# NUMERICAL METHODS FOR THE EXIT TIME OF A PIECEWISE-DETERMINISTIC MARKOV PROCESS

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## Abstract

We present a numerical method to compute the survival function and the moments of the exit time for a piecewise-deterministic Markov process (PDMP). Our approach is based on the quantization of an underlying discrete-time Markov chain related to the PDMP. The approximation we propose is easily computable and is even flexible with respect to the exit time we consider. We prove the convergence of the algorithm and obtain bounds for the rate of convergence in the case of the moments. We give an academic example and a model from the reliability field to illustrate the results of the paper.

*Keywords:* Exit time; piecewise-deterministic Markov process; quantization; numerical method

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## 1. Introduction

The aim of this paper is to propose a practical numerical method to approximate the survival function and the moments of the exit time for a piecewise-deterministic Markov process based on the quantization of a discrete-time Markov chain naturally embedded within the continuous-time process.

Piecewise-deterministic Markov processes (PDMPs) were introduced by Davis [5] as a general class of stochastic models. PDMPs are a family of Markov processes involving deterministic motion punctuated by random jumps. The motion depends on three local characteristics, namely the flow $\Phi$, the jump rate $\lambda$, and the transition measure $Q$, which specifies the post-jump location. Starting from the point $x$, the motion of the process follows the flow $\Phi(x, t)$ until the first jump time $T_1$, which occurs either spontaneously in a Poisson-like fashion with rate $\lambda(\Phi(x, t))$ or when the flow $\Phi(x, t)$ hits the boundary of the state space. In either case, the location of the process at the jump time $T_1$ is selected by the transition measure $Q(\cdot, \Phi(x, T_1))$ and the motion restarts from this new point $X(T_1)$ denoted by $Z_1$. We similarly define the time $S_2$ until the next jump; the next jump time is $T_2 = T_1 + S_2$, the next post-jump location $Z_2 = X(T_2)$, and so on. Thus, associated to the PDMP we have discrete-time Markov chains $(Z_n, T_n)_{n \in \mathbb{N}}$, given by the post-jump locations and the jump times, and $(Z_n, S_n)_{n \in \mathbb{N}}$, given by the post-jump locations and the inter-jump times. Suitable choices of the state space and the local characteristics $\Phi$, $\lambda$, and $Q$ provide stochastic models covering a great number of problems in operations research; see, for example, [4], [5], and the corrosion model presented in this paper.

Numerical computation of the moments of the exit time for a Markov process has been studied by Helmes *et al*. [8]. Starting from an assumption related to the generator of the process, they derived a system of linear equations satisfied by the moments. In addition to these equations, they included finitely many Hausdorff moment conditions that are also linear constraints. This optimization problem is a standard linear programming problem for which a lot of efficient software is available. Lasserre and Prieto-Rumeau [9] introduced a similar method, but they improved the efficiency of the algorithm by replacing the Hausdorff moment conditions with semidefinite positivity constraints of some moment matrices. Nevertheless, their approach cannot be applied to PDMPs because the assumption related to the generator of the process is generally not satisfied. In [5, Section 33] Davis gave an iterative method to compute the mean exit time for a PDMP, but his approach involved solving a large set of ordinary differential equations whose forms are very problem specific, depending on the behaviour of the process at the boundary of the state space. Besides, and in the context of applications to reliability, it seems important to also study the distribution of the exit time.

There exists extensive literature on quantization methods for random variables and processes. The interested reader is referred to, e.g. [7], [10], and the references therein. Quantization methods have been developed recently for numerical probability or optimal stochastic control problems with applications in finance (see, e.g. [1]–[3] and [10]). The quantization of a Markov chain $(\Theta_n)_{n \in \mathbb{N}}$ consists in finding, for each $n$, an optimally designed discretization of the state space of $\Theta_n$ providing the best possible $L^p$-approximation by a random variable $\widehat{\Theta}_n$ taking its values in a grid $\Gamma_n$ of finite and fixed size as well as a transition measure of the quantized chain $(\widehat{\Theta}_n)_{n \in \mathbb{N}}$. As explained for instance in [10, Section 3], provided that the Markov kernel is Lipschitz, bounds for the rate of $L^p$-convergence of the quantized process towards the original process are obtained.

In the present work, we consider a PDMP $(X_t)_{t \geq 0}$ with state space $E$ and we present approximation methods to compute the moments and the survival function of the exit time from a set $U \subset E$, given that the PDMP exits the set $U$ before the $N$th jump time $T_N$. Roughly speaking, we estimate the moments and the survival function for $\tau \wedge T_N$. In our approach, the first step consists in expressing the $j$th moment (respectively the survival function) as the last term of some sequence $(p_{k,j})_{k \leq N}$ (respectively $(p_k)_{k \leq N}$) satisfying a recursion $p_{k+1,j} = \psi(p_{k,j})$ (respectively $p_{k+1} = \psi(p_k)$) specifically built within our paper.

In this context, a natural way to deal with these problems is to follow the idea developed in [6], namely to write the recursions in terms of an underlying discrete-time Markov chain and to replace it by its quantized approximation. The definitions of $(p_{k,j})_k$ and $(p_k)_k$ involve some discontinuities related to indicator functions, but, as in [6], we show that they occur with small enough probabilities. However, an important feature that distinguishes the present work from [6] and which prevents a straightforward application of the ideas developed therein, is that an additional important difficulty appears in the definitions of the sequences $(p_{k,j})_k$ and $(p_k)_k$. Indeed, the mapping $\psi$ such that $p_{k+1,j} = \psi(p_{k,j})$ and $p_{k+1} = \psi(p_k)$ is not Lipschitz continuous. One of the main results of this paper is to overcome this difficulty by deriving new and important properties of the Markov chain $(Z_n, T_n)_{n \in \mathbb{N}}$, combined with a sharp feature of the quantization algorithm. We are able to prove the convergence of the approximation scheme. Moreover, in the case of the moments, we even obtain bounds for the rate of convergence. It is important to stress that these assumptions are quite reasonable with regards to the applications.

An important advantage of our method is that it is flexible. Indeed, as pointed out in [1], a quantization-based method is 'obstacle free', which means, in our case, that it produces, once and for all, a discretization of the process independently of the set $U$. Consequently, the

approximation schemes for both the moments and the distribution of the exit time are flexible with respect to $U$. Indeed, if we are interested in the exit time from a new set $U'$, it will be possible, provided that $U'$ satisfies the same assumptions as $U$, to obtain in a very simple way the moments and the distribution of this new exit time. Indeed, the quantization grids are computed only once, stored offline, and may therefore serve many purposes.

The paper is organized as follows. We first recall the definition of a PDMP and state our assumptions. In Section 3, we introduce the moments and the distribution problems, and present recursive methods to solve them. Section 4 contains the main contributions of this paper, namely the approximation schemes, the proofs of convergence, and bounds for the rates of convergence. Two numerical examples are developed in Section 5 and the advantages of our approach are discussed in Section 6.

## 2. Definitions and assumptions

For any metric space $X$, we denote by $\mathcal{B}(X)$ its Borel $\sigma$-field and by $B(X)$ the set of real-valued, bounded, and measurable functions defined on $X$. For $a, b \in \mathbb{R}$, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

### 2.1. Definition of a PDMP

In this section we define a PDMP and introduce some general assumptions. Let $M$ be a finite set, called the set of the modes, that represents the different regimes of evolution of the PDMP ($M$ is supposed to be a finite space although it could be countable); for each $m \in M$, the process evolves in $E_m$, an open subset of $\mathbb{R}^{d(m)}$ (where $d: M \to \mathbb{N}^*$). Let

$$E = \{(m, \xi), \, m \in M, \, \xi \in E_m\}.$$

This is the state space of the process $(X_t)_{t \in \mathbb{R}^+} = (m_t, \xi_t)_{t \in \mathbb{R}^+}$. Let $\partial E$ be its boundary, let $\bar{E}$ be its closure, and, for any subset $Y$ of $E$, let $Y^c$ denote its complement.

Define on $E$ the following distance: for $x = (m, \xi)$ and $x' = (m', \xi') \in E$,

$$|x - x'| = \begin{cases} +\infty & \text{if } m \neq m', \\ |\xi - \xi'| & \text{otherwise.} \end{cases}$$

Moreover, for any $x \in E$ and $Y \subset E$, denote by $d(x, Y)$ the distance between the point $x$ and the set $Y$, i.e. $d(x, Y) = \inf_{y \in Y} |x - y|$.

A PDMP is defined by its local characteristics $(\Phi_m, \lambda_m, Q_m)_{m \in M}$.

- For each $m \in M$, $\Phi_m : \mathbb{R}^{d(m)} \times \mathbb{R} \to \mathbb{R}^{d(m)}$ is a continuous function called the flow in mode $m$. For all $t \in \mathbb{R}$, $\Phi_m(\cdot, t)$ is an homeomorphism and $t \to \Phi_m(\cdot, t)$ is a group, i.e. for all $\xi \in \mathbb{R}^{d(m)}$, $\Phi_m(\xi, t + s) = \Phi_m(\Phi_m(\xi, s), t)$. For all $x = (m, \xi) \in E$, define the deterministic exit time from $E$ by

$$t^*(x) = \inf\{t > 0 \text{ such that } \Phi_m(\xi, t) \in \partial E_m\}.$$

  Here and throughout, we use the convention that $\inf \varnothing = +\infty$.

- For all $m \in M$, the jump rate $\lambda_m : \bar{E}_m \to \mathbb{R}^+$ is measurable and satisfies the following condition:

  for all $(m, \xi) \in E$, there exists $\varepsilon > 0$ such that $\displaystyle\int_0^\varepsilon \lambda_m(\Phi_m(\xi, t)) \, \mathrm{d}t < +\infty.$

- For all $m \in M$, $Q_m$ is a Markov kernel on $(\mathcal{B}(\bar{E}), \bar{E}_m)$ which satisfies the following condition:

$$\text{for all } \xi \in \bar{E}_m, \quad Q_m(E \setminus \{(m, \xi)\}, \xi) = 1.$$

From these characteristics, it can be shown (see [5, p. 57]) that there exists a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, (P_x)_{x \in E})$ on which a process $(X_t)_{t \in \mathbb{R}^+}$ is defined. Its motion, starting from a point $x \in E$, may be constructed as follows. Let $T_1$ be a nonnegative random variable with survival function

$$P_x(T_1 > t) = \begin{cases} e^{-\Lambda(x,t)} & \text{if } 0 \leq t < t^*(x), \\ 0 & \text{if } t \geq t^*(x), \end{cases}$$

where, for $x = (m, \xi) \in E$ and $t \in [0, t^*(x)]$,

$$\Lambda(x, t) = \int_0^t \lambda_m(\Phi_m(\xi, s)) \, ds.$$

We then choose an $E$-valued random variable $Z_1$ with distribution $Q_m(\cdot, \Phi_m(\xi, T_1))$. The trajectory of $X_t$ for $t \leq T_1$ is

$$X_t = \begin{cases} (m, \Phi_m(\xi, t)) & \text{if } t < T_1, \\ Z_1 & \text{if } t = T_1. \end{cases}$$

Starting from the point $X_{T_1} = Z_1$, we select the next inter-jump time $T_2 - T_1$ and the next post-jump location $Z_2$ in a similar way.

Davis showed (see [5]) that the process so defined is a strong Markov process $(X_t)_{t \geq 0}$ with jump times $(T_n)_{n \in \mathbb{N}}$ (with $T_0 = 0$). The process $(\Theta_n)_{n \in \mathbb{N}} = (Z_n, T_n)_{n \in \mathbb{N}}$, where $Z_n = X_{T_n}$ is the post-jump location and $T_n$ is the $n$th jump time, is clearly a discrete-time Markov chain. Besides, we denote by $S_n = T_n - T_{n-1}$ and $S_0 = 0$ the inter-jump times.

The following assumption about the jump times is standard (see, for example, [5, Section 24]).

**Assumption 2.1.** *For all $(x, t) \in E \times \mathbb{R}^+$, $E_x[\sum_k \mathbf{1}_{\{T_k < t\}}] < +\infty$.*

Assumption 2.1 implies that $T_k \to +\infty$ almost surely (a.s.) when $k \to +\infty$.

For notational convenience, any function $h$ defined on $E$ will be identified with its component functions $h_m$ defined on $E_m$. Thus, we write

$$h(x) = h_m(\xi) \quad \text{when} \quad x = (m, \xi) \in E.$$

We also define a generalized flow $\Phi \colon E \times \mathbb{R}^+ \to E$ such that

$$\Phi(x, t) = (m, \Phi_m(\xi, t)) \quad \text{when} \quad x = (m, \xi) \in E.$$

## 2.2. Notation

For any function $w$ in $B(\bar{E})$, we introduce

$$Qw(x) = \int_E w(y) Q(dy, x), \qquad C_w = \sup_{x \in \bar{E}} |w(x)|,$$

and, for any Lipschitz continuous function $w$ in $B(\bar{E})$, we denote its Lipschitz constant by

$$[w] = \sup_{x \neq y \in \bar{E}} \frac{|w(x) - w(y)|}{|x - y|},$$

with the convention that $1/\infty = 0$.

**Remark 2.1.** For $w \in B(\bar{E})$ and from the definition of the distance on $E$, we have $[w] = \sup_{m \in M} [w_m]$.

## 3. Exit time

For all $m \in M$, let $U_m$ be a Borel subset of $E_m$ and let $U = \{(m, \xi), \, m \in M, \, \xi \in U_m\}$. We are interested in the exit time from $U$, denoted by $\tau$ and given by

$$\tau = \inf\{s \geq 0 \text{ such that } X_s \notin U\}.$$

Denote by $\mu$ the distribution of the initial state of the process $Z_0$. Since the present paper concerns numerical computations, the following assumption appears natural.

**Assumption 3.1.** *The process starts in $U$ and eventually leaves it a.s., i.e. the support of $\mu$ is included in $U$ and $P_\mu(\tau < +\infty) = 1$.*

The aim of this paper is to provide approximation schemes for the survival function and moments of the process. Our method has a high practical interest because it will provide numerical approximations as soon as the process can be simulated. Our approach is based on a recursive computation using the underlying discrete-time Markov chain $(Z_n, T_n)_{n \in \mathbb{N}}$. Therefore, we will study $\tau \wedge T_N$ rather than $\tau$ for some $N \in \mathbb{N}$ called the computation horizon. Indeed, thanks to Assumption 2.1, when $N$ goes to $\infty$, we have

$$\tau \wedge T_N \to \tau \quad P_\mu\text{-a.s.}$$

One may approximate $\tau$ by $\tau \wedge T_N$ if $N$ is chosen such that $P_\mu(\tau > T_N)$ is small enough (the choice of $N$ will be discussed in Section 3.3) because the evolution of the process beyond $T_N$ will have little impact on the law or the moments of the exit time. In the rest of this section we present the two problems we are interested in and describe the recursive methods we use to solve them.

**Definition 3.1.** Let us define $u^*(x)$ for all $x \in U$ to be the time for the flow starting from the point $x$ to exit from $U$, i.e.

$$u^*(x) = \inf\{s \geq 0 \text{ such that } \Phi(x, s) \notin U\}.$$

We now introduce some technical assumptions that will be in force throughout the paper. The first three assumptions will be crucial, while the two last assumptions can be made without loss of generality.

**Assumption 3.2.** *The function $u^*$ is*

(a) *Lipschitz continuous,*

(b) *bounded by $C_{u^*}$.*

**Assumption 3.3.** *For all $m \in M$, the set $U_m$ is convex.*

**Assumption 3.4.** *For $\alpha > 0$, let $U^\alpha = \{x \in E \text{ such that } d(x, \partial U) \leq \alpha\}$. There exist $C > 0$ and $\beta > 0$ such that, for all $k \in \{0, \ldots, N\}$, $P_\mu(Z_k \in U^\alpha) \leq C\alpha^\beta$.*

**Remark 3.1.** Assumption 3.4 can be checked in most of the applications. We will see, in the examples developed in Section 5, how it can be derived quite generally when $Z_k$ has a bounded density. Moreover, it could be replaced by the following assumption, similar to an hypothesis

introduced in [5, Section 24] and presented as quite general in applications: there exists $\varepsilon > 0$ such that, for all $x \in U$, $Q(U^\varepsilon, x) = 0$, where $U^\varepsilon = \{x \in E$ such that $d(x, \partial U) \leq \varepsilon\}$, i.e. for all $k \in \{0, \ldots, N\}$, $P_\mu(Z_k \in U^\varepsilon) = 0$.

**Assumption 3.5.** *The process cannot go back to $U$ once it has left it, i.e. for all $z \in U^c$, $P_z(\text{there exists } t \geq 0, \ X_t \in U) = 0$.*

**Assumption 3.6.** *The function $t^*$ is bounded by $C_{t^*}$.*

In our discussion, Assumption 3.5 does not imply any loss of generality and Assumption 3.6 stems from Assumption 3.2(b). Indeed, if any of the two previous assumptions is not satisfied by the process $(X_t)_{t \in \mathbb{R}^+}$, we introduce the process *killed at time $\tau$*, denoted by $(\tilde{X}_t)_{t \in \mathbb{R}^+}$ and defined by

$$\tilde{X}_t = \begin{cases} X_t & \text{for } t < \tau, \\ \Delta & \text{for } t \geq \tau, \end{cases}$$

where $\Delta$ denotes a cemetery state. The state space of the killed process is $\tilde{E} = U \cup \{\Delta\}$ and Assumption 3.5 is fulfilled since the killed process remains in $\Delta$ after leaving $U$. In addition, $\tilde{t}^*$, the deterministic exit time from $\tilde{E}$ for the killed process, equals $u^*$, which is bounded and Lipschitz continuous according to Assumption 3.2.

### 3.1. Distribution

The first goal of this paper is to compute an approximation for the law of the exit time $\tau$. More precisely, we intend to approximate $P_\mu(\tau > s \mid \tau \leq T_N)$ for $s > 0$.

Our approach is of huge practical interest because we will see that, after some initial computations, any value of the survival function of $\tau$ may be quickly obtained. More importantly, our approach is even flexible with respect to $U$ in the sense that the survival function of the exit time $\tau'$ from a new set $U' \subset U$ will also be directly available (provided that Assumptions 3.2–3.5 are still fulfilled by $U'$).

**Definition 3.2.** For all $s > 0$, define the sequences $(p_k(s))_{k \geq 0}$, $(q_k)_{k \geq 0}$, and $(r_k(s))_{k \geq 0}$ as follows:

$$p_k(s) = P_\mu(\tau > s \mid \tau \leq T_k),$$
$$q_k = P_\mu(\tau \leq T_k),$$
$$r_k(s) = P_\mu(\{\tau > s\} \cap \{T_k < \tau \leq T_{k+1}\}).$$

**Remark 3.2.** The conditional probability $p_k(s)$ does not exist when $q_k = 0$. We then choose to extend the sequence by setting $p_k(s) = 0$.

Our objective is to approximate $p_N(s)$, where $N$ represents the computation horizon. The following proposition provides a recursion for the sequence $(p_k)_{k \leq N}$; note that $p_N$ may be computed as soon as the sequences $(q_k)_{k \leq N}$ and $(r_k)_{k \leq N-1}$ are known.

**Proposition 3.1.** *Under Assumption 3.1, for all $k \in \mathbb{N}$ and $s > 0$, $p_0(s) = 0$ and*

$$p_{k+1}(s) = \begin{cases} \dfrac{p_k(s)q_k + r_k(s)}{q_{k+1}} & \text{if } q_{k+1} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* First, recall that $T_0 = 0$ so that we have $p_0 = 0$ since the process starts in $U$ according to Assumption 3.1. Then, let $k \in \mathbb{N}$ such that $q_{k+1} \neq 0$ and note that $\{\tau \leq T_{k+1}\} = \{\tau \leq T_k\} \cup \{T_k < \tau \leq T_{k+1}\}$. Then we have

$$
\begin{aligned}
p_{k+1}(s) &= \frac{P_\mu(\{\tau > s\} \cap \{\tau \leq T_{k+1}\})}{P_\mu(\tau \leq T_{k+1})} \\
&= \frac{P_\mu(\{\tau > s\} \cap \{\tau \leq T_k\}) + P_\mu(\{\tau > s\} \cap \{T_k < \tau \leq T_{k+1}\})}{q_{k+1}} \\
&= \frac{p_k(s)q_k + r_k(s)}{q_{k+1}},
\end{aligned}
$$

completing the proof.

Now, before turning to computations, let us present the second problem we are interested in.

### 3.2. Moments

Our second goal is to approximate the moments of the exit time from $U$, i.e. for all $j \in \mathbb{N}$, we are interested in $E_\mu[\tau^j \mid \tau \leq T_N]$. This is a very classical problem and some results are already available. First, it is possible to use a Monte Carlo method, and we will point out why the method we propose is more efficient and flexible. Furthermore, Helmes *et al.* [8] introduced a numerical method for computing the moments of the exit time based on linear programming. Lasserre and Prieto-Rumeau [9] improved this method by using semidefinite positivity moment conditions. These methods are quite efficient, but they require an assumption related to the generator of the process which is generally not fulfilled by the PDMP. The method we introduce now is based on the use of the Markov chain $(\Theta_n)_{n \in \mathbb{N}} = (Z_n, T_n)_{n \in \mathbb{N}}$ associated to the continuous-time process $(X_t)_{t \in \mathbb{R}^+}$.

**Definition 3.3.** For all $j \in \mathbb{N}$, introduce the sequences $(p_{k,j})_{k \geq 0}$ and $(r_{k,j})_{k \geq 0}$ defined as follows:

$$
p_{k,j} = E_\mu[\tau^j \mid \tau \leq T_k], \qquad r_{k,j} = E_\mu[\tau^j \mathbf{1}_{\{T_k < \tau \leq T_{k+1}\}}].
$$

Our objective is to approximate $p_{N,j}$, where $N$ still represents the computation horizon. Similarly to the previous section, the sequence $(p_{k,j})_{k \leq N}$ satisfies a recursion whose parameters are the sequences $(q_k)_{k \leq N}$, previously introduced, and $(r_{k,j})_{k \leq N-1}$.

**Proposition 3.2.** *Under Assumption 3.1, we have, for all $k$, $j \in \mathbb{N}$, $p_{0,j} = 0$ and*

$$
p_{k+1,j} = \begin{cases} \dfrac{p_{k,j}q_k + r_{k,j}}{q_{k+1}} & \text{if } q_{k+1} \neq 0, \\ 0 & \text{otherwise.} \end{cases}
$$

*Proof.* The proof is similar to that of Proposition 3.1.

Before turning to the approximation method itself, let us discuss the crucial question of the computation horizon.

### 3.3. The computation horizon

In this subsection we study more precisely the construction of the process $(X_t)$ in order to obtain some results concerning the jump times $(T_k)_{k \in \mathbb{N}}$. For this purpose, we introduce, in this section only, two additional hypotheses.

**Assumption 3.7.** *The jump rate $\lambda$ is bounded by $C_\lambda$.*

**Assumption 3.8.** *There exists $\varepsilon > 0$ such that, for all $x \in E$, $Q(x, A_\varepsilon) = 1$, where $A_\varepsilon = \{x \in E$ such that $t^*(x) \geq \varepsilon\}$. Roughly speaking, the jumps cannot send the process too close to the boundary of $E$.*

Assumption 3.7 is satisfied in a large majority of applications; Assumption 3.8 is quite general too and was introduced in [5, Section 24].

Let $(\Omega, \mathcal{A}, P)$ be a probability space on which is defined a sequence $(\Pi_k)_{k \in \mathbb{N}}$ of independent random variables with uniform distribution on $[0; 1]$. Let $x = (m, \xi) \in E$ and $\omega \in \Omega$, and let us focus on the construction of the trajectory $\{X_t(\omega), \, t > 0\}$ of the process starting from the point $x$. Let

$$
F(t, x) = \begin{cases} 1 & \text{if } t \leq 0, \\ \exp\left(-\int_0^t \lambda(m, \Phi_m(\xi, s)) \, \mathrm{d}s\right) & \text{if } 0 \leq t < t^*(x), \\ 0 & \text{if } t \geq t^*(x). \end{cases}
$$

It is the survival function of the first jump time $T_1$. Define its generalized inverse by

$$
\Psi(u, x) = \begin{cases} \inf\{t \geq 0 \colon F(t, x) \leq u\}, \\ +\infty, & \text{if the above set is empty.} \end{cases}
$$

Let $S_1(\omega) = T_1(\omega) = \Psi(\Pi_1(\omega), x)$ and, for all $t < T_1(\omega)$,

$$
X_t(\omega) = (m, \Phi_m(\xi, t)).
$$

If $T_1(\omega) < +\infty$, choose $X_{T_1}$ with distribution $Q(\cdot, \Phi_m(\xi, T_1))$. Assume that the trajectory is constructed until time $T_k$. If $T_k(\omega) < +\infty$, let

$$
S_{k+1}(\omega) = \Psi(\Pi_k(\omega), X_{T_k}), \qquad T_{k+1}(\omega) = T_k(\omega) + S_{k+1}(\omega).
$$

If $T_{k+1}(\omega) < +\infty$, choose $X_{T_{k+1}}$ with distribution $Q(\cdot, \Phi_{m_{T_k}}(\xi_{T_k}, S_{k+1}))$. The trajectory is finally constructed by induction.

With the same notation as above, we state the following lemma.

**Lemma 3.1.** *Let $H$ be a survival function such that, for all $t \in \mathbb{R}$ and all $x \in E$, $H(t) \leq F(t, x)$. There exists a sequence of independent random variables $(\tilde{S}_k)_{k \in \mathbb{N}}$ with distribution $H$ and such that, for all $K \in \mathbb{R}$ and $N \in \mathbb{N}$,*

$$
P_\mu(T_N < K) \leq P_\mu(\tilde{T}_N < K),
$$

*where $\tilde{T}_N = \sum_{k=0}^N \tilde{S}_k$.*

*Proof.* Let $H$ be such a survival function, and let $\tilde{\Psi}$ be its generalized inverse, i.e.

$$
\tilde{\Psi}(u) = \begin{cases} \inf\{t \geq 0 \colon H(t) \leq u\}, \\ +\infty, & \text{if the above set is empty.} \end{cases}
$$

The assumption made on $H$ yields, for all $x \in E$, $\tilde{\Psi}(u) \leq \Psi(u, x)$. Let, for all $k \in \mathbb{N}$ and all $\omega \in \Omega$,

$$
\tilde{S}_k(\omega) = \tilde{\Psi}(\Pi_k(\omega)).
$$

Note that we are using the same $\Pi_k$ as in the definition of $S_k$, allowing us to write $\tilde{S}_k \leq S_k$ a.s. and, therefore, $\tilde{T}_k \leq T_k$ a.s. The result follows.

Similarly to Davis [5, Section 33], we approximated $\tau$ by $\tau \wedge T_N$ since $\tau \wedge T_N \to \tau$ as $N \to +\infty$ thanks to Assumption 2.1. It is therefore necessary to choose $N$ large enough such that $P_\mu(T_N < \tau)$ is small. It is difficult to estimate this probability for a general process because the links between $\tau$ and the jump times are largely problem dependent. For instance, the geometry of $U$ can be very complex. Therefore, $N$ will generally be estimated through simulations. Indeed, we can compute $P_\mu(T_N < \tau)$ for some fixed $N$ thanks to a Monte Carlo method and increase the value of $N$ until this probability becomes small enough. However, we introduce another method to bound this probability that may prove useful in applications. First, note that, for any $K > 0$,

$$\{T_N < \tau\} \subset \{T_N < K\} \cup \{\tau > K\}.$$

This implies that

$$P_\mu(T_N < \tau) \leq P_\mu(T_N < K) + P_\mu(\tau > K).$$

This will prove especially useful whenever $\tau$ is bounded, which happens quite often in applications, because there exists a $K$ such that $P_\mu(\tau > K) = 0$. When $\tau$ is not bounded, it is sometimes possible to obtain $K$ such that $P_\mu(\tau > K)$ is small.

**Example 3.1.** (*A crack propagation model.*) We adapt here an example studied by Chiquet and Limnios [4], which models a crack propagation. Here $Y_t$ is a real-valued process representing the crack size and satisfying

$$Y_0 > 0, \qquad \dot{Y}_t = A_t Y_t \quad \text{for all } t \geq 0,$$

where $A_t$ is a Markov process with state space $\{\alpha, \beta\}$, $0 < \alpha \leq \beta$. We are interested in the time $\tau$ before the crack size reaches a critical size $y_c$. Consider the PDMP $X_t = (A_t, Y_t)$, where $A_t$ represents the mode at time $t$. It is possible to bound the exit time by considering the slowest flow: we clearly have, for all $t \geq 0$, $Y_t \geq Y_0 e^{\alpha t}$ and, thus,

$$P_\mu\left(\tau > \frac{1}{\alpha} \ln\left(\frac{y_c}{Y_0}\right)\right) = 0.$$

We now intend to bound $P_\mu(T_N < K)$ for a fixed $K > 0$. Let

$$H(t) = \begin{cases} 1 & \text{if } t \leq 0, \\ e^{-C_\lambda t} & \text{if } 0 \leq t < \varepsilon, \\ 0 & \text{if } t \geq \varepsilon. \end{cases}$$

Distribution $H$ represents, roughly speaking, the worst distribution of the inter-jump times $S_k$ in the sense that it is the distribution that gives the most frequent jumps. Indeed, denote by $F_k$ the survival function of $S_k$. We have $H \leq F_k$ for all $k \in \mathbb{N}$. Therefore, Lemma 3.1 provides a random variable $\tilde{T}_N = \sum_{k=0}^{N} \tilde{S}_k$, where the $\tilde{S}_k$ are independent and have survival function $H$, such that

$$P_\mu(T_N < K) \leq P_\mu(\tilde{T}_N < K).$$

We now bound $P_\mu(\tilde{T}_N < K)$. Standard computations yield $E_\mu[\tilde{T}_N] = Nm$ and $\text{var}_\mu[\tilde{T}_N] = N\sigma^2$, where

$$m := E_\mu[\tilde{S}_1] = \frac{1}{C_\lambda}(1 - e^{-C_\lambda \varepsilon}),$$

$$\sigma^2 := \text{var}_\mu[\tilde{S}_1] = \frac{1}{C_\lambda^2}(1 - 2C_\lambda \varepsilon e^{-C_\lambda \varepsilon} - e^{-2C_\lambda \varepsilon}).$$

Assume now that $N$ is such that $Nm > K$, and note that

$$P_\mu(\tilde{T}_N < K) \leq P_\mu(|\tilde{T}_N - E_\mu[\tilde{T}_N]| > E_\mu[\tilde{T}_N] - K).$$

Tchebychev's inequality yields

$$P_\mu(\tilde{T}_N < K) \leq \frac{N\sigma^2}{(Nm - K)^2}.$$

The right-hand side term goes to 0 when $N$ goes to $\infty$.

Finally, when $\tau$ is bounded with a high probability and when Assumptions 3.7 and 3.8 are fulfilled, we are able to choose $N$ a priori such that $P_\mu(T_N < \tau)$ is small. These conditions are satisfied in a large class of applications.

## 4. Approximation scheme

### 4.1. The quantization algorithm

First, we describe the quantization procedure for a random variable and recall some important properties that will be used in the sequel. There exists extensive literature on quantization methods for random variables and processes. We do not pretend to present here an exhaustive panorama of these methods. However, the interested reader is referred to, e.g. [1], [7], [10], and the references therein. Consider $X$, an $\mathbb{R}^q$-valued random variable such that $\|X\|_p < \infty$, where $\|X\|_p$ denotes the $L_p$-norm of $X$, i.e. $\|X\|_p = (E[|X|^p])^{1/p}$.

Let $K$ be a fixed integer. The optimal $L_p$-quantization of the random variable $X$ consists in finding the best possible $L_p$-approximation of $X$ by a random vector $\widehat{X}$ taking at most $K$ values: $\widehat{X} \in \{x^1, \ldots, x^K\}$. This procedure consists of the following two steps.

1. Find a finite weighted grid $\Gamma \subset \mathbb{R}^q$ with $\Gamma = \{x^1, \ldots, x^K\}$.

2. Set $\widehat{X} = \widehat{X}^\Gamma$, where $\widehat{X}^\Gamma = \text{proj}_\Gamma(X)$, a Borel nearest-neighbour projection on $\Gamma$.

The asymptotic properties of the $L_p$-quantization are given by the following result; see, e.g. [10].

**Theorem 4.1.** *If* $E[|X|^{p+\eta}] < +\infty$ *for some* $\eta > 0$ *then we have*

$$\lim_{K \to \infty} K^{p/q} \min_{|\Gamma| \leq K} \|X - \widehat{X}^\Gamma\|_p^p = J_{p,q} \int |h|^{q/(q+p)}(u) \, du,$$

*where the law of* $X$ *is* $P_X(du) = h(u)\lambda_q(du) + \nu$ *with* $\nu \perp \lambda_q$, $J_{p,q}$ *a constant, and* $\lambda_q$ *the Lebesgue measure in* $\mathbb{R}^q$.

Note that $X$ needs to have finite moments up to the order $p + \eta$ to ensure the above convergence. There exists a similar procedure for the optimal quantization of a Markov chain $\{X_k\}_{k \in \mathbb{N}}$. There are two approaches to provide the quantized approximation of a Markov chain. The first approach, based on the quantization at each time $k$ of the random variable $X_k$, is called the *marginal quantization*. The second approach, which enhances the preservation of the Markov property, is called the *Markovian quantization*. Note that, for the latter, the quantized Markov process is not homogeneous. These two methods are described in detail in [10, Section 3]. In this work, we use the marginal quantization approach for simplicity reasons.

Our approximation methods are based on the quantization of the underlying discrete-time Markov chain $(\Theta_k)_{k \leq N} = (Z_k, T_k)_{k \leq N}$. The quantization algorithm provides, for each time

step $0 \leq k \leq N$, a finite grid $\Gamma_k$ of $E \times \mathbb{R}^+$ as well as the transition matrices $(\widehat{Q}_k)_{0 \leq k \leq N-1}$ from $\Gamma_k$ to $\Gamma_{k+1}$. Let $p \geq 1$ such that, for all $k \leq N$, $Z_k$ and $T_k$ have finite moments at least up to order $p$ and let $\mathrm{proj}_{\Gamma_k}$ be the nearest-neighbour projection from $E \times \mathbb{R}^+$ onto $\Gamma_k$. The quantized process $(\widehat{\Theta}_k)_{k \leq N} = (\widehat{Z}_k, \widehat{T}_k)_{k \leq N}$ with value for each $k$ in the finite grid $\Gamma_k$ of $E \times \mathbb{R}^+$ is then defined by

$$(\widehat{Z}_k, \widehat{T}_k) = \mathrm{proj}_{\Gamma_k}(Z_k, T_k).$$

In practice, we begin with the computation of the quantization grids, which merely requires us to be able to simulate the process. These grids are computed only once and may be stored offline. Our schemes are then based on the following simple idea: we replace the process by its quantized approximation within the different recursions. The results are obtained in a very simple way since the quantized process has finite state space.

**Remark 4.1.** In addition, we recall a technical property of the quantization algorithm proved by Bouton and Pagès in [3]: the quantized process evolves within the convex hull of the support of the law of the original process. Therefore, and it will be required below, it follows from Assumption 3.3 that if $Z_k \in U$ a.s. for some $k \in \{0, \ldots, N\}$ then $\widehat{Z}_k \in U$ a.s.

### 4.2. Approximation scheme of the distribution and proof of convergence

We already noted in Proposition 3.1 that $p_N(s) = \mathrm{P}_\mu(\tau > s \mid \tau \leq T_N)$ may be computed as soon as the sequences $(q_k)_{k \leq N}$ and $(r_k)_{k \leq N-1}$ are known. Therefore, we find expressions of these sequences depending on the Markov chain $(Z_k, T_k)_{k \leq N}$, which we replace by the quantized process $(\widehat{Z}_k, \widehat{T}_k)_{k \leq N}$ in order to define their quantized approximations $(\widehat{q}_k)_{k \leq N}$ and $(\widehat{r}_k)_{k \leq N-1}$.

First, note that $\{T_k < \tau\} = \{Z_k \in U\}$ and $\{\tau \leq T_k\} = \{Z_k \notin U\}$ thanks to Assumption 3.5. Moreover, on $\{Z_k \in U, \; Z_{k+1} \notin U\}$, we have $\tau = (T_k + u^*(Z_k)) \wedge T_{k+1}$ a.s., where $u^*(x)$ is the deterministic exit time from $U$ starting from the point $x$ (see Definition 3.1), and we have

$$q_k = \mathrm{E}_\mu[\mathbf{1}_{U^c}(Z_k)], \qquad r_k(s) = \mathrm{E}_\mu[\mathbf{1}_{\{(T_k+u^*(Z_k)) \wedge T_{k+1} > s\}} \mathbf{1}_U(Z_k) \mathbf{1}_{U^c}(Z_{k+1})]. \tag{4.1}$$

The above equations are crucial in our discussion and, from now on, we will use them without referring to Assumption 3.5.

Before turning to the approximation scheme itself, let us state some properties of the sequence $(q_k)_{k \leq N}$ that will be important in the following proofs. Indeed, the sequence $(q_k)_k$ increases since $\{\tau \leq T_k\} \subset \{\tau \leq T_{k+1}\}$ for all $k \leq N-1$. Moreover, note that $q_0 = 0$ and $\lim_{n \to +\infty} q_n = 1$ thanks to Assumption 3.1. Therefore, there exists an index, denoted by $\tilde{k} \geq 1$, such that

- for all $k < \tilde{k}$, we have $q_k = 0$,

- for all $k \geq \tilde{k}$, we have $q_k > 0$.

We denote by $\tilde{q} = q_{\tilde{k}}$ the first positive value of the sequence so that $q_k \geq \tilde{q}$ for all $k \geq \tilde{k}$. Then we obtain the following definition.

**Definition 4.1.** Let

$$\tilde{k} = \inf\{k \geq 0 \text{ such that } q_k > 0\}, \qquad \tilde{q} = q_{\tilde{k}},$$

i.e. $\tilde{q}$ is the first strictly positive value of the sequence $(q_k)_{k \in \{0, \ldots, N\}}$.

We now naturally define the quantized approximations of the previous sequences.

**Definition 4.2.** For all $s > 0$, define the sequences $(\widehat{q}_k)_{k \in \{0,\dots,N\}}$ and $(\widehat{r}_k)_{k \in \{0,\dots,N-1\}}$ by

$$\widehat{q}_k = \mathrm{E}_\mu[\mathbf{1}_{U^c}(\widehat{Z}_k)],$$

$$\widehat{r}_k(s) = \mathrm{E}_\mu[\mathbf{1}_{\{(\widehat{T}_k + u^*(\widehat{Z}_k)) \wedge \widehat{T}_{k+1} > s\}} \mathbf{1}_U(\widehat{Z}_k) \mathbf{1}_{U^c}(\widehat{Z}_{k+1})].$$

It is important to note that both $\widehat{q}_k$ and $\widehat{r}_k(s)$ may be computed easily from the quantization algorithm. Indeed, we have

$$\widehat{q}_k = \sum_{\substack{\theta = (z,t) \in \Gamma_k \\ z \notin U}} \mathrm{P}(\widehat{\Theta}_k = \theta),$$

$$\widehat{r}_k(s) = \sum_{\substack{\theta = (z,t) \in \Gamma_k \\ z \in U}} \sum_{\substack{\theta' = (z',t') \in \Gamma_{k+1} \\ z' \notin U}} \mathbf{1}_{\{(t + u^*(z)) \wedge t' > s\}} \mathrm{P}(\widehat{\Theta}_k = \theta) \widehat{Q}_k(\theta; \theta').$$

Recall from Proposition 3.1 that the sequence $(p_k)_{k \leq N}$ satisfies a recursion that depends on the two parameters $(q_k)_{k \leq N}$ and $(r_k)_{k \leq N-1}$, which we are now able to approximate. Hence, replacing them by their quantized approximations within the same recursion leads to a new sequence, denoted by $(\widehat{p}_k)_{k \leq N}$. The rest of this section is dedicated to the proof of the convergence of $(\widehat{p}_k)_{k \leq N}$ towards $(p_k)_{k \leq N}$. This convergence is far from trivial because, on the one hand, the definitions of the sequences $(q_k)_{k \leq N}$ and $(r_k)_{k \leq N-1}$ contain many indicator functions that are not Lipschitz continuous and, on the other hand, the recursive function giving $p_{k+1}$ from $p_k, q_k, q_{k+1}$, and $r_k$ is not Lipschitz continuous either.

**Definition 4.3.** For all $s > 0$ and all $k \in \{0, \dots, N-1\}$, let $\widehat{p}_0(s) = 0$ and

$$\widehat{p}_{k+1}(s) = \begin{cases} \dfrac{\widehat{p}_k(s)\widehat{q}_k + \widehat{r}_k(s)}{\widehat{q}_{k+1}} & \text{if } \widehat{q}_{k+1} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The two following propositions will be necessary to prove the convergence of the approximation scheme. They respectively state the convergences of $(\widehat{q}_k)_{k \leq N}$ and $(\widehat{r}_k)_{k \leq N-1}$ towards $(q_k)_{k \leq N}$ and $(r_k)_{k \leq N-1}$.

**Proposition 4.1.** *Under Assumptions 3.4 and 3.5, for all $k \in \{0, \dots, N\}$, $\widehat{q}_k$ converges towards $q_k$ when the quantization error $\|\Theta_k - \widehat{\Theta}_k\|_p$ goes to $0$. More precisely, the error is bounded by*

$$|q_k - \widehat{q}_k| \leq C^{p/(p+\beta)} \left( \left(\frac{\beta}{p}\right)^{p/(p+\beta)} + \left(\frac{p}{\beta}\right)^{\beta/(p+\beta)} \right) \|Z_k - \widehat{Z}_k\|_p^{p\beta/(p+\beta)},$$

*where $C$ and $\beta$ are defined in Assumption 3.4.*

*Proof.* For all $k \in \{0, \dots, N\}$, (4.1) yields

$$|q_k - \widehat{q}_k| = |\mathrm{E}_\mu[\mathbf{1}_U(Z_k) - \mathbf{1}_U(\widehat{Z}_k)]|.$$

The difference between the indicator functions is nonzero if and only if $Z_k$ and $\widehat{Z}_k$ are on either side of $\partial U$. Therefore, in this case, for all $\alpha > 0$, if $|Z_k - \widehat{Z}_k| \leq \alpha$ then $d(Z_k, \partial U) \leq \alpha$. Hence, either $|Z_k - \widehat{Z}_k| > \alpha$ or $Z_k \in U^\alpha$. The Markov inequality and Assumption 3.4 yield

$$\mathrm{E}_\mu |\mathbf{1}_U(Z_k) - \mathbf{1}_U(\widehat{Z}_k)| \leq \mathrm{P}_\mu(|Z_k - \widehat{Z}_k| > \alpha) + \mathrm{P}_\mu(Z_k \in U^\alpha)$$

$$\leq \frac{\|Z_k - \widehat{Z}_k\|_p^p}{\alpha^p} + C\alpha^\beta.$$

This bound reaches a minimum when

$$\alpha = \left( \frac{p \|Z_k - \widehat{Z}_k\|_p^p}{\beta C} \right)^{1/(p+\beta)},$$

and the result follows.

**Proposition 4.2.** *Under Assumptions 3.2(a), 3.4, and 3.5, for all $k \in \{0, \ldots, N-1\}$ and almost every $s > 0$ with respect to the Lebesgue measure on $\mathbb{R}$,*

$$\widehat{r}_k(s) \to r_k(s)$$

*when the quantization errors $\|\Theta_l - \widehat{\Theta}_l\|_p$ for $l \in \{k, k+1\}$ go to 0.*

*Proof.* Let $k \in \{0, \ldots, N-1\}$ and $s > 0$. Equation (4.1) yields

$$|r_k(s) - \widehat{r}_k(s)| \le A + B,$$

where

$$A = |\mathrm{E}_\mu[(\mathbf{1}_{\{(T_k + u^*(Z_k)) \wedge T_{k+1} > s\}} - \mathbf{1}_{\{(\widehat{T}_k + u^*(\widehat{Z}_k)) \wedge \widehat{T}_{k+1} > s\}}) \, \mathbf{1}_U(Z_k) \, \mathbf{1}_{U^c}(Z_{k+1})]|,$$

$$B = |\mathrm{E}_\mu[\mathbf{1}_{\{(\widehat{T}_k + u^*(\widehat{Z}_k)) \wedge \widehat{T}_{k+1} > s\}} (\mathbf{1}_U(Z_k) \, \mathbf{1}_{U^c}(Z_{k+1}) - \mathbf{1}_U(\widehat{Z}_k) \, \mathbf{1}_{U^c}(\widehat{Z}_{k+1}))]|.$$

In the $A$ term, we crudely bound $\mathbf{1}_U(Z_k)$ and $\mathbf{1}_{U^c}(Z_{k+1})$ by 1 and turn to the difference between the two indicator functions. This difference is nonzero if and only if $(T_k + u^*(Z_k)) \wedge T_{k+1}$ and $(\widehat{T}_k + u^*(\widehat{Z}_k)) \wedge \widehat{T}_{k+1}$ are on either side of $s$, implying that they both belong to $[s - \eta; s + \eta]$, where $\eta = |(T_k + u^*(Z_k)) \wedge T_{k+1} - (\widehat{T}_k + u^*(\widehat{Z}_k)) \wedge \widehat{T}_{k+1}|$. Then we have

$$|\mathbf{1}_{\{(T_k + u^*(Z_k)) \wedge T_{k+1} > s\}} - \mathbf{1}_{\{(\widehat{T}_k + u^*(\widehat{Z}_k)) \wedge \widehat{T}_{k+1} > s\}}| \le \mathbf{1}_{\{|(T_k + u^*(Z_k)) \wedge T_{k+1} - s| \le \eta\}}$$

so that

$$A \le \mathrm{P}_\mu(|(T_k + u^*(Z_k)) \wedge T_{k+1} - s| \le \eta).$$

The following discussion consists in noting that either $\eta$ and the probability that $(T_k + u^*(Z_k)) \wedge T_{k+1}$ belongs to the interval $[s - \eta; s + \eta]$ are small, or $\eta$ is large, but this happens with a small probability too when the quantization error goes to 0. For all $\alpha > 0$, we have

$$A \le \mathrm{P}_\mu(|(T_k + u^*(Z_k)) \wedge T_{k+1} - s| \le \eta, \ \eta \le \alpha) + \mathrm{P}_\mu(\eta > \alpha)$$

$$\le \mathrm{P}_\mu(|(T_k + u^*(Z_k)) \wedge T_{k+1} - s| \le \alpha) + \mathrm{P}_\mu(\eta > \alpha)$$

$$\le |\varphi_k(s + \alpha) - \varphi_k(s - \alpha)| + \frac{\|\eta\|_p^p}{\alpha^p},$$

where $\varphi_k$ denotes the distribution function of $(T_k + u^*(Z_k)) \wedge T_{k+1}$. Let $\varepsilon > 0$, and assume that $s$ is not an atom of this distribution, so that there exists $\alpha_1 > 0$ such that $|\varphi_k(s + \alpha_1) - \varphi_k(s - \alpha_1)| \le \varepsilon$. Besides, thanks to Assumption 3.2(a), the Lipschitz continuity condition on $u^*$, we have $\eta \le |T_k - \widehat{T}_k| + [u^*]|Z_k - \widehat{Z}_k| + |T_{k+1} - \widehat{T}_{k+1}|$. Moreover, since the quantization error goes to 0, we may assume that $\|\eta\|_p \le \alpha_1 \varepsilon^{1/p}$. Setting $\alpha = \alpha_1$ in the previous computations yields

$$A \le |\varphi_k(s + \alpha_1) - \varphi_k(s - \alpha_1)| + \frac{\|\eta\|_p^p}{\alpha_1^p} \le 2\varepsilon.$$

Note that the set of atoms of the distribution function of $(T_k + u^*(Z_k)) \wedge T_{k+1}$ is at most countable, so the previous discussion is true for almost every $s > 0$ with respect to the Lebesgue measure. Let us now bound the $B$ term:

$$
\begin{aligned}
B &\leq \mathrm{E}_\mu \left| \mathbf{1}_U(Z_k)\, \mathbf{1}_{U^c}(Z_{k+1}) - \mathbf{1}_U(\widehat{Z}_k)\, \mathbf{1}_{U^c}(\widehat{Z}_{k+1}) \right| \\
&\leq \mathrm{E}_\mu[\mathbf{1}_{U^c}(Z_{k+1})|\mathbf{1}_U(Z_k) - \mathbf{1}_U(\widehat{Z}_k)|] + \mathrm{E}_\mu[\mathbf{1}_U(\widehat{Z}_k)|\mathbf{1}_{U^c}(Z_{k+1}) - \mathbf{1}_{U^c}(\widehat{Z}_{k+1})|] \\
&\leq |q_k - \widehat{q}_k| + |q_{k+1} - \widehat{q}_{k+1}|,
\end{aligned}
$$

which goes to 0 thanks to Proposition 4.1.

The convergence of the approximation scheme of the distribution of the exit time is now a straightforward consequence of the following proposition.

**Proposition 4.3.** *We assume that Assumptions 3.1, 3.3, 3.4, and 3.5 hold. Let $(\sigma_k)_{k \leq N-1}$ and $(\widehat{\sigma}_k)_{k \leq N-1}$ be two sequences of $[0, 1]$-valued real numbers. Let $(\pi_k)_{0 \leq k \leq N}$ and $(\widehat{\pi}_k)_{0 \leq k \leq N}$ be defined as follows: $\pi_0 = \widehat{\pi}_0 = 0$,*

$$
\pi_{k+1} = \begin{cases} \dfrac{\pi_k q_k + \sigma_k}{q_{k+1}} & \text{if } q_{k+1} \neq 0, \\ 0 & \text{otherwise,} \end{cases}
$$

$$
\widehat{\pi}_{k+1} = \begin{cases} \dfrac{\widehat{\pi}_k \widehat{q}_k + \widehat{\sigma}_k}{\widehat{q}_{k+1}} & \text{if } \widehat{q}_{k+1} \neq 0, \\ 0 & \text{otherwise.} \end{cases}
$$

*For $0 \leq k \leq N$, if the quantization error is such that, for all $l \leq k$,*

$$
C^{p/(p+\beta)} \left( \left( \frac{\beta}{p} \right)^{p/(p+\beta)} + \left( \frac{p}{\beta} \right)^{\beta/(p+\beta)} \right) \|Z_l - \widehat{Z}_l\|_p^{p\beta/(p+\beta)} \leq \frac{1}{2}\tilde{q},
$$

*then*

$$
\begin{aligned}
|\pi_k - \widehat{\pi}_k| &\leq \frac{2}{\tilde{q}} (\pi^{\sup}|q_{k-1} - \widehat{q}_{k-1}| + |\pi_{k-1} - \widehat{\pi}_{k-1}| + |\sigma_{k-1} - \widehat{\sigma}_{k-1}|) \\
&\quad + \frac{2(\pi^{\sup} + 1)}{\tilde{q}^2} |q_k - \widehat{q}_k|,
\end{aligned}
$$

*where $\pi^{\sup} = \max_{0 \leq k \leq N} \pi_k$.*

*Proof.* The difficulty with proving this result lies in the fact that the recursive function giving $\pi_{k+1}$ from $\pi_k, q_k, q_{k+1}$, and $\sigma_k$ is not Lipschitz continuous because of the division by $q_{k+1}$. To overcome this drawback, we will use the strictly positive lower bound for $q_k$ described earlier. Indeed, recall from Definition 4.1 that there exists a step $\tilde{k}$ such that $q_k \geq \tilde{q} > 0$ for all $k \geq \tilde{k}$ and $q_k = 0$ for all $k < \tilde{k}$. What is more, a similar bound will be derived for the quantized values $\widehat{q}_k$ thanks to the convergence of $\widehat{q}_k$ towards $q_k$.

We now prove by induction that $\widehat{\pi}_k$ converges towards $\pi_k$. First, we have $\widehat{\pi}_0 = \pi_0 = 0$. Then, let $k \in \{1, \dots, N\}$.

If $k < \tilde{k}$ then $q_k = 0$ and Assumption 3.3 yields $\widehat{q}_k = 0$ too. Indeed, $q_k = 0$ means that $Z_k \in U$ a.s. Since $U$ is a convex set, Remark 4.1 implies that $\widehat{Z}_k \in U$ a.s. too. In other words, $\widehat{q}_k = 0$. Finally, from the definitions, we have $\pi_k = \widehat{\pi}_k = 0$.

If $k \geq \tilde{k}$ then $q_k \geq \tilde{q} > 0$. In order to bound the error between $\pi_k$ and $\widehat{\pi}_k$, it is indeed necessary to have a strictly positive lower bound for $q_k$ because of the division by $q_k$ within the

recursion. Now we need to obtain the same kind of bound for $\widehat{q}_k$. This can be achieved thanks to Proposition 4.1, giving the convergence of $\widehat{q}_k$ towards $q_k$. Indeed, assume from now on that the number of points in the quantization grids is large enough such that the quantization error is sufficiently small to ensure that, for all $j = \tilde{k}, \ldots, N$, $|q_j - \widehat{q}_j| \leq \frac{1}{2}\tilde{q}$. Hence, the required lower bound is $\widehat{q}_k \geq \frac{1}{2}\tilde{q} > 0$. Therefore,

$$
\begin{aligned}
|\pi_k - \widehat{\pi}_k| &\leq \left| \frac{\pi_{k-1}q_{k-1} + \sigma_{k-1}}{q_k} - \frac{\widehat{\pi}_{k-1}\widehat{q}_{k-1} + \widehat{\sigma}_{k-1}}{\widehat{q}_k} \right| \\
&\leq \frac{\pi_{k-1}}{\widehat{q}_k}|q_{k-1} - \widehat{q}_{k-1}| + \frac{\widehat{q}_{k-1}}{\widehat{q}_k}|\pi_{k-1} - \widehat{\pi}_{k-1}| + \frac{1}{\widehat{q}_k}|\sigma_{k-1} - \widehat{\sigma}_{k-1}| \\
&\quad + |\pi_{k-1}q_{k-1} + \sigma_{k-1}|\frac{|q_k - \widehat{q}_k|}{q_k\widehat{q}_k} \\
&\leq \frac{\pi^{\sup}}{\widehat{q}_k}|q_{k-1} - \widehat{q}_{k-1}| + \frac{1}{\widehat{q}_k}|\pi_{k-1} - \widehat{\pi}_{k-1}| + \frac{1}{\widehat{q}_k}|\sigma_{k-1} - \widehat{\sigma}_{k-1}| \\
&\quad + (\pi^{\sup} + 1)\frac{|q_k - \widehat{q}_k|}{q_k\widehat{q}_k} \\
&\leq \frac{2}{\tilde{q}}(\pi^{\sup}|q_{k-1} - \widehat{q}_{k-1}| + |\pi_{k-1} - \widehat{\pi}_{k-1}| + |\sigma_{k-1} - \widehat{\sigma}_{k-1}|) \\
&\quad + \frac{2(\pi^{\sup} + 1)}{\tilde{q}^2}|q_k - \widehat{q}_k|,
\end{aligned}
$$

where $\pi^{\sup} = \max_{0 \leq k \leq N} \pi_k$.

**Remark 4.2.** Note that a bound for the rate of convergence of $\widehat{\pi}_k$ towards $\pi_k$ may be obtained as soon as a bound for the rate of convergence of $\widehat{\sigma}_k$ towards $\sigma_k$ and an upper bound for the sequence $(\pi_k)_{0 \leq k \leq N}$ are available.

We now state one of our main results, namely the convergence of the approximation scheme of the distribution of the exit time.

**Theorem 4.2.** *Under Assumptions 3.1, 3.2(a), 3.3, 3.4, and 3.5, for all $k \in \{0, \ldots, N\}$ and almost every $s > 0$ with respect to the Lebesgue measure on $\mathbb{R}$,*

$$
\widehat{p}_k(s) \to p_k(s)
$$

*when the quantization errors $\|\Theta_j - \widehat{\Theta}_j\|_p$ for $j \in \{0, \ldots, k\}$ go to 0.*

*Proof.* Let $s > 0$ such that $(\widehat{r}_k(s))_k$ converges towards $(r_k(s))_k$ and apply Proposition 4.3 with $(\sigma_k)_k = (r_k(s))_k$ and $(\widehat{\sigma}_k)_k = (\widehat{r}_k(s))_k$ so that $(\pi_k)_k = (p_k(s))_k$ and $(\widehat{\pi}_k)_k = (\widehat{p}_k(s))_k$. Finally, note that $(p_k(s))_k$ is bounded by 1.                                                                            ∎

**Remark 4.3.** It may be useful to note that, although it will be crucial in the moments approximation scheme, the boundedness condition on $u^*$ (Assumption 3.2(b)) was unnecessary in this section. Hence, the distribution approximation can be achieved without this hypothesis.

We now obtain an easily computable approximation for the survival function of the exit time. Let us now consider its moments. Of course, they can be derived from the distribution, but we present in the following subsection a method to approximate them directly. An important advantage of this method will be to provide a bound for the rate of convergence.

### 4.3. Approximation scheme of the moments and rate of convergence

Similarly to the distribution, the moments can be approximated thanks to the quantization of the process $(\Theta_k)_{k \leq N} = (Z_k, T_k)_{k \leq N}$. However, it is important to stress that we will be able to derive a rate of convergence for our approximation scheme. We note from Proposition 3.2 that, similarly to the case of the distribution, $p_{N,j} = \mathrm{E}_\mu[\tau^j \mid \tau \leq T_N]$ can be computed as soon as the sequences $(q_k)_{k \leq N}$ and $(r_{k,j})_{k \leq N-1}$ are known. The first sequence has already been approximated in the previous section, but we still need to find an expression for the second sequence, dependent on the Markov chain $(Z_k, T_k)_k$ to define its quantized approximation $(\widehat{r}_{k,j})_{k \leq N-1}$. Thanks to Assumption 3.5, the same arguments give

$$r_{k,j} = \mathrm{E}_\mu[((T_k + u^*(Z_k)) \wedge T_{k+1})^j \, \mathbf{1}_U(Z_k) \, \mathbf{1}_{U^c}(Z_{k+1})].$$

Hence, we can now naturally define the quantized approximation of the sequences $(r_{k,j})_{k \leq N-1}$ and $(p_{k,j})_{k \leq N}$.

**Definition 4.4.** For all $j \in \mathbb{N}$, define the sequence $(\widehat{r}_{k,j})_{k \in \{0,\dots,N-1\}}$ by

$$\widehat{r}_{k,j} = \mathrm{E}_\mu[((\widehat{T}_k + u^*(\widehat{Z}_k)) \wedge \widehat{T}_{k+1})^j \, \mathbf{1}_U(\widehat{Z}_k) \, \mathbf{1}_{U^c}(\widehat{Z}_{k+1})]$$

and the sequence $(\widehat{p}_{k,j})_{k \in \{0,\dots,N\}}$ by $\widehat{p}_{0,j} = 0$ and

$$\widehat{p}_{k+1,j} = \begin{cases} \dfrac{\widehat{p}_{k,j}\widehat{q}_k + \widehat{r}_{k,j}}{\widehat{q}_{k+1}} & \text{if } \widehat{q}_{k+1} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

As for $\widehat{q}_k$ and $\widehat{r}_k(s)$ defined in the previous section, $\widehat{r}_{k,j}$ may be computed easily from the quantization algorithm. Indeed, we have

$$\widehat{r}_{k,j} = \sum_{\substack{\theta=(z,t)\in\Gamma_k \\ z\in U}} \sum_{\substack{\theta'=(z',t')\in\Gamma_{k+1} \\ z'\notin U}} ((t + u^*(z)) \wedge t')^j \, \mathrm{P}(\widehat{\Theta}_k = \theta) \widehat{Q}_k(\theta; \theta').$$

The following proposition proves the convergence of $\widehat{r}_{k,j}$ towards $r_{k,j}$.

**Proposition 4.4.** *Under Assumptions 3.2(a), 3.4, 3.5, and 3.6, for all $k \in \{0, \dots, N-1\}$ and all $j \in \mathbb{N}$, $\widehat{r}_{k,j}$ converges towards $r_{k,j}$ when the quantization errors $\|\Theta_l - \widehat{\Theta}_l\|_p$ for $l \in \{k, k+1\}$ go to $0$. More precisely, the error is bounded by*

$$|r_{k,j} - \widehat{r}_{k,j}| \leq j((k+1)C_{t^*})^{j-1}(\|T_k - \widehat{T}_k\|_p + [u^*]\|Z_k - \widehat{Z}_k\|_p + \|T_{k+1} - \widehat{T}_{k+1}\|_p)$$
$$+ ((k+1)C_{t^*})^j(|q_k - \widehat{q}_k| + |q_{k+1} - \widehat{q}_{k+1}|).$$

*Proof.* Let $k \in \{0, \dots, N-1\}$ and $j \in \mathbb{N}$. We have

$$|r_{k,j} - \widehat{r}_{k,j}| \leq A + B,$$

where

$$A = |\mathrm{E}_\mu[(((T_k + u^*(Z_k)) \wedge T_{k+1})^j - ((\widehat{T}_k + u^*(\widehat{Z}_k)) \wedge \widehat{T}_{k+1})^j) \, \mathbf{1}_U(Z_k) \, \mathbf{1}_{U^c}(Z_{k+1})]|,$$
$$B = |\mathrm{E}_\mu[((\widehat{T}_k + u^*(\widehat{Z}_k)) \wedge \widehat{T}_{k+1})^j (\mathbf{1}_U(Z_k) \, \mathbf{1}_{U^c}(Z_{k+1}) - \mathbf{1}_U(\widehat{Z}_k) \, \mathbf{1}_{U^c}(\widehat{Z}_{k+1}))]|.$$

It follows from Assumption 3.6 that the inter-jump times $S_i$ are a.s. bounded by $C_{t^*}$, so $T_i \le i C_{t^*}$ a.s. and $(T_i + u^*(Z_i)) \wedge T_{i+1} \le (i+1)C_{t^*}$ a.s. By Remark 4.1, these bounds are equally true for the quantized process $\widehat{T}_i \le i C_{t^*}$ and $(\widehat{T}_i + u^*(\widehat{Z}_i)) \wedge \widehat{T}_{i+1} \le \widehat{T}_{i+1} \le (i+1)C_{t^*}$ a.s.

Let us first consider the term $A$. We crudely bound the indicator functions by 1. Moreover, define $\eta = |(T_k + u^*(Z_k)) \wedge T_{k+1} - (\widehat{T}_k + u^*(\widehat{Z}_k)) \wedge \widehat{T}_{k+1}|$ and note that the function $x \to x^j$ is Lipschitz continuous on any set $[0, M]$ with Lipschitz constant $j M^{j-1}$. Then

$$A \le \mathrm{E}_\mu[j((k+1)C_{t^*})^{j-1}\eta] \le j((k+1)C_{t^*})^{j-1}\|\eta\|_p,$$

and thanks to Assumption 3.2(a), the Lipschitz continuity condition on $u^*$, we have

$$A \le j((k+1)C_{t^*})^{j-1}(\|T_k - \widehat{T}_k\|_p + [u^*]\|Z_k - \widehat{Z}_k\|_p + \|T_{k+1} - \widehat{T}_{k+1}\|_p).$$

Moreover, the term $B$ is bounded by

$$B \le ((k+1)C_{t^*})^j \, \mathrm{E}_\mu \, |\mathbf{1}_U(Z_k)\,\mathbf{1}_{U^c}(Z_{k+1}) - \mathbf{1}_U(\widehat{Z}_k)\,\mathbf{1}_{U^c}(\widehat{Z}_{k+1})|$$
$$\le ((k+1)C_{t^*})^j(|q_k - \widehat{q}_k| + |q_{k+1} - \widehat{q}_{k+1}|).$$

Using Proposition 4.1 completes the proof.

We may now state the other important results of our paper, namely the convergence of the approximation scheme of the moments of the exit time with a bound for the rate of convergence.

**Theorem 4.3.** *Under Assumptions 3.1, 3.2(a), 3.3, 3.4, 3.5, and 3.6, for all $k \in \{0, \dots, N\}$ and all $j \in \mathbb{N}$, $\widehat{p}_{k,j}$ converges towards $p_{k,j}$ when the quantization errors $\|\Theta_j - \widehat{\Theta}_j\|_p$ for $j \in \{0, \dots, k\}$ go to 0.*

*More precisely, if the quantization error is such that, for all $l \le k$,*

$$C^{p/(p+q)}\left(\left(\frac{q}{p}\right)^{p/(p+q)} + \left(\frac{p}{q}\right)^{q/(p+q)}\right)\|Z_l - \widehat{Z}_l\|_p^{pq/(p+q)} \le \frac{1}{2}\tilde{q},$$

*then*

$$|p_{k,j} - \widehat{p}_{k,j}| \le \frac{2}{\tilde{q}}((NC_{t^*})^j|q_{k-1} - \widehat{q}_{k-1}| + |p_{k-1,j} - \widehat{p}_{k-1,j}| + |r_{k-1,j} - \widehat{r}_{k-1,j}|)$$
$$+ \frac{2((NC_{t^*})^j + 1)}{\tilde{q}^2}|q_k - \widehat{q}_k|.$$

**Remark 4.4.** The rate of convergence depends on the quantity $\tilde{q}$ whose exact value might be unknown in some complex applications. In that case, it may still be approximated through Monte Carlo simulations (see the examples in Section 5). Nevertheless, Theorems 4.2 and 4.3 prove the convergence of our approximation schemes regardless of the value of $\tilde{q}$.

*Proof of Theorem 4.3.* Let $j \in \mathbb{N}$, and apply Proposition 4.3 with $(\sigma_k)_k = (r_{k,j})_k$ and $(\widehat{\sigma}_k)_k = (\widehat{r}_{k,j})_k$ such that $(\pi_k)_k = (p_{k,j})_k$ and $(\widehat{\pi}_k)_k = (\widehat{p}_{k,j})_k$. Finally, according to Remark 4.2, a bound for the rate of convergence is obtained since the sequence $(p_{k,j})_{0 \le k \le N}$ is bounded by

$$p_{k,j} = \mathrm{E}_\mu[\tau^j \mid \tau \le T_k] \le \mathrm{E}_\mu[T_k^j \mid \tau \le T_k] \le \mathrm{E}_\mu[(kC_{t^*})^j \mid \tau \le T_k] \le (kC_{t^*})^j \le (NC_{t^*})^j.$$

This completes the proof.

## 5. Examples and numerical results

### 5.1. A Poisson process example

Let $N_t$ be a Poisson process with parameter $\lambda = 1$, and let $Y_t = t + N_t$. Here $(Y_t)_{t\geq 0}$ is a PDMP with state space $E = \mathbb{R}$; the inter-jump times $S_k$ have independent exponential distribution with parameter $\lambda = 1$; the flow is defined on $(\mathbb{R}^+)^2$ by $\Phi(x, t) = x + t$; and, finally, the post-jump locations satisfy, for all $x \in E$, $Q(\{x + 1\}, x) = 1$. An example of a trajectory of the process is represented in Figure 1. We are interested in the exit time problem for the process $(Y_t)_{t\geq 0}$. The study of this process is especially interesting because it is possible to compute the exact value of its distribution function in order to compare it with the numerical value given by our approximation scheme.

Let us turn now to the numerical simulations. Let $b = 10$, i.e. $U = (-\infty, 10)$. We may choose $N = 10$ since $Y_{T_N} = T_N + N_{T_N} = T_N + N \geq N$. Besides, it is clear that, for all $y \in (-\infty, 10)$, $u^*(y) = 10 - y$. Assumptions 3.2 and 3.3 are clearly satisfied and so is Assumption 3.4 thanks to the following lemma.

**Lemma 5.1.** *For all $\alpha > 0$ and all $k \in \{0, \ldots, N\}$,*

$$P_\mu(Z_k \in U^\alpha) \leq 2\alpha.$$

*Proof.* Since $Z_0 = 0$ a.s., $P_\mu(Z_0 \in U^\alpha) = P_\mu(Z_0 \in [10 - \alpha, 10 + \alpha]) = \mathbf{1}_{\{\alpha \geq 10\}} \leq \frac{1}{10}\alpha \leq 2\alpha$.

Now let $k \in \{1, \ldots, N\}$. Denote by $f_{\gamma(k,1)}$ the density of the distribution $\gamma(k, 1)$, and let its bound be denoted by

$$C_k = \frac{1}{(k-1)!}\left(\frac{k-1}{e}\right)^{k-1}.$$

Since $T_k$ has distribution $\gamma(k, 1)$, $Z_k = k + T_k$ has density $f_{Z_k}(\cdot) = f_{\gamma(k,1)}(\cdot - k)$, which is also bounded by $C_k$. Eventually, we have

$$P_\mu(Z_k \in U^\alpha) = P_\mu(Z_k \in [10 - \alpha, 10 + \alpha]) \leq 2C_k\alpha \leq 2\alpha.$$

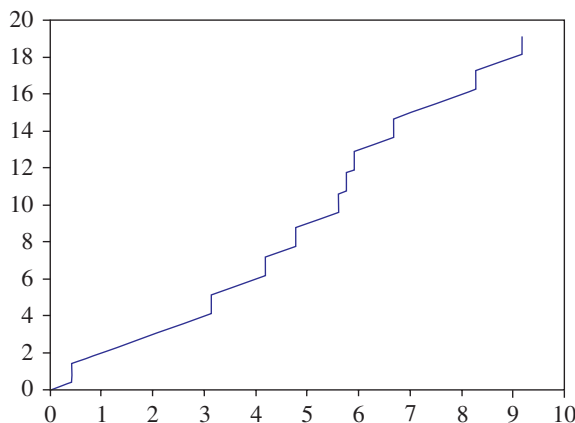Indeed, the sequence $(C_k)_k$ decreases so that, for all $k \in \{1, \ldots, N\}$, $C_k \leq C_1 = 1$. □



FIGURE 1: A trajectory of the process $(Y_t)$ drawn until the 10th jump time.

TABLE 1: Simulation results for the mean exit time.

| Number of points in the quantization grids | $\widehat{p}_{N,1}$ | Relative error to 5.125 (%) |
|---|---|---|
| 20 | 5.050 | 1.46 |
| 50 | 5.096 | 0.56 |
| 100 | 5.095 | 0.58 |
| 200 | 5.118 | 0.13 |
| 300 | 5.128 | 0.06 |
| 500 | 5.123 | 0.03 |

TABLE 2: Simulation results for the second moment.

| Number of points in the quantization grids | $\widehat{p}_{N,2}$ | Relative error to 27.5 (%) |
|---|---|---|
| 20 | 26.66 | 3.05 |
| 50 | 27.20 | 1.11 |
| 100 | 27.21 | 1.05 |
| 200 | 27.43 | 0.25 |
| 300 | 27.54 | 0.13 |
| 500 | 27.49 | 0.03 |

Moreover, Assumption 3.5 is satisfied since the process increases but Assumption 3.6 is not, because $t^*(x) = +\infty$ for all $x \in E$. However, as pointed out in Section 3, this can be solved by considering the process killed at time $\tau$.

*The mean exit time.* Table 1 displays the simulation results for the approximation of the mean exit time. For different numbers of points in the quantization grids, the value of $\widehat{p}_{N,1}$ which approximates the mean exit time is given. A reference value is obtained thanks to the Monte Carlo method ($10^6$ simulations): $E[\tau_{10}]_{\text{Monte Carlo}} = 5.125$.

*The second moment.* We present the results of the approximation of the second moment in Table 2. Our Monte Carlo reference value ($10^6$ simulations) is $E[\tau_{10}^2]_{\text{Monte Carlo}} = 27.5$.

For the first and second moments, the empirical convergence rates are presented in Figure 2. Through a regression model the empirical convergence is estimated as $-1.23$ for the first moment and $-1.39$ for the second moment. Note that they are roughly of the same order as the rate of convergence of the optimal quantizer (see Theorem 4.1), as here the dimension is 1.

*The exit time distribution.* As mentioned earlier, we can obtain the exact value of the survival function of the exit time.

**Proposition 5.1.** *Denote by* fl$(\cdot)$ *the floor function. For all* $s, b \in \mathbb{R}^+$, *we have*

$$P(\tau_b \geq s) = \begin{cases} P(T_{\text{fl}(b-s)+1} > s) & \text{for all } s \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 5.1.** Note that $T_k$ has distribution $\gamma(k, 1)$, so the right-hand side term in the above proposition can be computed easily.
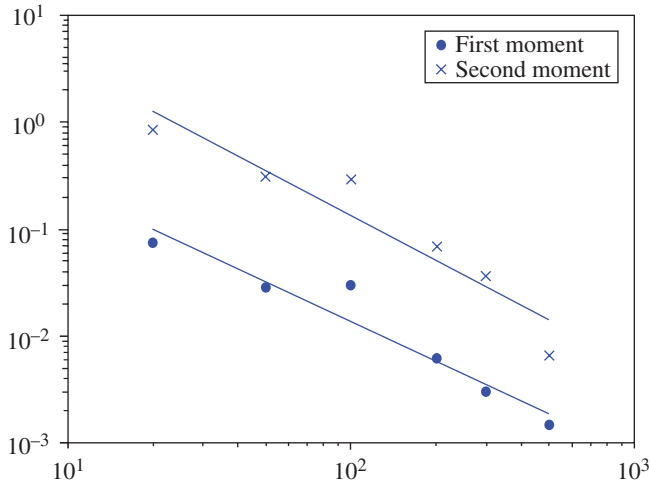
FIGURE 2: Logarithm of the error with respect to the logarithm of the number of points in the quantization grids for the first and second moments of the Poisson process.
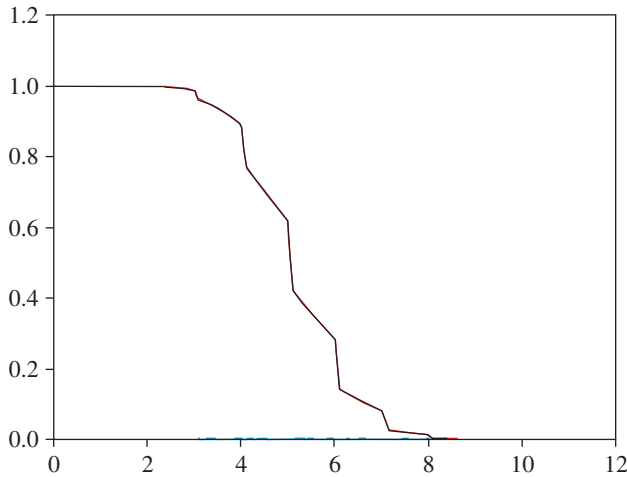


FIGURE 3: Survival function of $\tau_{10}$ and its quantized approximation with 500 points in the quantization grids. The functions appear indistinguishable.

*Proof of Proposition 5.1.* Let $s > 0$. Note that $Y_s \geq s$; thus, $\tau_b < s$ a.s. when $s > b$. Assume now that $s \leq b$. We have

$$P(\tau_b \geq s) = P(Y_s \leq b) = P(N_s \leq b - s) = P(N_s \leq \mathrm{fl}(b - s)) = P(T_{\mathrm{fl}(b-s)+1} \geq s).$$

This completes the proof.

Figure 3 shows both the exact survival function of the exit time and its quantized approximation. Table 3 contains the empirical error between the two functions. For the survival function, the empirical convergence rate is presented in Figure 4. Through a regression model the convergence rate is estimated as $-1.05$. Note that it is roughly of the same order as the rate of convergence of the optimal quantizer (see Theorem 4.1), as here the dimension is 1.

TABLE 3: Simulation results for the distribution.

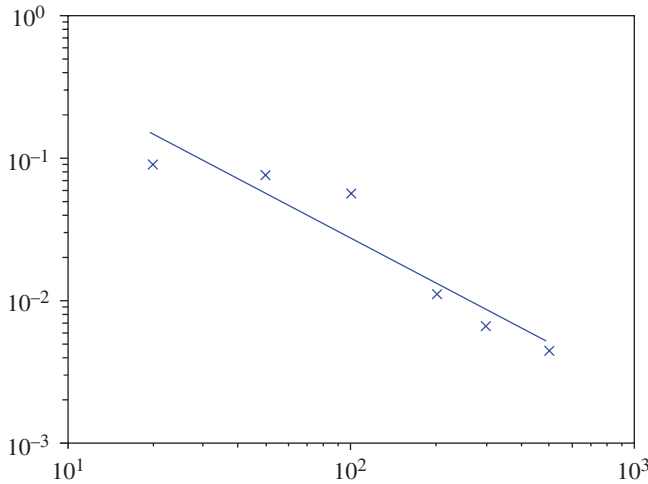| Number of points in the quantization grids | $\max_s |p_N(s) - \widehat{p}_N(s)|$ |
|:---:|:---:|
| 20 | 0.090 |
| 50 | 0.077 |
| 100 | 0.057 |
| 200 | 0.011 |
| 300 | 0.007 |
| 500 | 0.005 |



FIGURE 4: Logarithm of the error with respect to the logarithm of the number of points in the quantization grids for the survival function of the Poisson process.

**Remark 5.2.** We already insisted on the fact that our approach is flexible with respect to $U$. In this example we could very quickly obtain the mean exit time or the exit time distribution for a different set $U' = (-\infty, b']$ for any $0 < b' \leq b = 10$. Indeed, $P(\tau_{b'} > T_{10}) = 0$, so it is not necessary to compute new quantization grids.

**Remark 5.3.** Recall that the value of $T_k$ may be obtained from $Z_k$ since $T_k = Z_k - k$, so it is sufficient to quantize the process $(Z_k)_{k \leq N}$ instead of $(Z_k, T_k)_{k \leq N}$. The reduction of the dimension of the process that has to be quantized results in an improvement of the convergence rate and it appears that the approximations presented in the previous tables indeed converge very quickly.

*Convergence rate for the exit time distribution.* We note from the proof of Proposition 4.2 that a bound for the rate of convergence for the exit time distribution can be obtained as soon as, for all $k \in \{0, \ldots, N-1\}$, the survival function of $(T_k + u^*(Z_k)) \wedge T_{k+1}$ denoted $\varphi_k$ is piecewise Lipschitz continuous. Although it is difficult to state general assumptions under which this is true, the following proposition proves that the condition is fulfilled in our example.

**Proposition 5.2.** *For all $k \in \{0, \ldots, N-1\}$, the survival function $\varphi_k$ of $(T_k + u^*(Z_k)) \wedge T_{k+1}$ is Lipschitz continuous on $(-\infty; b - k)$ and on $(b - k; +\infty)$ with Lipschitz constant $[\varphi_k] \leq 1$.*

*Proof.* Let $k = 0$ and $s > 0$. We have

$$
\begin{aligned}
\varphi_0(s) &= \mathrm{P}_\mu((T_0 + u^*(Z_0)) \wedge T_1 > s) \\
&= \mathrm{P}_\mu(b \wedge T_1 > s) \\
&= \mathbf{1}_{\{b > s\}} \, \mathrm{P}_\mu(T_1 > s) \\
&= \mathbf{1}_{\{b > s\}} \, \mathrm{e}^{-s} \quad \text{(since } T_1 \text{ has exponential distribution with parameter 1).}
\end{aligned}
$$

Therefore, the function $\varphi_0$ is equal to 0 on $[b; +\infty)$ and is Lipschitz continuous with Lipschitz constant 1 on $(0; b)$.

Let $k \geq 1$ and $s > 0$, and recall that the random variables $(S_j)_{j \geq 0}$ are independent and have exponential distributions with parameter 1 so that, in particular, $T_k$ and $S_{k+1}$ are independent and $T_k$ has distribution $\gamma(k, 1)$. Moreover, recall that $Z_k = k + T_k$ and that $u^*(x) = b - x$. Then

$$
\begin{aligned}
\varphi_k(s) &= \mathrm{P}_\mu((T_k + u^*(Z_k)) \wedge T_{k+1} > s) \\
&= \int_{(\mathbb{R}^+)^2} \mathbf{1}_{\{(t + (b-k-t)) \wedge u > s\}} \, f_{\gamma(k,1)}(t) f_{\gamma(k+1,1)}(u) \, \mathrm{d}t \, \mathrm{d}u,
\end{aligned}
$$

where $f_{\gamma(j,1)}$ denotes the density function of the distribution $\gamma(j, 1)$ for $j \in \{k, k+1\}$.

Let $s' > s > 0$. We have

$$
\begin{aligned}
|\varphi_k(s') - \varphi_k(s)| &\leq \int_{(\mathbb{R}^+)^2} |\mathbf{1}_{\{(b-k) \wedge u > s'\}} - \mathbf{1}_{\{(b-k) \wedge u > s\}}| \, f_{\gamma(k,1)}(t) f_{\gamma(k+1,1)}(u) \, \mathrm{d}t \, \mathrm{d}u \\
&\leq \int_{(\mathbb{R}^+)^2} \mathbf{1}_{\{(b-k) \wedge u \in (s; s']\}} \, f_{\gamma(k,1)}(t) f_{\gamma(k+1,1)}(u) \, \mathrm{d}t \, \mathrm{d}u \\
&\leq \int_{(\mathbb{R}^+)^2} (\mathbf{1}_{\{b-k \in (s; s']\}} + \mathbf{1}_{\{u \in (s; s']\}}) f_{\gamma(k,1)}(t) f_{\gamma(k+1,1)}(u) \, \mathrm{d}t \, \mathrm{d}u \\
&\leq \mathbf{1}_{\{b-k \in [s; s']\}} + C_{f_{\gamma(k+1,1)}} |s' - s| \\
&\leq \mathbf{1}_{\{b-k \in [s; s']\}} + |s' - s| \quad \left(\text{since } C_{f_{\gamma(k+1,1)}} = \frac{1}{(k)!} \left(\frac{k}{\mathrm{e}}\right)^k \leq 1\right).
\end{aligned}
$$

If $s$ and $s'$ both belong to $(0; b - k)$ or if they both belong to $(b - k; +\infty)$, we have $|\varphi_k(s') - \varphi_k(s)| \leq |s' - s|$. The completes the proof.

Consequently, in this example, we are now able to state a bound for the rate of convergence of the exit time distribution approximation scheme. The following proposition is therefore an improvement over Proposition 4.2 and Theorem 4.2.

**Proposition 5.3.** *For all $k \in \{0, \dots, N - 1\}$, let $s > 0$ and assume that the quantization error is small enough to ensure that*

$$
\left(\frac{p}{2}\right)^{1/(p+1)} (\|T_k - \widehat{T}_k\|_p + \|Z_k - \widehat{Z}_k\|_p + \|T_{k+1} - \widehat{T}_{k+1}\|_p)^{p/(p+1)} < |b - k - s|.
$$

*Then we have*

$$
\begin{aligned}
|r_k(s) - \widehat{r}_k(s)| &\leq 2\left(\frac{p}{2}\right)^{1/(p+1)} \left(\frac{1}{p} + 1\right) \\
&\quad \times (\|T_k - \widehat{T}_k\|_p + \|Z_k - \widehat{Z}_k\|_p + \|T_{k+1} - \widehat{T}_{k+1}\|_p)^{p/(p+1)} \\
&\quad + |q_k - \widehat{q}_k| + |q_{k+1} - \widehat{q}_{k+1}|.
\end{aligned}
$$

*Moreover, for all $k \in \{0, \ldots, N\}$, if the quantization error is such that, for all $l \leq k$,*

$$2\left(\frac{p}{2}\right)^{1/(p+1)}\left(\frac{1}{p}+1\right)\|Z_l - \widehat{Z}_l\|_p^{p/(p+1)} \leq \frac{1}{2}\tilde{q},$$

*then we have*

$$|p_k(s) - \widehat{p}_k(s)| \leq \frac{2}{\tilde{q}}(|q_{k-1} - \widehat{q}_{k-1}| + |p_{k-1}(s) - \widehat{p}_{k-1}(s)| + |r_{k-1}(s) - \widehat{r}_{k-1}(s)|)$$
$$+ \frac{4}{\tilde{q}^2}|q_k - \widehat{q}_k|.$$

*Proof.* The proof follows directly from the proofs of Proposition 4.2 and Theorem 4.2. Simply note that the $A$ term may be bounded thanks to the piecewise Lipschitz continuity of the functions $\varphi_k$ on $(-\infty; b - k)$ and on $(b - k; +\infty)$. Let $s > 0$, $s \neq b - k$, and let $\alpha > 0$ such that $b - k \notin [s - \alpha; s + \alpha]$, i.e. $\alpha < |b - k - s|$. Then

$$A \leq |\varphi_k(s + \alpha) - \varphi_k(s - \alpha)| + \frac{\|\eta\|_p^p}{\alpha^p} \quad \text{(from the proof of Proposition 4.2)}$$
$$\leq 2[\varphi_k]\alpha + \frac{\|\eta\|_p^p}{\alpha^p},$$

which reaches a minimum when $\alpha = (p\|\eta\|_p^p/2[\varphi_k])^{1/(p+1)}$. Note that $[\varphi_k] = 1$ and $[u^*] = 1$. $\qquad \square$

**Remark 5.4.** We can calculate the exact value of $\tilde{q}$ that is the first nonnegative value of the sequence $(P_\mu(Z_k \notin U))_k$. We have $\tilde{q} = P_\mu(Z_1 \notin (-\infty; 10)) = P_\mu(T_1 \geq 9) = e^{-9}$ because $T_1$ has an exponential distribution with parameter 1.

### 5.2. A corrosion model example

Let us consider the structure of aluminium corroded successively in three different environments. Corrosion is prevented by some protection until a random time $\gamma$ when corrosion starts. Then, in each environment $i \in \{1; 2; 3\}$, the loss of thickness satisfies

$$d_i(t) = \rho_i(t - \gamma + \eta_i(e^{-(t-\gamma)/\eta_i} - 1))\,\mathbf{1}_{\{t \geq \gamma\}},$$

where $\rho_i$ is the corrosion rate ($\rho_i$ has a uniform distribution on an interval that depends on the environment $i$) and $\eta_i$ is a constant transition time. The structure goes from environment 1 to environment 2, then from 2 to 3, from 3 to 1, and so on. It remains in environment $i$ for a time $T_i$, which has an exponential distribution with parameter $\lambda_i$. When the loss of thickness reaches 0.2 mm, the piece is said to be unusable; this will be the exit criterion. Table 4 gives the values of the different parameters.

The loss of thickness will be represented by a PDMP whose modes are the different environments. Let $M = \{(i, j): i \in \{1, 2, 3\}, j \in \{0, 1\}\}$. For $m = (i, j) \in M$, $i$ represents the environment and $j$ is worth 1 if the protection $\gamma$ is still active and 0 otherwise. For each $m \in M$, let $E_m = \mathbb{R}^4$ and, for $\xi \in E_m$, $\xi$ represents the family $(d, s, \rho, \gamma)$, where $d$ is the corroded thickness and $s$ is the time since the last jump. The set $U_m$ will therefore be, for all $m \in M$, $U_m = (-\infty; 0.2] \times \mathbb{R}^3$. This set is convex, so Assumption 3.3 is satisfied. Finally, the

TABLE 4: Numerical values of the parameters of the corrosion model.

| | Environment | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| $\lambda_i$ (h$^{-1}$) | $(17\,520)^{-1}$ | $(131\,400)^{-1}$ | $(8760)^{-1}$ |
| $\eta_i$ (h) | 30 000 | 200 000 | 40 000 |
| $\rho_i$ (mm/h) | $[10^{-6}, 10^{-5}]$ | $[10^{-7}, 10^{-6}]$ | $[10^{-6}, 10^{-5}]$ |
| $\gamma$ (h) | Weibull distribution with $\alpha = 2.5$ and $\beta = 11\,800$ | | |

flow in mode $m = (i, j)$ is

$$\Phi_{(i,0)}\left(\begin{pmatrix} d \\ s \\ \rho \\ 0 \end{pmatrix}, t\right) = \begin{pmatrix} d + d_m(t+s) - d_m(s) \\ t+s \\ \rho \\ 0 \end{pmatrix},$$

$$\Phi_{(i,1)}\left(\begin{pmatrix} 0 \\ s \\ \rho \\ \gamma \end{pmatrix}, t\right) = \begin{pmatrix} 0 \\ t+s \\ \rho \\ (\gamma - t)\,\mathbf{1}_{\{\gamma \geq t\}} \end{pmatrix}.$$

The parameters $d$ and $\gamma$ evolve continuously between the jumps, but $\rho$ is chosen independently after each jump and is constant along the flow.

Let us consider the approximation of the distribution and of the mean exit time. Consider the first moment. We note that $E_\mu[\tau] = E_\mu[\gamma] + E_\mu[\tau']$, where $\gamma$ has Weibull distribution and $\tau'$ represents the exit time in the case of a process without initial protection against corrosion (i.e. $\gamma = 0$). Therefore, it is sufficient to check whether $\tau'$ satisfies the required assumptions. Hence, let $\gamma = 0$ and note that $u^*$ is then bounded since $\rho \geq 10^{-7}$ and $\eta \leq 200\,000$, so $d_m(t) \geq 10^{-7}(t - 200\,000)$ and eventually $u^* \leq 0.2 \times 10^7 + 200\,000 = 2.2 \times 10^6$ h. Denote this bound by $C_{u^*}$. Consider the distribution. Assumption 3.2(b) (the boundedness condition on $u^*$) is not required according to Remark 4.3. Moreover, from the proofs of Propositions 4.2 and 4.4, it follows that Assumption 3.2(a) (the Lipschitz continuity condition on $u^*$) becomes useless in this example thanks to Lemma 5.3. Assumption 3.4 follows from Lemma 5.2 below. Eventually, Assumption 3.5 is satisfied, but Assumption 3.6 is not. However, considering the process killed at time $\tau$ solves this issue.

**Lemma 5.2.** *For all $\alpha > 0$ and all $k \in \{0, \ldots, N\}$,*

$$P_\mu(Z_k \in U^\alpha) \leq 5\alpha.$$

*Proof.* For notational convenience, let $M_k$, $D_k$, $R_k$, and $G_k$ denote the values of $m$, $d$, $\rho$, and $\gamma$ after the $k$th jump, so $Z_k = (M_k, D_k, R_k, G_k)$. Note now that

$$P_\mu(Z_k \in U^\alpha) = P_\mu(|D_k - 0.2| \leq \alpha).$$

We therefore study more precisely the law of $D_k$. Let $K = \inf\{k \geq 0 \text{ such that } G_k = 0\}$; $K$ is the jump that occurs at the end of the protection period against corrosion. Define $F(s) = s + \eta(e^{-s/\eta} - 1)$. Then we have

$$D_k = \begin{cases} 0 & \text{for } k \leq K, \\ D_{k-1} + R_k F(S_k) & \text{for } k > K. \end{cases}$$

Let us now prove that, for all $k$, the random variable $R_k F(S_k)$ has a bounded density. Recall that $R_k$ has a uniform distribution on $[a_k; b_k] \subset [10^{-7}; 10^{-5}]$ and that $S_k$ has an exponential distribution with parameter $\lambda_k$. Now let $h$ be a real, bounded, measurable function. Then

$$\mathrm{E}_\mu[h(R_k F(S_k))] = \int_0^{+\infty} \int_{a_k}^{b_k} h(\rho F(s)) \frac{1}{b_k - a_k} \lambda_k e^{-\lambda_k s} \, \mathrm{d}\rho \, \mathrm{d}s.$$

Introduce the transformation

$$u = \rho, \qquad v = \rho F(s),$$

whose Jacobian is worth $(1/u)(F^{-1})'(v/u)$, so

$$\mathrm{E}_\mu[h(R_k F(S_k))] = \int_0^{+\infty} h(v) \left( \int_{a_k}^{b_k} \frac{\lambda_k e^{-\lambda_k F^{-1}(v/u)} (F^{-1})'(v/u)}{(b_k - a_k) u} \, \mathrm{d}u \right) \mathrm{d}v.$$

Hence, we obtain the density of the random variable $R_k F(S_k)$ and integration by parts yields

$$\int_{a_k}^{b_k} \frac{\lambda_k e^{-\lambda_k F^{-1}(v/u)} (F^{-1})'(v/u)}{(b_k - a_k) u} \, \mathrm{d}u = \frac{1}{b_k - a_k} \int_{a_k}^{b_k} u \frac{\lambda_k e^{-\lambda_k F^{-1}(v/u)} (F^{-1})'(v/u)}{u^2} \, \mathrm{d}u$$

$$= \frac{1}{b_k - a_k} \left( [u e^{-\lambda_k F^{-1}(v/u)}]_{a_k}^{b_k} - \int_{a_k}^{b_k} e^{-\lambda_k F^{-1}(v/u)} \, \mathrm{d}u \right).$$

Finally, the density of the random variable $R_k F(S_k)$ is bounded by

$$\left| \int_{a_k}^{b_k} \frac{\lambda_k e^{-\lambda_k F^{-1}(v/u)} (F^{-1})'(v/u)}{(b_k - a_k) u} \, \mathrm{d}u \right| \leq \frac{a_k + b_k}{b_k - a_k} + 1 \leq \frac{2 b_k}{b_k - a_k} \leq 2.$$

Let $j \in \mathbb{N}$. We now study the distribution of the random variables $(D_k)_{k \in \mathbb{N}}$ conditionally on the event $\{K = j\}$. An induction argument shows that, conditionally on the event $\{K = j\}$, the random variable $D_k$ has distribution $\delta_0$ for $k \leq j$ and has a density $\psi_k$ bounded by 2 for $k > j$. Indeed, in the second case, the density of $D_k$ may be obtained by convolution since $D_{k-1}$ and $R_k F(S_k)$ are independent random variables. Therefore, for $k \leq j$,

$$\mathrm{P}_\mu(|D_k - 0.2| \leq \alpha \mid K = j) = \mathbf{1}_{\{\alpha \geq 0.2\}} \leq 5\alpha$$

since $D_k = 0$ for $k \leq j$ and, for $k > j$,

$$\mathrm{P}_\mu(|D_k - 0.2| \leq \alpha \mid K = j) = \int_{0.2-\alpha}^{0.2+\alpha} \psi_k(v) \, \mathrm{d}v \leq 4\alpha$$

since $\psi_k \leq 2$. Eventually,

$$\mathrm{P}_\mu(Z_k \in U^\alpha) = \mathrm{P}_\mu(|D_k - 0.2| \leq \alpha) = \sum_{j \in \mathbb{N}} \mathrm{P}_\mu(|D_k - 0.2| \leq \alpha \mid K = j) \, \mathrm{P}_\mu(K = j) \leq 5\alpha.$$

This completes the proof.

**Lemma 5.3.** *For all $k \in \mathbb{N}$, let*

$$\eta_k = |((T_k + u^*(Z_k)) \wedge T_{k+1}) - ((\widehat{T}_k + u^*(\widehat{Z}_k)) \wedge \widehat{T}_{k+1})|.$$

*We have, for all $\alpha > 0$,*

$$\|\eta_k\|_p \leq \|T_k - \widehat{T}_k\|_p + 2\|T_{k+1} - \widehat{T}_{k+1}\|_p + \left([u^*]_{\alpha/2} + \frac{4C_{u^*}}{\alpha}\right)\|Z_k - \widehat{Z}_k\|_p + 10C_{u^*}\alpha^{1/p},$$

*where $[u^*]_\alpha = (1 + C_{u^*} + 4 \times 10^5)/(10^{-7}(1 - e^{-\alpha/2}))$.*

*Proof.* Let $\alpha > 0$. Let $\tilde{U}_\alpha = [0, 0.2 - \alpha] \times \{0\} \times [10^{-7}; 10^{-5}] \times \{0\}$. We will prove that the function $u^*(d, 0, \rho, 0)$ is Lipschitz continuous on this set. The function $u^*(d, 0, \rho, 0)$ satisfies the following equivalent equations:

$$d + d_m(u^*) = 0.2 \quad \Longleftrightarrow \quad d + \rho(u^* + \eta(e^{-u^*/\eta} - 1)) = 0.2.$$

The implicit equation satisfied by $u^*$ yields, on the set $\tilde{U}_\alpha$, $u^* \geq \alpha/\rho_{\max} = 10^5\alpha$. This lower bound will be crucial to prove the Lipschitz continuity. Let $d, d' \leq 0.2 - \alpha$, and define $u = u^*(d, 0, \rho, 0)$ and $u' = u^*(d', 0, \rho, 0)$. Note that $d + d_m(u) = d' + d_m(u')$ because they are both equal to 0.2. Consequently, $|d_m(u) - d_m(u')| = |d' - d|$ and, noting that $\eta \leq 2 \times 10^5$, we have

$$\begin{aligned}
|d - d'| &= \rho|u - u' + \eta(e^{-u/\eta} - e^{-u'/\eta})| \\
&\geq \rho(1 - e^{-u \wedge u'/\eta})|u - u'| \\
&\geq 10^{-7}(1 - e^{-\alpha/2})|u - u'|,
\end{aligned}$$

which proves the Lipschitz continuity of $u^*$ with respect to $d$ on $\tilde{U}_\alpha$.

Similarly, let $\rho, \rho' \in [10^{-7}; 10^{-5}]$, and define $u = u^*(d, 0, \rho, 0)$ and $u' = u^*(d, 0, \rho', 0)$. Note that $d + \rho(u + \eta(e^{-u/\eta} - 1)) = d + \rho'(u' + \eta(e^{-u'/\eta} - 1))$ because they are both equal to 0.2. Subtracting $d + \rho(u' + \eta(e^{-u'/\eta} - 1))$ from both terms yields

$$\rho|u - u' + \eta(e^{-u/\eta} - e^{-u'/\eta})| = |\rho - \rho'||u' + \eta(e^{-u'/\eta} - 1)|.$$

A lower bound for the left-hand side term has already been computed, while the right hand-side is easily bounded by $(C_{u^*} + 4 \times 10^5)|\rho - \rho'|$, since $\eta \leq 2 \times 10^5$, so we have

$$(C_{u^*} + 4 \times 10^5)|\rho - \rho'| \geq 10^{-7}(1 - e^{-\alpha/2})|u - u'|,$$

which proves the Lipschitz continuity of $u^*$ with respect to $\rho$ on $\tilde{U}_\alpha$. Eventually, for all $\alpha > 0$, the function $u^*$ is Lipschitz continuous on $\tilde{U}_\alpha$ with Lipschitz constant $[u^*]_\alpha = (1 + C_{u^*} + 4 \times 10^5)/10^{-7}(1 - e^{-\alpha/2})$.

Let $k \in \mathbb{N}$. We now intend to bound $\|\eta_k\|_p$. Define, as in the proof of Lemma 5.2, the random variable $K = \inf\{k \geq 0 \text{ such that } G_k = 0\}$; $K$ is the jump that occurs at the end of the protection period against corrosion.

First, note that, on the event $\{k \leq K\}$ (i.e. when protection against corrosion is still active), we have $Z_k \in E_{(i,1)}$ for some $i \in \{1, 2, 3\}$ and, since the projection defining $\widehat{Z}_k$ from $Z_k$ ensures that they are in the same mode, we also have $\widehat{Z}_k \in E_{(i,1)}$. Moreover, $u^*(x) = +\infty$ for all $x \in E_{(i,1)}$, so

$$\|\eta_k \mathbf{1}_{\{k \leq K\}}\|_p = \|(T_{k+1} - \widehat{T}_{k+1})\mathbf{1}_{\{k \leq K\}}\|_p \leq \|T_{k+1} - \widehat{T}_{k+1}\|_p.$$

Furthermore, if $Z_k = \Delta$, where $\Delta$ denotes the cemetery state, then $\widehat{Z}_k = \text{proj}_{\Gamma_k}(Z_k) = \Delta$ too and we have $\eta_k = 0$, so

$$\begin{aligned}
\|\eta_k \mathbf{1}_{\{k > K\}}\|_p &\leq \|\eta_k \mathbf{1}_{\{k > K\}} \mathbf{1}_{\{Z_k \neq \Delta\}}\|_p \\
&\leq \|T_k - \widehat{T}_k\|_p + \|T_{k+1} - \widehat{T}_{k+1}\|_p + \|(u^*(Z_k) - u^*(\widehat{Z}_k))\mathbf{1}_{\{k > K\}}\mathbf{1}_{\{Z_k \neq \Delta\}}\|_p.
\end{aligned}$$

Eventually, we intend to bound the last term of the previous sum; therefore, we consider the event $\{k > K\} \cap \{Z_k \neq \Delta\}$. On the one hand, the random variables $Z_k$ and $\widehat{Z}_k$ both belong to $E_{(i,0)}$ for some $i \in \{1, 2, 3\}$. On the other hand, although $U_m = (-\infty; 0.2] \times \mathbb{R}^3$ for all $m \in M$, we actually have $Z_k \in [0; 0.2] \times \{0\} \times [10^{-7}; 10^{-5}] \times \mathbb{R}^+$ a.s. and, according to Remark 4.1, $\widehat{Z}_k \in [0; 0.2] \times \{0\} \times [10^{-7}; 10^{-5}] \times \mathbb{R}^+$ a.s. too. Combining the two previous remarks, we have $Z_k \in \tilde{U}$ and $\widehat{Z}_k \in \tilde{U}$, where $\tilde{U} = [0; 0.2] \times \{0\} \times [10^{-7}; 10^{-5}] \times \{0\}$. Finally, let $\alpha > 0$ and note that $\tilde{U} \subset \tilde{U}_\alpha \cup U^\alpha$. We have

$$\|(u^*(Z_k) - u^*(\widehat{Z}_k)) \mathbf{1}_{\{k \geq K\}} \mathbf{1}_{\{Z_k \neq \Delta\}}\|_p \leq A + B,$$

where

$$A = \|(u^*(Z_k) - u^*(\widehat{Z}_k)) \mathbf{1}_{\{Z_k \in \tilde{U}_\alpha\}} \mathbf{1}_{\{k \geq K\}}\|_p, \quad B = \|(u^*(Z_k) - u^*(\widehat{Z}_k)) \mathbf{1}_{\{Z_k \in U^\alpha\}} \mathbf{1}_{\{k \geq K\}}\|_p.$$

The term $B$ is easily bounded thanks to Lemma 5.2: $B \leq 2C_{u^*} \, \mathrm{P}_\mu(Z_k \in U^\alpha)^{1/p} \leq 10 C_{u^*} \alpha^{1/p}$. We now turn to the term $A$ and use the Lipschitz continuity of $u^*$ on $\tilde{U}_\beta$ for any $\beta > 0$. We have

$$\begin{aligned}
A &\leq \|(u^*(Z_k) - u^*(\widehat{Z}_k)) \mathbf{1}_{\{Z_k \in \tilde{U}_\alpha\}} \mathbf{1}_{\{\widehat{Z}_k \in \tilde{U}_{\alpha/2}\}} \mathbf{1}_{\{k \geq K\}}\|_p \\
&\quad + \|(u^*(Z_k) - u^*(\widehat{Z}_k)) \mathbf{1}_{\{Z_k \in \tilde{U}_\alpha\}} \mathbf{1}_{\{\widehat{Z}_k \notin \tilde{U}_{\alpha/2}\}} \mathbf{1}_{\{k \geq K\}}\|_p \\
&\leq [u^*]_{\alpha/2} \|Z_k - \widehat{Z}_k\|_p + 2C_{u^*} \|\mathbf{1}_{\{Z_k \in \tilde{U}_\alpha\}} \mathbf{1}_{\{\widehat{Z}_k \notin \tilde{U}_{\alpha/2}\}}\|_p.
\end{aligned}$$

Note now that $\mathbf{1}_{\{Z_k \in \tilde{U}_\alpha\}} \mathbf{1}_{\{\widehat{Z}_k \notin \tilde{U}_{\alpha/2}\}} \leq \mathbf{1}_{\{|Z_k - \widehat{Z}_k| \geq \alpha/2\}}$, so, finally,

$$A \leq [u^*]_{\alpha/2} \|Z_k - \widehat{Z}_k\|_p + 2C_{u^*} \left( \mathrm{P}_\mu \left( |Z_k - \widehat{Z}_k| \geq \frac{\alpha}{2} \right) \right)^{1/p}$$

$$\leq [u^*]_{\alpha/2} \|Z_k - \widehat{Z}_k\|_p + 4C_{u^*} \frac{\|Z_k - \widehat{Z}_k\|_p}{\alpha},$$

completing the proof.

*The mean exit time.* Simulation results for the approximation of the mean exit time are given in Table 5. In order to have a value of reference, a Monte Carlo method ($10^6$ simulations) yields the value $\mathrm{E}[\tau]_{\text{Monte Carlo}} = 526 \times 10^3$ h. For the first moment, the empirical convergence rate is presented in Figure 5. Through a regression model the empirical convergence rate is estimated as $-0.38$. Note that it is roughly of the same order as the rate of convergence of the optimal quantizer (see Theorem 4.1), as here the dimension is 4.

TABLE 5: Simulation results for the mean exit time.

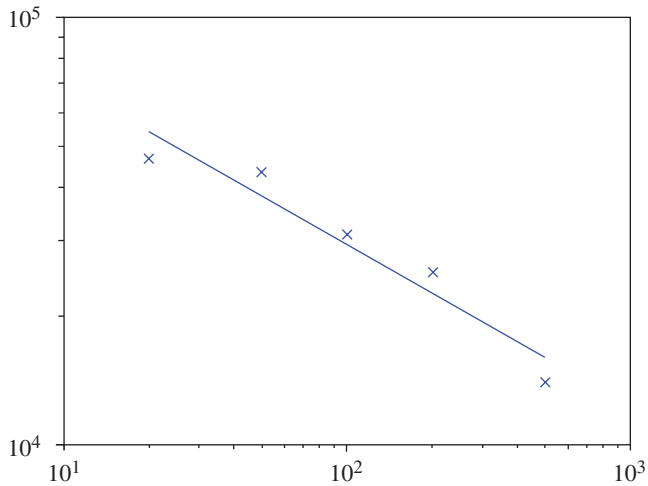| Number of points in the quantization grids | $\widehat{p}_{N,1}$ ($\times 10^3$ h) | Relative error to $526 \times 10^3$ h (%) |
|---|---|---|
| 20 | 572 | 8.7 |
| 50 | 569 | 8.2 |
| 100 | 557 | 5.9 |
| 200 | 551 | 4.8 |
| 500 | 539 | 2.5 |

FIGURE 5: Logarithm of the error with respect to the logarithm of the number of points in the quantization grids for the first moment of the corrosion process.
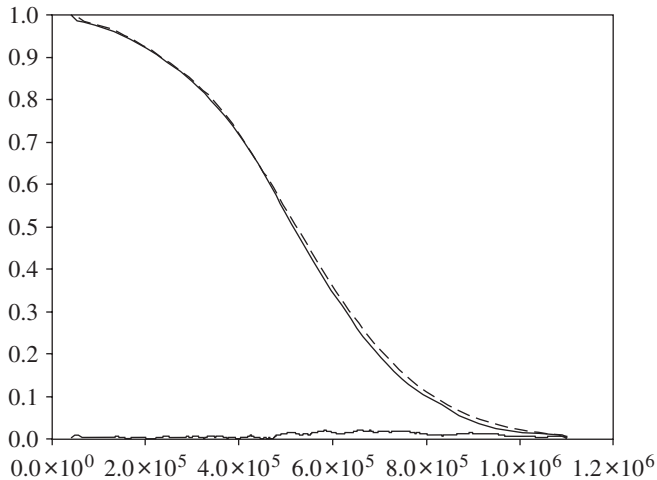


FIGURE 6: Survival function of $\tau$ obtained using Monte Carlo simulations (*dashed line*) and the quantized approximation (*solid line*), and the error with 500 points in the quantization grids.

*The exit time distribution.* Considering the approximation scheme for the exit time distribution, we note that the quantized value $\widehat{p}_N(s)$ is not necessarily smaller than 1. Therefore, it appears natural to replace $\widehat{p}_N(s)$ by $\widehat{p}_N(s) \wedge 1$. This does not change the convergence theorem and can only improve the approximation error. It is equally possible, and this is done in the results below, to replace $\widehat{p}_N(s)$ by $\widehat{p}_N(s)/\widehat{p}_N(0)$ since $\widehat{p}_N(0)$ goes to 1.

Figure 6 presents the survival function of $\tau$ obtained using Monte Carlo simulations (dashed line) and our approximation scheme (solid line), and the error. Table 6 contains the empirical error for different numbers of points in the quantization grids. For the survival function, the empirical convergence rate is presented in Figure 7. Through a regression model the empirical

TABLE 6: Simulation results for the distribution.

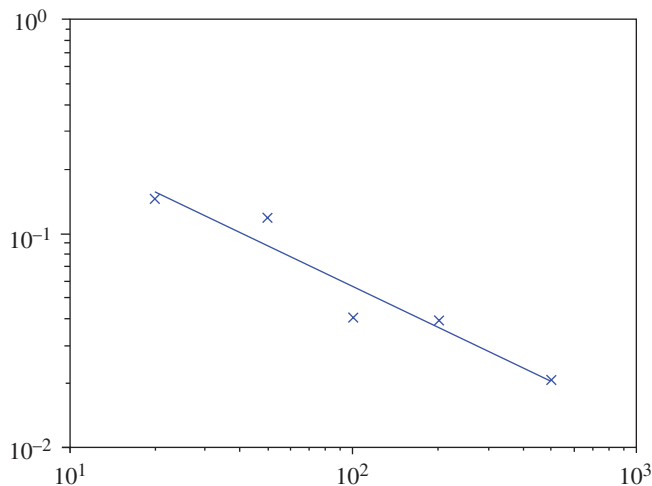| Number of points in the quantization grids | $\max_s \lvert p_N(s) - \widehat{p}_N(s) \rvert$ |
|:---:|:---:|
| 20 | 0.145 |
| 50 | 0.119 |
| 100 | 0.040 |
| 200 | 0.039 |
| 500 | 0.020 |



FIGURE 7: Logarithm of the error with respect to the logarithm of the number of points in the quantization grids for the survival of the corrosion process.

convergence rate is estimated as $-0.63$. Note that it is roughly of the same order as the rate of convergence of the optimal quantizer (see Theorem 4.1), as here the dimension is 4.

The convergence of the approximation scheme in the corrosion model appears to be slightly slower than in the previous example. This is due to the higher dimension of the process that has to be quantized, which is 4 in the case of the corrosion model and 1 in the case of the Poisson process.

**Remark 5.5.** Using Monte Carlo simulations, we can approximate the value of $\tilde{q}$. We have $\tilde{q} \simeq 0.0187$ for $10^7$ histories.

## 6. Advantages and practical interest of our approach

Let us describe the practical interest of our approach.

- The quantization grids only have to be computed once and can be used for several purposes. Moreover, once they are obtained, the procedures leading to $\widehat{p}_N(s)$ and to $\widehat{p}_{N,j}$ can be achieved very simply since we only have to compute finite sums.

- Concerning the distribution, since $\widehat{p}_N(s)$ can be computed almost instantly for any value of $s$, the whole survival function can be obtained very quickly. Similarly, concerning

the moments, $\widehat{p}_{N,j}$ can be computed very quickly for any $j$, so any moment is almost instantly available.

- Furthermore, in both cases, one may decide to change the set $U$ and consider the exit time $\tau'$ from a new set $U'$. This will yield new sequences $(\widehat{q}_k)_k$, $(\widehat{r}_{k,j})_k$, and $(\widehat{p}_{k,j})_k$ in the case of the $j$th moment approximation or new sequences $(\widehat{q}_k)_k$, $(\widehat{r}_k(s))_k$, and $(\widehat{p}_k(s))_k$ if we are interested in the distribution. These new sequences are obtained quickly and easily since the quantized process remains the same and we only have to compute finite sums. Of course, the set $U'$ must be such that Assumptions 3.2–3.5 remain true and such that $\mathrm{P}_\mu(T_N < \tau')$ remains small without changing the computation horizon $N$. This last condition is fulfilled if, for instance, $U' \subset U$. This flexibility is an important advantage of our method over, for instance, a Monte Carlo method.

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