finite index M such that the lattice $\mathfrak{L}(M)$ is modular.*

It is well known that a special role among modular subgroups is played by permutable subgroups: a subgroup X of a group G is said to be *permutable* if $XH = HX$ for each subgroup H of G, and a group is *quasihamiltonian* if all its subgroups are permutable. Moreover, a group G is called a *BQF-group* if there exists a positive integer k such that every subgroup X of G contains a permutable subgroup Y of G with $|X : Y| \leqslant k$; it was

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proved in [3] that any locally finite BQF-group contains a quasihamiltonian subgroup of finite index. Since a subgroup X of a group G is permutable if and only if X is ascendant in G and it is a modular element of the lattice $\mathfrak{L}(G)$ (see [17]), in primary locally finite groups modular subgroups coincide with permutable subgroups, and hence the above result applies in particular to primary locally finite BMF-groups.

It was shown in [2] that any locally finite CF-group has the property BCF, and we leave here as an open question whether a similar result holds for locally finite MF-groups.

In Section 3, among other results it will be proved that torsion-free locally (soluble-by-finite) MF-groups are Abelian. Actually, the hypothesis that the group is locally (soluble-by-finite) can be replaced here by a much weaker assumption, but it cannot be completely avoided. In fact, there exists a torsion-free group $G = \langle a, b\rangle$ such that $Z(G) = \langle a\rangle \cap \langle b\rangle$ is infinite cyclic and $G/Z(G)$ is a Tarski group (see [1, proof of Theorem 2]); clearly all subgroups of G are normal-by-finite, and so G is even a CF-group.

Most of our notation is standard and can be found in [13]; moreover, we shall use the monograph [15] as a general reference for results on subgroup lattices.

2. Periodic groups

A group G is called a *P^*-group* if it is the semidirect product of an Abelian normal subgroup A of prime exponent by a cyclic group $\langle x\rangle$ of prime-power order such that x induces on A a power automorphism of prime order (recall that a *power automorphism* of a group G is an automorphism mapping every subgroup of G onto itself). It is easy to see that the subgroup lattice of any P^*-group is modular, and Iwasawa [10, 11] proved that a locally finite group has modular subgroup lattice if and only if it is a direct product

$$G = \operatorname*{Dr}_{i\in I} G_i,$$

where each G_i is either a P^*-group or a primary locally finite group with modular subgroup lattice, and elements of different factors have coprime orders. Recall also that a group G is said to be a *P-group* if either it is Abelian of prime exponent or $G = \langle x\rangle \ltimes A$ is a P^*-group with the subgroup $\langle x\rangle$ of prime order. Finally, a subgroup X of a group G is said to be *P-embedded* in G if G/X_G is a periodic group and the following conditions are satisfied:

1. $G/X_G = \left(\operatorname*{Dr}_{i\in I}(K_i/X_G)\right) \times L/X_G$, where each K_i/X_G is a non-Abelian P-group and elements from different factors have coprime orders;
2. $X/X_G = \left(\operatorname*{Dr}_{i\in I}(Q_i/X_G)\right) \times \left((X\cap L)/X_G\right)$, where each Q_i/X_G is a non-normal Sylow subgroup of K_i/X_G;
3. $X \cap L$ is a permutable subgroup of G.

All P-embedded subgroups are modular, and it can be proved that every modular subgroup of a locally finite group is either permutable or P-embedded (see [19, Theorem 3.2 and Theorem E]).

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The following easy lemma suggests that properties of power automorphisms can be used in the study of modular-by-finite subgroups.

LEMMA 2.1. *Let G be a group, and let X be a modular subgroup of G. If N is a normal subgroup of G such that $X \cap N = \{1\}$, then every subgroup of N is normalised by X.*

PROOF: Let y be any element of N. Then $\langle y \rangle = \langle y, X \rangle \cap N$ is a normal subgroup of $\langle y, X \rangle$, and hence X normalises all subgroups of N. ∎

Our next result shows in particular that if G is a locally finite MF-group and R is the Hirsch-Plotkin radical of G, then all Sylow subgroups of G/R are finite.

LEMMA 2.2. *Let G be a locally finite MF-group, and let S be a Sylow p-subgroup of G (where p is a prime number). Then $S/O_p(G)$ is finite.*

PROOF: Let S_0 be the modular core of S in $\mathfrak{L}(G)$. As the index $|S : S_0|$ is finite, it can be assumed without loss of generality that $O_p(G)$ is properly contained in S_0. Then S_0 is not permutable in G, and so it must be P-embedded. Clearly, $O_p(G)$ is the core of S_0 in G, and hence we have

$$G/O_p(G) = \Big(\operatorname*{Dr}_{i \in I}\big(K_i/O_p(G)\big)\Big) \times L/O_p(G),$$

where each $K_i/O_p(G)$ is a non-Abelian P-group, elements from different factors have coprime orders and $S_0 \cap L$ is permutable in G. It follows that $S_0 \cap L = O_p(G)$ and there is an index i such that S_0 is contained in K_i. Therefore $\big|S_0 : O_p(G)\big| = p$ and so $S/O_p(G)$ is finite. ∎

LEMMA 2.3. *Let G be an MF-group, and let R be the Hirsch-Plotkin radical of G. Then there exist only finitely many prime numbers p such that R contains a p-subgroup which is not permutable in G.*

PROOF: Let ω be the set of all prime numbers p such that R contains a p-subgroup H_p which is not permutable in G, and put $H = \operatorname*{Dr}_{p \in \omega} H_p$. The modular core H_0 of H in $\mathfrak{L}(G)$ is permutable in G, and $H_0 = \operatorname*{Dr}_{p \in \omega}(H_0 \cap H_p)$; then for each $p \in \omega$ the subgroup $H_0 \cap H_p$ is likewise permutable in G (see [15, Lemma 6.2.16]), and hence none of the H_p's is contained in H_0. As the index $|H : H_0|$ is finite, it follows that the set ω must be finite. ∎

LEMMA 2.4. *Let G be a locally finite MF-group which is generated by cyclic modular subgroups. Then G contains a finite normal subgroup N such that the lattice $\mathfrak{L}(G/N)$ is modular.*

PROOF: Let X be any subgroup of G, and let X_0 be the modular core of X in $\mathfrak{L}(G)$. As the index $|X : X_0|$ is finite, there are finitely many elements $g_1, \dots, g_t$ of G such that the subgroups $\langle g_1 \rangle, \dots, \langle g_t \rangle$ are modular and

$$X \leqslant X_1 = \langle X_0, g_1, \dots, g_t \rangle;$$

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clearly, the subgroup X_1 is modular in G and the index $|X_1 : X|$ is finite. Therefore each subgroup of G has finite index in a modular subgroup, and hence G contains a finite normal subgroup N such that $\mathfrak{L}(G/N)$ is modular (see [**5**]). $\square$

Lemma 2.5. *Let G be a locally finite BMF-group, and let R be the Hirsch-Plotkin radical of G. If the factor group G/R is countable and π is any finite set of prime numbers, there exists a normal subgroup of finite index H of G such that $H = A \times K$, where A is an Abelian π-subgroup and K is a π'-subgroup.*

Proof: For each prime number p, let R_p be the unique Sylow p-subgroup of R. Clearly, the subgroups of R_p are boundedly permutable-by-finite, so that R_p is quasihamiltonian-by-finite and hence even Abelian-by-finite (see [**15**], Theorem 2.4.14). Thus R_p contains an Abelian characteristic subgroup of finite index B_p. Put

$$B = \operatorname*{Dr}_{p\in\mathbb{P}} B_p \quad \text{and} \quad A = \operatorname*{Dr}_{p\in\pi} B_p.$$

It follows from Lemma 2.2 that the Sylow subgroups of G/B are finite, and so the subgroup $L/B = O_{\pi'}(G/B)$ has finite index in G/B (see [**6**, Theorem 3.5.15 and Corollary 2.5.13]); moreover, since B is contained in $AC_L(A)$, the group $L/AC_L(A)$ is countable and so there exists a subgroup U of L such that $AU = L$ and $A \cap U = \{1\}$ (see [**6**, Theorem 2.4.5]). Let U_0 be the modular core of U in $\mathfrak{L}(G)$; then U_0 normalises all subgroups of A by Lemma 2.1, so that $U_0/C_{U_0}(A)$ is isomorphic to a group of power automorphisms of A and hence it is finite. As $K = C_{U_0}(A)$ is a π'-group and $H = A \times K$ has finite index in G, the lemma is proved. $\square$

We can now prove the main result of this section.

Proof of the Theorem: Let k be a positive integer such that $\|X : \operatorname{core}_{\mathfrak{L}(G)} X\| \leqslant k$ for each subgroup X of G. Consider the Hirsch-Plotkin radical R of G, and suppose first that the factor group G/R is countable. By Lemma 2.3 the set ω of all prime numbers p such that R contains a p-subgroup which is not permutable in G is finite, and it follows from Lemma 2.5 that there exists a normal subgroup of finite index H of G such that $H = A \times K$, where A is an Abelian ω-subgroup and K is an ω'-subgroup. Clearly,

$$R(H) = H \cap R = A \times (K \cap R)$$

and each subgroup of $K \cap R$ is permutable in G; thus each subgroup of $R(H)$ is permutable in H, and replacing G by H it can be assumed without loss of generality that all subgroups of R are permutable in G.

Let L be the subgroup generated by all cyclic modular subgroups of G, and assume by contradiction that G/L is infinite, so that it contains an infinite Abelian subgroup U/L by the Hall-Kulatilaka-Kargapolov Theorem. As $R \leqslant L$, the Sylow subgroups of G/L are finite by Lemma 2.2, and so the set of primes $\pi(U/L)$ is infinite. Consider $k+1$ distinct primes $p_1, \dots, p_k, p_{k+1}$ in $\pi(U/L)$, and for each $i = 1, \dots, k, k+1$ let u_iL be an element

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of order p_i in U/L. If $u = u_1 \dots u_k u_{k+1}$, the coset uL has obviously order $p_1 \dots p_k p_{k+1}$. On the other hand, we have $\|\langle u \rangle : \mathrm{core}_{\mathfrak{L}(G)}\langle u \rangle\| \leqslant k$, so that $\|\langle u \rangle : \langle u \rangle \cap L\| \leqslant k$ and this contradiction shows that the group G/L is finite. By Lemma 2.4 there exists a finite normal subgroup N of L such that $\mathfrak{L}(L/N)$ is a modular lattice, and another application of Lemma 2.5 for $\pi = \pi(N)$ yields that G contains a normal subgroup of finite index V such that $V = B \times W$, where B is an Abelian π-subgroup and W is a π'-subgroup. Thus $B \leqslant R \leqslant L$ and hence $L \cap V = B \times (L \cap W)$; clearly the lattice $\mathfrak{L}(L \cap W)$ is modular, so that also the subgroup of finite index $L \cap V$ of G has modular subgroup lattice.

We shall now prove that the group G/R must be countable. If Q/R is any countable subgroup of G/R, it follows from the first part of the proof that Q contains a subgroup of finite index with modular subgroup lattice, so that in particular Q is metabelian-by-finite. Since the class of soluble-by-finite groups is countably recognisable (see [**7**, Proposition 2.6]), we obtain that G itself is soluble-by-finite; thus G/R is countable because its Sylow subgroups are finite by Lemma 2.2. The theorem is proved. □

It follows directly from the above theorem that locally finite BMF-groups are metabelian-by-finite.

3. Non-periodic groups

It is well known that any non-normal maximal modular subgroup of a group is actually a maximal subgroup; on the other hand, the consideration of Tarski groups shows that maximal modular subgroups may have infinite index. Following Stonehewer [**19**], we shall denote by $\mathfrak{X}$ the class of all groups in which any maximal modular subgroup has finite index, and by $\overline{\mathfrak{X}} = \mathfrak{X}^S$ the class of groups in which all subgroups are $\mathfrak{X}$-groups. It follows from the definition that if G is an $\overline{\mathfrak{X}}$-group and M is a modular subgroup of G such that the interval $[G/M]$ is finite, then also the index $|G : M|$ is finite; thus modular subgroups of $\overline{\mathfrak{X}}$-groups are *permodular* (see [**19**, Corollary 2.2] and [**15**, Lemma 6.2.5]). Moreover, the join of finitely many finite modular subgroups of any $\overline{\mathfrak{X}}$-group is likewise finite. It was proved by Stonehewer (see [**18**, Theorem 1]) that the class $\overline{\mathfrak{X}}$ is very large, as for instance it contains all locally (soluble-by-finite) groups.

Clearly, the infinite dihedral group has the property BMF, so that in particular the set of all elements of finite order of a BMF-group need not be a subgroup. However, we shall prove that an MF-group in the class $\overline{\mathfrak{X}}$ is periodic-by-Abelian-by-finite, provided that its periodic homomorphic images are locally finite.

Lemma 3.1. *Let G be an $\overline{\mathfrak{X}}$-group, and let $\langle a_1 \rangle, \dots, \langle a_t \rangle$ be infinite cyclic modular subgroups of G. Then the group $\langle a_1, \dots, a_t \rangle$ is central-by-finite.*

Proof: Consider the equivalence relation $\sim$ in the set $\{a_1, \dots, a_t\}$, defined by the position $a_i \sim a_j$ if and only if $\langle a_i \rangle \cap \langle a_j \rangle \neq \{1\}$, and let $C_1, \dots, C_s$ be the equivalence classes; put $X_k = \langle C_k \rangle$ for $k = 1, \dots, s$, so that $[X_h, X_k] = \{1\}$ if $h \neq k$ (see [**19**, Theorem

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1.3]). For each $k \leqslant s$, the intersection

$$N_k = \bigcap_{a \in C_k} \langle a \rangle$$

is a non-trivial central subgroup of X_k, and the factor group X_k/N_k is finite, since it is generated by finitely many finite modular subgroups. As $\langle X_1, \dots, X_s \rangle = \langle a_1, \dots, a_t \rangle$, it follows that $\langle a_1, \dots, a_t \rangle / Z(\langle a_1, \dots, a_t \rangle)$ is finite. ∎

Let G be an $\overline{\mathfrak{X}}$-group, and let $M = M(G)$ be the join of all infinite cyclic modular subgroups of G. Then M is locally (central-by-finite) by Lemma 3.1; in particular, the commutator subgroup M' of M is periodic, and hence the set T of all elements of finite order of M is a normal subgroup of G. Let x be an element of finite order of M, and let $\langle a \rangle$ be any infinite cyclic modular subgroup of G. Then $\langle x \rangle = \langle x, a \rangle \cap T$ is normal in $\langle x, a \rangle$, and hence $\langle x \rangle$ is a normal subgroup of M; note also that, if x has order 4, then $ax = xa$ because the group $\langle x, a \rangle / \langle a^2 \rangle$ cannot be dihedral of order 8, and hence x even belongs to $Z(M)$. Therefore all subgroups of T are normal in M and T is Abelian, since it does not contain subgroups isomorphic to the quaternion group of order 8.

Lemma 3.2. *Let G be an $\overline{\mathfrak{X}}$-group with no periodic non-trivial normal subgroups. Then $M(G)$ is a torsion-free Abelian group and every infinite cyclic modular subgroup of G is normal.*

Proof: As the commutator subgroup of $M(G)$ is periodic, we have that $M(G)$ is a torsion-free Abelian group. Let $\langle a \rangle$ be any infinite cyclic modular subgroup of G, so that also the normal closure $\langle a \rangle^G$ is torsion-free. If x is any element of finite order of G, then $\langle a, x \rangle \cap \langle a \rangle^G = \langle a \rangle$, and so $\langle a \rangle^x = \langle a \rangle$. Consider now an element of infinite order b of G. If $\langle a \rangle \cap \langle b \rangle = \{1\}$, it is well known that b normalises $\langle a \rangle$ (see [**19**, Theorem 1.3]). Suppose finally that $\langle a \rangle \cap \langle b \rangle \neq \{1\}$; then $\langle a, b \rangle$ is central-by-finite, so that the commutator $[a, b]$ is an element of finite order of $\langle a \rangle^G$, and hence $[a, b] = 1$. Therefore $\langle a \rangle$ is a normal subgroup of G. ∎

Lemma 3.3. *Let G be an $\overline{\mathfrak{X}}$-group with no periodic non-trivial normal subgroups. If G is an MF-group, then $G/C_G(M(G))$ has order at most 2.*

Proof: By Lemma 3.2 the subgroup $M(G)$ is torsion-free Abelian and all infinite cyclic modular subgroups of G are normal. Assume that G contains two infinite cyclic normal subgroups $\langle a \rangle$ and $\langle b \rangle$ and an element x such that $a^x = a$ and $b^x = b^{-1}$; then $\langle ab \rangle$ does not contain non-trivial normal subgroups of G, a contradiction because it must contain a modular subgroup of finite index. It follows that each element of G induces on $M(G)$ either the identity or the inversion map, and hence $G/C_G(M(G))$ has order at most 2. ∎

We can now prove the following result concerning torsion-free MF-groups.

Theorem 3.4. *Let G be a torsion-free $\overline{\mathfrak{X}}$-group whose periodic homomorphic images are locally finite. If G is an MF-group, then it is Abelian.*

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PROOF: By Lemma 3.2 and Lemma 3.3 the subgroup $M(G)$ is Abelian and $\left|G/C_G(M(G))\right| \leqslant 2$. Assume that G is not Abelian, so that by Schur's theorem the group $G/Z(G)$ is not locally finite, and hence even not periodic. Let u be an element of infinite order of G such that $\langle u\rangle \cap Z(G) = \{1\}$, and let n be a positive integer for which $\langle u^n\rangle$ is a modular (and so even normal) subgroup of G. There exists an element v of G such that $[u^n, v] \neq 1$, so that in particular $(u^n)^v = u^{-n}$ and $\langle u^n\rangle \cap \langle v\rangle = \{1\}$. Clearly $u^n v^2 = v^2 u^n$, and hence the cyclic subgroup $\langle u^n v^2\rangle$ has trivial core in G, which is impossible by Lemma 3.2. This contradiction shows that the group G is Abelian. ∎

LEMMA 3.5. *Let G be an $\overline{\mathfrak{X}}$-group, and let X be a modular subgroup of G. If X has finitely many conjugates in G, then X^G/X_G is finite.*

PROOF: Let $X^{g_1}, \dots, X^{g_t}$ be the conjugates of X in G, and put $H = \langle X, g_1, \dots, g_t\rangle$. Then $X^G = X^H$ and so the index $|X^G : X|$ is finite (see [**15**, Lemma 6.2.8]); it follows that X^G/X_G is a finite group. ∎

It was proved by Iwasawa that if G is a group with modular subgroup lattice and there exist elements of infinite order a and b of G such that $\langle a\rangle \cap \langle b\rangle = \{1\}$, then G is Abelian (see [**15**, Lemma 2.4.10]). On the other hand, a non-periodic group is Abelian if and only if all its cyclic subgroups are normal; in the case of MF-groups, we have the following corresponding result.

THEOREM 3.6. *Let G be an $\overline{\mathfrak{X}}$-group whose periodic homomorphic images are locally finite. If G is an MF-group, then it contains a subgroup C of index at most 2 whose commutator subgroup is periodic and all periodic subgroups of C are normal-by-finite in G. Moreover, if G contains elements of infinite order a and b such that $\langle a\rangle \cap \langle b\rangle = \{1\}$, then every cyclic subgroup of G is normal-by-finite.*

PROOF: Let T be the largest periodic normal subgroup of G, and put $\overline{G} = G/T$. Then by Lemma 3.3 the centraliser $\overline{C} = C/T = C_{\overline{G}}(M(\overline{G}))$ has index at most 2 in $\overline{G}$. Since G is an MF-group, the factor group $\overline{G}/M(\overline{G})$ is periodic, and so even locally finite. Thus $\overline{C}/Z(\overline{C})$ is locally finite, and $\overline{C}'$ is a locally finite normal subgroup of $\overline{G}$, so that $\overline{C}' = \{1\}$ and C' is periodic. Let H be any periodic subgroup of C, and let X be a subgroup of finite index of H which is modular in G. Since all elements of infinite order of G normalise X (see [**19**, Corollary 2.2]), the subgroup X is normal in C and so it has finitely many conjugates in G; thus X^G/X_G is finite by Lemma 3.5, and hence H is normal-by-finite in G.

Suppose now that there exist elements of infinite order a and b of G such that $\langle a\rangle \cap \langle b\rangle = \{1\}$. Let $\langle x\rangle$ be an infinite cyclic subgroup of C, and let X be a non-trivial subgroup of $\langle x\rangle$ which is modular in G. Consider now any other element of infinite order y of C, and let Y be a non-trivial subgroup of $\langle y\rangle$ which is modular in G. If $\langle x\rangle \cap \langle y\rangle = \{1\}$, we have also $X \cap \langle y\rangle = \{1\}$, and hence $X^y = X$ (see [**19**, Corollary 2.2]). Suppose now that $\langle x\rangle \cap \langle y\rangle \neq \{1\}$, so that by hypothesis there exists an element of infinite order c of

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C with

$$\langle x\rangle \cap \langle c\rangle = \langle y\rangle \cap \langle c\rangle = \{1\}.$$

Then $Y^c = Y$ and the index of Y in $\langle y, c\rangle = \langle y, yc\rangle$ is infinite. As G' is locally finite, the product yc has infinite order and $Y \cap \langle yc\rangle = \{1\}$, so that $X \cap \langle yc\rangle = \{1\}$, and hence $X^c = X^{yc} = X$; it follows that $X^y = X$ also in this case. Since C is generated by its elements of infinite order, the subgroup X is normal in C, and so $\langle x\rangle$ is normal-by-finite in C. Therefore all cyclic subgroups of C are normal-by-finite in C. Let g be any element of infinite order of G, and let $\langle u\rangle$ be the core of $\langle g\rangle \cap C$ in C. By hypothesis $\langle u\rangle$ contains a non-trivial subgroup $\langle v\rangle$ which is modular in G; clearly $\langle v\rangle$ is also normal in C, and it follows from Lemma 3.5 that $\langle v\rangle^G/\langle v\rangle_G$ is finite. Therefore the subgroup $\langle g\rangle$ is normal-by-finite in G. The theorem is proved. $\square$

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