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SPECTRAL ANALYSIS OF MARKOV KERNELS AND APPLICATION TO THE CONVERGENCE RATE OF DISCRETE RANDOM WALKS

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Abstract

Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain on a measurable space $\mathbb{X}$ with transition kernel P, and let $V: \mathbb{X}\to[1, +\infty)$. The Markov kernel P is here considered as a linear bounded operator on the weighted-supremum space $\mathcal{B}_V$ associated with V. Then the combination of quasicompactness arguments with precise analysis of eigenelements of P allows us to estimate the geometric rate of convergence $\rho_V(P)$ of $\{X_n\}_{n\in\mathbb{N}}$ to its invariant probability measure in operator norm on $\mathcal{B}_V$. A general procedure to compute $\rho_V(P)$ for discrete Markov random walks with identically distributed bounded increments is specified.

Keywords: V-geometric ergodicity; quasicompactness; drift condition; birth-and-death Markov chain

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1. Introduction

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space with a σ-field $\mathcal{X}$, and let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space $\mathbb{X}$ and transition kernels $\{P(x, \cdot): x \in \mathbb{X}\}$. Let $V: \mathbb{X}\to[1, +\infty)$. Assume that $\{X_n\}_{n\geq 0}$ has an invariant probability measure π such that $\pi(V) := \int_{\mathbb{X}} V(x)\pi(\mathrm{d}x) < \infty$. This paper is based on the connection between spectral properties of the Markov kernel P and the so-called V-geometric ergodicity [12] which is the following convergence property for some constants $c_\rho > 0$ and $\rho \in (0, 1)$:

$$\sup_{|f|\leq V}\sup_{x\in\mathbb{X}} \frac{|\mathbb{E}[f(X_n) \mid X_0 = x] - \pi(f)|}{V(x)} \leq c_\rho \rho^n. \tag{1.1}$$

Let us introduce the weighted-supremum Banach space $(\mathcal{B}_V, \|\cdot\|_V)$ composed of measurable functions $f: \mathbb{X}\to\mathbb{C}$ such that

$$\|f\|_V := \sup_{x\in\mathbb{X}} \frac{|f(x)|}{V(x)} < \infty.$$

Then (1.1) reads as $\|P^n f - \pi(f)\mathbf{1}_{\mathbb{X}}\|_V \leq c_\rho\rho^n$ for any $f \in \mathcal{B}_V$ such that $\|f\|_V \leq 1$, and there is great interest in obtaining upper bounds for *the convergence rate* $\rho_V(P)$ defined by

$$\rho_V(P) := \inf\Big\{\rho \in (0, 1), \sup_{\|f\|_V\leq 1} \|P^n f - \pi(f)\mathbf{1}_{\mathbb{X}}\|_V = O(\rho^n)\Big\}. \tag{1.2}$$

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For irreducible and aperiodic discrete Markov chains, criteria for the V-geometric ergodicity are well known from the literature using either the equivalence between geometric ergodicity and V-geometric ergodicity of $\mathbb{N}$-valued Markov chains [5, Proposition 2.4], or the strong drift condition. For instance, when $\mathbb{X} := \mathbb{N}$ (with $\lim_n V(n) = +\infty$), the strong drift condition is

$$PV \le \varrho V + b\,\mathbf{1}_{\{0,1,\dots,n_0\}}$$

for some $\varrho < 1$, $b < \infty$, and $n_0 \in \mathbb{N}$ (see [12]). Estimating $\rho_V(P)$ from the parameters ϱ, b, and n_0 is a difficult issue. This often leads to unsatisfactory bounds, except for stochastically monotone P (see [1], [10], [13], and the references therein).

In this work we present a new procedure to study the convergence rate $\rho_V(P)$ under the following weak drift condition.

(WD) There exist $N \in \mathbb{N}^*$, $d \in (0, +\infty)$, and $\delta \in (0, 1)$ such that $P^N V \le \delta^N V + d\,\mathbf{1}_{\mathbb{X}}$.

The V-geometric ergodicity clearly implies (WD). Conversely, such a condition with $N = 1$ was introduced in [12, Lemma 15.2.8] as an alternative to the drift condition [12, Condition (V4)] to obtain the V-geometric ergodicity under suitable assumptions on V. Note that, under condition (WD), the following real number $\delta_V(P)$ is well defined:

$$\delta_V(P) := \inf\{\delta \in [0, 1) \colon \text{there exist } N \in \mathbb{N}^* \text{ and } d \in (0, +\infty) \text{ such that } P^N V \le \delta^N V + d\,\mathbf{1}_{\mathbb{X}}\}.$$

A spectral analysis of P is presented in Section 2 using quasicompactness. More specifically, when the Markov kernel P has an invariant probability distribution, the connection between the V-geometric ergodicity and the quasicompactness of P is made explicit in Proposition 2.1. Namely, P is V-geometrically ergodic if and only if P is a power-bounded quasicompact operator on $\mathcal{B}_V$ for which $\lambda = 1$ is a simple eigenvalue and the unique eigenvalue of modulus 1. In this case, if $r_{\text{ess}}(P)$ denotes the essential spectral radius of P on $\mathcal{B}_V$ (see (2.2)) and if $\mathcal{V}$ denotes the set of eigenvalues λ of P such that $r_{\text{ess}}(P) < |\lambda| < 1$, then the convergence rate $\rho_V(P)$ is given by (Proposition 2.1):

$$\rho_V(P) = r_{\text{ess}}(P) \quad \text{if } \mathcal{V} = \varnothing \quad \text{and} \quad \rho_V(P) = \max\{|\lambda|,\ \lambda \in \mathcal{V}\} \quad \text{if } \mathcal{V} \ne \varnothing. \tag{1.3}$$

Interesting bounds for generalized eigenfunctions $f \in \mathcal{B}_V \cap \ker(P - \lambda I)^p$ associated with $\lambda \in \mathcal{V}$ are presented in Proposition 2.2. Property (1.3) is relevant to study the convergence rate $\rho_V(P)$ provided that first an accurate bound of $r_{\text{ess}}(P)$ is known and second the above set $\mathcal{V}$ is available. Bounds of $r_{\text{ess}}(P)$ related to drift conditions can be found in [4] and [17] under various assumptions (see Subsection 2.1). In view of our applications, let us just mention that $r_{\text{ess}}(P) = \delta_V(P)$ in the case in which $\mathbb{X} := \mathbb{N}$ and $\lim_n V(n) = +\infty$ (see Proposition 3.1). However, even if the state space is discrete, finding the above set $\mathcal{V}$ is difficult.

In Section 3, the above spectral analysis is applied to compute the rate of convergence $\rho_V(P)$ of discrete random walks (RWs). In particular, a complete solution is presented for RWs with identically distributed (i.d.) bounded increments. In fact, Proposition 3.3 allows us to formulate an algebraic elimination procedure to compute ρ_V (see Corollary 4.1). To the best of our knowledge, this general result is new. Note that it requires neither reversibility nor stochastic monotonicity of P.

This procedure is illustrated in Section 4. First we consider the case of the birth-and-death Markov kernel P defined by $P(0, 0) := a$ and $P(0, 1) := 1 - a$ for some $a \in (0, 1)$ and by

$$P(n, n-1) := p, \qquad P(n, n) := r, \qquad P(n, n+1) := q, \quad \text{for all } n \ge 1,$$

where $p, q, r \in [0, 1]$ are such that $p + r + q = 1$ and $p > q > 0$. An explicit formula for $\rho_V(P)$ with respect to $V := \{(p/q)^{n/2}\}_{n\in\mathbb{N}}$ is given in Proposition 4.1. When $r := 0$, such a result has been obtained for $a < p$ in [14] and [1, Section 8.4] using Kendall's theorem, and for $a \geq p$ in [10] using the stochastic monotony of P. Our method gives a unified and simpler computation of $\rho_V(P)$ which moreover encompasses the case $r \neq 0$. For general RWs with i.d. bounded increments, the elimination procedure requires the use of symbolic computations. The second example illustrates this point with the nonreversible RW defined for all $n \geq 2$ by

$$P(n, n-2) = a_{-2}, \qquad P(n, n-1) = a_{-1}, \qquad P(n, n) = a_0, \qquad P(n, n+1) = a_1,$$

for any nonnegative a_i satisfying $a_{-2}+a_{-1}+a_0+a_1 = 1$, $a_{-2} > 0$, and $2a_{-2}+a_{-1} > a_1 > 0$, and any finitely many boundary transition probabilities. In Section 5, specific examples of RWs on $\mathbb{X} := \mathbb{N}$ with unbounded increments considered in the literature are investigated.

To conclude this introduction, we mention a point which can be a source of confusion in a first reading. In this paper we are concerned with the convergence rate (1.2) with respect to some weighted-supremun Banach space $\mathcal{B}_V$. Thus, we do not consider here the decay parameter or the convergence rate of ergodic Markov chains in the usual Hilbert space $\mathbb{L}^2(\pi)$ which is related to spectral properties of the transition kernel with respect to this space. In particular, for birth-and-death Markov chains, we cannot compare our results with those of [16] on the $\ell^2(\pi)$-spectral gap and the decay parameter. A detailed discussion is provided in Remark 4.2.

2. Quasicompactness on $\mathcal{B}_V$ and V-geometric ergodicity

We assume that P satisfies (WD) . Then P continuously acts on $\mathcal{B}_V$, and iterating (WD) shows that P is power bounded on $\mathcal{B}_V$, namely, $\sup_{n\geq 1} \|P^n\|_V < \infty$, where $\|\cdot\|_V$ also stands for the operator norm on $\mathcal{B}_V$. Thus, we have $r(P) := \lim_n \|P^n\|_V^{1/n} = 1$ since P is Markov.

2.1. From quasicompactness on $\mathcal{B}_V$ to V-geometric ergodicity

Let I denote the identity operator on $\mathcal{B}_V$. Recall that P is said to be quasicompact on $\mathcal{B}_V$ if there exist $r_0 \in (0, 1)$, $m \in \mathbb{N}^*$, $\lambda_i \in \mathbb{C}$, and $p_i \in \mathbb{N}^*$ $(i = 1, \dots, m)$ such that

$$\mathcal{B}_V = \bigoplus_{i=1}^{m} \ker(P - \lambda_i I)^{p_i} \oplus H, \tag{2.1a}$$

where the λ_i are such that

$$|\lambda_i| \geq r_0 \quad \text{and} \quad 1 \leq \dim \ker(P - \lambda_i I)^{p_i} < \infty, \tag{2.1b}$$

and H is a closed P-invariant subspace such that

$$\inf_{n\geq 1}\Big(\sup_{h\in H,\, \|h\|\leq 1} \|P^n h\|\Big)^{1/n} < r_0. \tag{2.1c}$$

Concerning the essential spectral radius of P, denoted by $r_{\mathrm{ess}}(P)$, here it is enough to have in mind that if P is quasicompact on $\mathcal{B}_V$ then we have (see, for instance, [3])

$$r_{\mathrm{ess}}(P) := \inf\{r_0 \in (0, 1) \text{ such that (2.1a)–(2.1c) hold}\}. \tag{2.2}$$

As mentioned in the introduction, the essential spectral radius of Markov kernels acting on $\mathcal{B}_V$ is studied in [4] and [17]. For instance, under condition (WD), the following result

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is proved in [4]: if P^ℓ is compact from $\mathcal{B}_0$ to $\mathcal{B}_V$ for some $\ell \geq 1$, where $(\mathcal{B}_0, \|\cdot\|_0)$ is the Banach space composed of bounded measurable functions $f : \mathbb{X} \to \mathbb{C}$ equipped with the supremum norm $\|f\|_0 := \sup_{x \in \mathbb{X}} |f(x)|$, then P is quasicompact on $\mathcal{B}_V$ with

$$r_{\mathrm{ess}}(P) \leq \delta_V(P).$$

Moreover, the equality $r_{\mathrm{ess}}(P) = \delta_V(P)$ holds in many situations, in particular in the discrete-state case with $V(n) \to \infty$ (see Proposition 3.1).

Next we give a result which makes explicit the relationship between the quasicompactness of P and the V-geometric ergodicity of the Markov chain $\{X_n\}_{n \in \mathbb{N}}$ with transition kernel P. Moreover, we provide an explicit formula for $\rho_V(P)$ in terms of the spectral elements of P. Note that, for any $r_0 \in (r_{\mathrm{ess}}(P), 1)$, the set of all the eigenvalues λ of P such that $r_0 \leq |\lambda| \leq 1$ is finite (use (2.2)).

Proposition 2.1. *Let P be a transition kernel which has an invariant probability measure π such that $\pi(V) < \infty$. The following two assertions are equivalent:*

(a) *P is V-geometrically ergodic,*

(b) *P is a power-bounded quasicompact operator on $\mathcal{B}_V$, for which $\lambda = 1$ is a simple eigenvalue (i.e. $\ker(P - I) = \mathbb{C}\,\mathbf{1}_{\mathbb{X}}$) and the unique eigenvalue of modulus 1.*

Under any of these conditions, we have $\rho_V(P) \geq r_{\mathrm{ess}}(P)$. In fact, for $r_0 \in (r_{\mathrm{ess}}(P), 1)$, denoting the set of all the eigenvalues λ of P such that $r_0 \leq |\lambda| < 1$ by $\mathcal{V}_{r_0}$, we have

- *either $\rho_V(P) \leq r_0$ when $\mathcal{V}_{r_0} = \varnothing$,*
- *or $\rho_V(P) = \max\{|\lambda|,\ \lambda \in \mathcal{V}_{r_0}\}$ when $\mathcal{V}_{r_0} \neq \varnothing$.*

Moreover, if $\mathcal{V}_{r_0} = \varnothing$ for all $r_0 \in (r_{\mathrm{ess}}(P), 1)$ then $\rho_V(P) = r_{\mathrm{ess}}(P)$.

The V-geometric ergodicity of P obviously implies that P is quasicompact on $\mathcal{B}_V$ with $\rho_V(P) \geq r_{\mathrm{ess}}(P)$ (see, e.g. [7]). This follows from (2.2) using $H := \{f \in \mathcal{B}_V : \pi(f) = 0\}$ in (2.1a)–(2.1c). The property that P has a spectral gap on $\mathcal{B}_V$ in the recent paper [8] corresponds here to the quasicompactness of P (which is classical terminology in spectral theory). The spectral gap in [8] corresponds to the value $1 - \rho_V(P)$. Then, [8, Proposition 1.1]) is another formulation, under ψ-irreducibility and aperiodicity assumptions, of the equivalence of properties (a) and (b) of Proposition 2.1 (see also [8, Lemma 2.1]). Details on the proof of Proposition 2.1 are provided in [2]. For general quasicompact Markov kernels on $\mathcal{B}_V$, the result [17, Theorem 4.6] also provides interesting additional material on peripheral eigenelements. The next subsection completes the previous spectral description by providing bounds for the generalized eigenfunctions associated with eigenvalues λ such that $\delta \leq |\lambda| \leq 1$, with δ given in (WD).

2.2. Bound on generalized eigenfunctions of *P*

Proposition 2.2. *Assume that the weak drift condition (WD) holds. If $\lambda \in \mathbb{C}$ is such that $\delta \leq |\lambda| \leq 1$, with δ given in (WD), and if $f \in \mathcal{B}_V \cap \ker(P - \lambda I)^p$ for some $p \in \mathbb{N}^*$, then there exists $c \in (0, +\infty)$ such that*

$$|f| \leq c\, V^{\ln|\lambda|/\ln\delta}(1 + \ln V)^{p(p-1)/2}.$$

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Thus, if λ is an eigenvalue such that $|\lambda| = 1$ then any associated eigenfunction f is bounded on $\mathbb{X}$. By contrast, if $|\lambda|$ is close to $\delta_V(P)$ then $|f| \le cV^{\beta(\lambda)}$ with $\beta(\lambda)$ close to 1. The proof of Proposition 2.2 is based on the following lemma.

Lemma 2.1. *Let $\lambda \in \mathbb{C}$ be such that $\delta \le |\lambda| \le 1$. Then, for all $f \in \mathcal{B}_V$ and all $x \in \mathbb{X}$, there exists $c \in (0, +\infty)$ such that*

$$|\lambda|^{-n(x)}|(P^{n(x)} f)(x)| \le cV(x)^{\ln|\lambda|/\ln\delta} \tag{2.3}$$

with, for any $x \in \mathbb{X}$, $n(x) := \lfloor -\ln V(x)/\ln\delta \rfloor$, where $\lfloor\cdot\rfloor$ denotes the integer part function.

Proof. First note that the iteration of (WD) gives

$$P^{kN} V \le \delta^{kN} V + d\Bigl(\sum_{j=0}^{k-1} \delta^{jN}\Bigr) \mathbf{1}_{\mathbb{X}} \le \delta^{kN} V + \frac{d}{1-\delta^N} \mathbf{1}_{\mathbb{X}} \quad \text{for all } k \ge 1.$$

Let $g \in \mathcal{B}_V$ and $x \in \mathbb{X}$. Using the last inequality, the positivity of P, and $|g| \le \|g\|_V V$, we obtain, with $b := d/(1-\delta^N)$,

$$|(P^{kN} g)(x)| \le (P^{kN}|g|)(x) \le \|g\|_V (P^{kN} V)(x) \le \|g\|_V (\delta^{kN} V(x) + b) \quad \text{for all } k \ge 1. \tag{2.4}$$

This inequality is also fulfilled with $k = 0$. Next, let $f \in \mathcal{B}_V$ and $n \in \mathbb{N}$. Writing $n = kN + r$, with $k \in \mathbb{N}$ and $r \in \{0, 1, \ldots, N-1\}$, and applying (2.4) to $g := P^r f$, we obtain, with $\xi := \max_{0\le\ell\le N-1} \|P^\ell f\|_V$ (use $P^n f = P^{kN}(P^r f)$),

$$|(P^n f)(x)| \le \xi[\delta^{kN} V(x) + b] \le \xi[\delta^{-r}(\delta^n V(x) + b)] \le \xi\delta^{-N}(\delta^n V(x) + b). \tag{2.5}$$

Using the inequality

$$-\frac{\ln V(x)}{\ln\delta} - 1 \le n(x) \le -\frac{\ln V(x)}{\ln\delta}$$

and the fact that $\ln\delta \le \ln|\lambda| \le 0$, inequality (2.5) with $n := n(x)$ gives

$$\begin{aligned}
|\lambda|^{-n(x)}|(P^{n(x)} f)(x)| &\le \xi\delta^{-N}((\delta|\lambda|^{-1})^{n(x)} V(x) + b|\lambda|^{-n(x)}) \\
&= \xi\delta^{-N}(\mathrm{e}^{n(x)(\ln\delta - \ln|\lambda|)}\mathrm{e}^{\ln V(x)} + b\mathrm{e}^{-n(x)\ln|\lambda|}) \\
&\le \xi\delta^{-N}(\mathrm{e}^{(\ln V(x)/\ln\delta + 1)(\ln|\lambda| - \ln\delta)}\mathrm{e}^{\ln V(x)} + b\mathrm{e}^{\ln V(x)\ln|\lambda|/\ln\delta}) \\
&= \xi\delta^{-N}(\mathrm{e}^{\ln|\lambda|\ln V(x)/\ln\delta}\mathrm{e}^{\ln|\lambda| - \ln\delta} + bV(x)^{\ln|\lambda|/\ln\delta}) \\
&= \xi\delta^{-N}(\mathrm{e}^{\ln|\lambda| - \ln\delta} + b)V(x)^{\ln|\lambda|/\ln\delta}.
\end{aligned}$$

This gives inequality (2.3) with $c := \xi\delta^{-N}(\mathrm{e}^{\ln|\lambda| - \ln\delta} + b)$.

Proof of Proposition 2.2. If $f \in \mathcal{B}_V \cap \ker(P - \lambda I)$ then $|\lambda|^{-n(x)}|(P^{n(x)} f)(x)| = |f(x)|$, so that (2.3) gives the expected conclusion when $p = 1$. Next, let us proceed by induction. Assume that the conclusion of Proposition 2.2 holds for some $p \ge 1$. Let $f \in \mathcal{B}_V \cap \ker(P - \lambda I)^{p+1}$. We can write

$$P^n f = (P - \lambda I + \lambda I)^n f = \lambda^n f + \sum_{k=1}^{\min(n,p)} \binom{n}{k} \lambda^{n-k}(P - \lambda I)^k f. \tag{2.6}$$

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For $k \in \{1, \ldots, p\}$, we have $f_k := (P - \lambda I)^k f \in \ker(P - \lambda I)^{p+1-k} \subset \ker(P - \lambda I)^p$; thus, from the induction hypothesis there exists $c' \in (0, +\infty)$ such that, for all $k \in \{1, \ldots, p\}$ and all $x \in \mathbb{X}$,

$$|f_k(x)| \leq c' V(x)^{\ln|\lambda|/\ln\delta}(1 + \ln V(x))^{p(p-1)/2}. \tag{2.7}$$

Now, it follows, from (2.6) (with $n := n(x)$), (2.7), and Lemma 2.1, that, for all $x \in \mathbb{X}$,

$$\begin{aligned}
|f(x)| &\leq |\lambda|^{-n(x)}|(P^{n(x)} f)(x)| \\
&\quad + c' V(x)^{\ln|\lambda|/\ln\delta}(1 + \ln V(x))^{p(p-1)/2}|\lambda|^{-\min(n,p)} \sum_{k=1}^{\min(n,p)} \binom{n(x)}{k} \\
&\leq c V(x)^{\ln|\lambda|/\ln\delta} + c_1 V(x)^{\ln|\lambda|/\ln\delta}(1 + \ln V(x))^{p(p-1)/2} n(x)^p \\
&\leq c_2 V(x)^{\ln|\lambda|/\ln\delta}(1 + \ln V(x))^{p(p-1)/2+p}
\end{aligned}$$

with some constants $c_1, c_2 \in (0, +\infty)$ independent of x. This gives the expected result.

3. Spectral properties of discrete RWs

In the sequel, the state space $\mathbb{X}$ is discrete. For the sake of simplicity, we assume that $\mathbb{X} := \mathbb{N}$. Let $P = (P(i, j))_{i,j\in\mathbb{N}^2}$ be a Markov kernel on $\mathbb{N}$. The function $V : \mathbb{N} \to [1, +\infty)$ is assumed to satisfy

$$\lim_n V(n) = +\infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} \frac{(PV)(n)}{V(n)} < \infty.$$

We focus first on the estimation of $r_{\text{ess}}(P)$ from condition (WD).

Proposition 3.1. *Let $\mathbb{X} := \mathbb{N}$. The following two conditions are equivalent:*

(a) *condition (WD) holds with V,*

(b) *$L := \inf_{N\geq 1}(\ell_N)^{1/N} < 1$, where $\ell_N := \limsup_{n\to+\infty} (P^N V)(n)/V(n)$.*

In this case, P is power bounded and quasicompact on $\mathcal{B}_V$ with $r_{\text{ess}}(P) = \delta_V(P) = L$.

The proof of the equivalence (a) $\Leftrightarrow$ (b), as well as the equality $\delta_V(P) = L$, is straightforward (see [2, Corollary 4]). That P is quasicompact on $\mathcal{B}_V$ under (WD) in the discrete case, with $r_{\text{ess}}(P) \leq \delta_V(P)$, can be derived from [4] or [17] (see Subsection 2.1 and use the fact that the injection from $\mathcal{B}_0$ to $\mathcal{B}_V$ is compact when $\mathbb{X} := \mathbb{N}$ and $\lim_n V(n) = +\infty$). The equality $r_{\text{ess}}(P) = \delta_V(P)$ can be proved by combining the results [4], [17] (see [2, Corollary 1] for details).

In Sections 3 and 4, sequences of the special form $V_\gamma := \{\gamma^n\}_{n\in\mathbb{N}}$ for some $\gamma \in (1, +\infty)$ will be considered. The associated weighted-supremum space $\mathcal{B}_\gamma \equiv \mathcal{B}_{V_\gamma}$ is defined by

$$\mathcal{B}_\gamma := \Big\{\{f(n)\}_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sup_{n\in\mathbb{N}} \gamma^{-n}|f(n)| < \infty\Big\}.$$

3.1. Quasicompactness of RWs with bounded state-dependent increments

Let us fix $c, g, d \in \mathbb{N}^*$, and assume that the kernel P satisfies the following conditions:

$$\sum_{j=0}^{c} P(i, j) = 1 \quad \text{for all } i \in \{0, \dots, g-1\}, \tag{3.1a}$$

$$P(i, j) = \begin{cases} a_{j-i}(i) & \text{if } i - g \le j \le i + d, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for all } i \ge g \text{ and all } j \in \mathbb{N}, \tag{3.1b}$$

where $(a_{-g}(i), \dots, a_d(i)) \in [0, 1]^{g+d+1}$ satisfies $\sum_{k=-g}^{d} a_k(i) = 1$ for all $i \ge g$. This kind of kernel arises, for instance, from the time discretization of Markovian queueing models. Note that more general models and their use in queueing theory are discussed in [6]. In particular, conditions for (non)positive recurrence are provided.

Proposition 3.2. *Assume that, for every $k \in \mathbb{Z}$ such that $-g \le k \le d$, $\lim_n a_k(n) = a_k \in [0, 1]$, and that there exists $\gamma \in (1, +\infty)$ such that*

$$\phi(\gamma) := \sum_{k=-g}^{d} a_k \gamma^k < 1. \tag{3.2}$$

Then P satisfies condition (WD) with $\delta = \phi(\gamma)$. Moreover, P is power bounded and quasi-compact on $\mathcal{B}_\gamma$ with $r_{\mathrm{ess}}(P) = L = \phi(\gamma)$.

Lemma 3.1. *When a_{-g} and a_d are positive, condition (3.2) is equivalent to*

(NERI) $\sum_{k=-g}^{d} k a_k < 0$.

Then there exists a unique real number $\gamma_0 > 1$ such that $\phi(\gamma_0) = 1$ and

$$\phi(\gamma) < 1 \quad \text{for all } \gamma \in (1, \gamma_0),$$

and there exists a unique $\widehat{\gamma}$ such that

$$\widehat{\delta} := \phi(\widehat{\gamma}) = \min_{\gamma \in (1,\infty)} \phi(\gamma) = \min_{\gamma \in (1,\gamma_0)} \phi(\gamma) < 1.$$

Condition (NERI) means that the expectation of the probability distribution of the random increment is negative. Although the results of the paper on RWs with i.d. bounded increments involving condition (NERI) and $a_{-g}, a_d > 0$ will be valid for $\gamma \in (1, \gamma_0)$, only this value $\widehat{\gamma}$ is considered in the statements. Note that the essential spectral radius $r_{\mathrm{ess}}(P_{|\mathcal{B}_{\widehat{\gamma}}})$ of P with respect to $\mathcal{B}_{\widehat{\gamma}}$, which will be denoted by $\widehat{r}_{\mathrm{ess}}(P)$ in the sequel, is the smallest value of $r_{\mathrm{ess}}(P_{|\mathcal{B}_\gamma})$ on $\mathcal{B}_\gamma$ for $\gamma \in (1, \gamma_0)$. When $\gamma \nearrow \gamma_0$, the essential spectral radius $r_{\mathrm{ess}}(P_{|\mathcal{B}_\gamma}) \nearrow 1$ since the space $\mathcal{B}_\gamma$ becomes large. When $\gamma \searrow 1$, then $r_{\mathrm{ess}}(P_{|\mathcal{B}_\gamma}) \nearrow 1$ since $\mathcal{B}_\gamma$ becomes close to the space $\mathcal{B}_0$ of bounded functions. In this case, the geometric ergodicity is lost since the RWs are typically not uniformly ergodic (i.e. $V \equiv 1$) due the nonquasicompactness of P on $\mathcal{B}_0$.

Example 3.1. (*State-dependent birth-and-death Markov chains.*) When $c = g = d := 1$ in (3.1a)–(3.1b), we obtain the standard class of state-dependent birth-and-death Markov chains:

$$P(0, 0) := r_0, \qquad P(0, 1) := q_0$$
$$P(n, n-1) := p_n, \qquad P(n, n) := r_n, \qquad P(n, n+1) := q_n, \quad \text{for all } n \ge 1.$$

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Here $(p_0, q_0) \in [0, 1]^2$, $p_0 + q_0 = 1$, and $(p_n, r_n, q_n) \in [0, 1]^3$, $p_n + r_n + q_n = 1$. Assume that

$$\lim_n p_n := p, \qquad \lim_n r_n := r, \qquad \lim_n q_n := q.$$

If $\gamma \in (1, +\infty)$ is such that $\phi(\gamma) := p/\gamma + r + q\gamma < 1$ then it follows from Proposition 3.2 that $r_{\mathrm{ess}}(P) = p/\gamma + r + q\gamma$. The conditions $\gamma > 1$ and $p/\gamma + r + q\gamma < 1$ are equivalent to the following conditions (use $r = 1 - p - q$ for (i)):

(i) either $q > 0$, $q - p < 0$ (i.e. (NERI)), and $1 < \gamma < \gamma_0 = p/q$;

(ii) or $q = 0$, $p > 0$, and $\gamma > 1$.

(i) When $p > q > 0$ and $1 < \gamma < \gamma_0$, P is power bounded and quasicompact on $\mathcal{B}_\gamma$ with $r_{\mathrm{ess}}(P) = \phi(\gamma)$. Set $\widehat{\gamma} := \sqrt{\gamma_0} = \sqrt{p/q} \in (1, \gamma_0)$. Then $\min_{\gamma>1} \phi(\gamma) = \phi(\widehat{\gamma}) = r + 2\sqrt{pq}$ and the essential spectral radius $\widehat{r}_{\mathrm{ess}}(P)$ on $\mathcal{B}_{\widehat{\gamma}}$ satisfies $\widehat{r}_{\mathrm{ess}}(P) = r + 2\sqrt{pq}$.

(ii) When $q := 0$, $p > 0$ and $\gamma > 1$, $r_{\mathrm{ess}}(P) = \phi(\gamma) = p/\gamma + r$.

Remark 3.1. If c is allowed to be $+\infty$ in condition (3.1a), that is,

$$\sum_{j\geq 0} P(i, j)\gamma^j < \infty \quad \text{for all } i \in \{0, \ldots, g-1\}, \tag{3.3}$$

then the conclusions of Proposition 3.2 and Example 3.1 are still valid under the additional condition (3.3).

Proof of Proposition 3.2. Set $\phi_n(\gamma) := \sum_{k=-g}^{d} a_k(n)\gamma^k$. We have $(PV_\gamma)(n) = \phi_n(\gamma)V_\gamma(n)$ for each $n \geq g$. Thus, $\ell_1 = \lim_n \phi_n(\gamma) = \phi(\gamma)$. Now assume that

$$\ell_{N-1} := \frac{\lim_n (P^{N-1}V)(n)}{V(n)} = \phi(\gamma)^{N-1} \quad \text{for some } N \geq 1.$$

Since

$$(P^N V)(i) = \sum_{j=-g}^{d} a_j(i)(P^{N-1}V)(i+j) \quad \text{for all } i \geq Ng,$$

we obtain

$$\frac{(P^N V)(i)}{V(i)} = \sum_{j=-g}^{d} a_j(i)\gamma^j \frac{(P^{N-1}V)(i+j)}{\gamma^{i+j}} \to \phi(\gamma)\phi(\gamma)^{N-1} \quad \text{as } i \to +\infty.$$

Hence, $\ell_N = \phi(\gamma)^N$, and $\phi(\gamma) = L = r_{\mathrm{ess}}(P)$ from Proposition 3.1.

Proof of Lemma 3.1. Since the second derivative of ϕ is positive on $(0, +\infty)$, ϕ is convex on $(0, +\infty)$. When a_{-g} and a_d are positive, then $\lim_{t\to 0^+} \phi(t) = \lim_{t\to+\infty} \phi(t) = +\infty$ and, since $\phi(1) = 1$, condition (3.2) is equivalent to $\phi'(1) < 0$, that is, (NERI) holds. The other properties of $\phi(\cdot)$ are immediate.

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3.2. Spectral analysis of RW with i.d. bounded increments

Let $P := (P(i, j))_{(i,j)\in\mathbb{N}^2}$ be the transition kernel of an RW with i.d. bounded increments. Specifically, we assume that there exist some positive integers $c, g, d \in \mathbb{N}^*$ such that

$$\sum_{j=0}^{c} P(i, j) = 1 \quad \text{for all } i \in \{0, \dots, g-1\}, \tag{3.4a}$$

$$P(i, j) = \begin{cases} a_{j-i} & \text{if } i - g \le j \le i + d, \\ 0 & \text{otherwise}, \end{cases} \quad \text{for all } i \ge g \text{ and all } j \in \mathbb{N}, \tag{3.4b}$$

$$a_{-g} > 0, \qquad a_d > 0, \qquad \sum_{k=-g}^{d} a_k = 1 \quad \text{for } (a_{-g}, \dots, a_d) \in [0, 1]^{g+d+1}. \tag{3.4c}$$

Let us assume that condition (NERI) holds. We know from Lemma 3.1 and Proposition 3.2 that P is quasicompact on $\mathcal{B}_{\widehat{\gamma}}$ with

$$\widehat{r}_{\text{ess}}(P) = \widehat{\delta} := \phi(\widehat{\gamma}) < 1,$$

where $\phi(\cdot)$ is given by (3.2).

For any $\lambda \in \mathbb{C}$, we denote by $\mathcal{E}_\lambda$ the set of complex roots of $E_\lambda(\cdot)$, where $E_\lambda(\cdot)$ denotes the following polynomial of degree $N := d + g$:

$$E_\lambda(z) := z^g(\phi(z) - \lambda) = \sum_{k=-g}^{d} a_k z^{g+k} - \lambda z^g \quad \text{for all } z \in \mathbb{C}.$$

Since $E_\lambda(0) = a_{-g} > 0$, we have, for any $\lambda \in \mathbb{C}$,

$$z \in \mathcal{E}_\lambda \iff \mathbb{E}_\lambda(z) = 0 \iff \lambda = \phi(z).$$

In Proposition 3.3 below we investigate the eigenvalues of P on $\mathcal{B}_{\widehat{\gamma}}$ which belong to the annulus

$$\Lambda := \{\lambda \in \mathbb{C} : \widehat{\delta} < |\lambda| < 1\}.$$

To that end, for any $\lambda \in \Lambda$, we introduce the following subset $\mathcal{E}_\lambda^-$ of $\mathcal{E}_\lambda$:

$$\mathcal{E}_\lambda^- := \{z \in \mathbb{C} : E_\lambda(z) = 0, \ |z| < \widehat{\gamma}\}.$$

If $\mathcal{E}_\lambda^- = \varnothing$, we set $N(\lambda) := 0$. If $\mathcal{E}_\lambda^- \neq \varnothing$ then $N(\lambda)$ is defined as

$$N(\lambda) := \sum_{z \in \mathcal{E}_\lambda^-} m_z,$$

where m_z denotes the multiplicity of z as a root of $E_\lambda(\cdot)$. Finally, for any $z \in \mathbb{C}$, we set $z^{(1)} := \{z^n\}_{n\in\mathbb{N}}$, and, for any $k \ge 2$, $z^{(k)} \in \mathbb{C}^{\mathbb{N}}$ is defined by

$$z^{(k)}(n) := n(n-1)\cdots(n-k+2)\, z^{n-k+1} \quad \text{for all } n \in \mathbb{N}.$$

Proposition 3.3. *Assume that assumptions (3.4a)–(3.4c) and (NERI) hold. Then there exists $\eta \ge 1$ such that, for all $\lambda \in \Lambda$,*

$$N(\lambda) = \eta.$$

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Moreover, the following two assertions are equivalent:

(a) $\lambda \in \Lambda$ *is an eigenvalue of* P *on* $\mathcal{B}_{\widehat{\gamma}}$,

(b) *there exists a nonzero* $\{\alpha_{\lambda,z,k}\}_{z\in\mathcal{E}_\lambda^-,\,1\le k\le m_z} \in \mathbb{C}^\eta$ *such that*

$$f := \sum_{z\in\mathcal{E}_\lambda^-}\sum_{k=1}^{m_z}\alpha_{\lambda,z,k}z^{(k)} \in \mathbb{C}^{\mathbb{N}} \tag{3.5}$$

satisfies the boundary equations $\lambda f(i) = (Pf)(i)$ *for all* $i = 0,\dots,g-1$.

The first step in the elimination procedure of Section 4 is to substitute f of the form (3.5) into the boundary equations. This gives a linear system in $\alpha_{\lambda,z,k}$. Since Λ is infinite, that $N(\lambda)$ does not depend on λ is crucial to initialize this procedure. To specify the value of η, it is sufficient to compute $N(\lambda)$ for some (any) $\lambda \in \Lambda$.

Remark 3.2. Under condition (NERI), $\phi(\cdot)$ is decreasing from $(1,\widehat{\gamma})$ to $(\widehat{\delta},1)$, so we have, for all $\lambda \in (\widehat{\delta},1)$, $\phi^{-1}(\lambda) \in (1,\widehat{\gamma})$. Since $\phi^{-1}(\lambda) \in \mathcal{E}_\lambda$, we obtain

$$N(\lambda) \ge 1 \quad \text{for all } \lambda \in (\widehat{\delta},1). \tag{3.6}$$

Remark 3.3. Let condition (NERI) be satisfied. Set $\mathcal{E}_\lambda^+ := \{z \in \mathbb{C}\colon E_\lambda(z) = 0,\ |z| > \widehat{\gamma}\}$. Then

$$\mathcal{E}_\lambda = \mathcal{E}_\lambda^- \sqcup \mathcal{E}_\lambda^+ \quad \text{for all } \lambda \in \Lambda.$$

In other words, for any $\lambda \in \Lambda$, $E_\lambda(\cdot)$ has no root of modulus $\widehat{\gamma}$. Indeed, consider $\lambda \in \Lambda$ and $z \in \mathcal{E}_\lambda$, and assume that $|z| = \widehat{\gamma}$. Since $\lambda = \phi(z)$, we obtain the inequality $|\lambda| \le \phi(|z|) = \phi(\widehat{\gamma})$, which is impossible since $\phi(\widehat{\gamma}) = \widehat{\delta}$ and $\lambda \in \Lambda$.

Remark 3.4. Proposition 3.3(b) does not mean that the dimension of the eigenspace $\ker(P - \lambda I)$ associated with λ is η. We shall see in Subsection 4.2 that we can have $\eta = 2$ when $g = 2$, $d = 1$, and $c = 2$ in (3.4a)–(3.4c), while $\dim\ker(P-\lambda I) \le 1$ since $Pf = \lambda f$ and $f(0) = 0$ clearly imply that $f = 0$ (by induction).

The following surprising lemma, based on Remark 3.3, is used to derive Proposition 3.3.

Lemma 3.2. *Under condition (NERI), the function* $N(\cdot)$ *is constant on* Λ.

Proof. Since Λ is connected and $N(\cdot)$ is $\mathbb{N}$-valued, it suffices to prove that $N(\cdot)$ is continuous on Λ. Note that the set $\bigcup_{\lambda\in\Lambda}\mathcal{E}_\lambda$ is bounded in $\mathbb{C}$ since the coefficients of $E_\lambda(\cdot)$ are obviously uniformly bounded in $\lambda \in \Lambda$. Now let $\lambda \in \Lambda$ and assume that $N(\cdot)$ is not continuous at λ. Then there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \in \Lambda^{\mathbb{N}}$ such that $\lim_n \lambda_n = \lambda$ and

(a) either $N(\lambda_n) \ge N(\lambda) + 1$ for all $n \ge 0$,

(b) or $N(\lambda_n) \le N(\lambda) - 1$ for all $n \ge 0$.

For any $n \ge 0$, let us denote the roots of $E_{\lambda_n}(\cdot)$ by $z_1(\lambda_n),\dots,z_N(\lambda_n)$, and suppose for convenience that they are listed by increasing modulus, and by increasing argument when they have the same modulus. Applying Remark 3.3 to λ_n, we obtain

$$\begin{aligned} &|z_i(\lambda_n)| < \widehat{\gamma} \quad \text{for all } i \in \{1,\dots,N(\lambda_n)\} \\ \text{and}\quad &|z_i(\lambda_n)| > \widehat{\gamma} \quad \text{for all } i \in \{N(\lambda_n)+1,\dots,N\}. \end{aligned}$$

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By passing to a subsequence, we may suppose that, for every $1 \leq i \leq N$, the sequence $\{z_i(\lambda_n)\}_{n\in\mathbb{N}}$ converges to some $z_i \in \mathbb{C}$. Note that

$$\mathcal{E}_\lambda = \{z_1, z_2, \dots, z_N\},$$

where z_i is repeated in this list with respect to its multiplicity m_{z_i}, since

$$E_\lambda(z) = \lim_n E_{\lambda_n}(z) = \lim_n a_d \prod_{i=1}^{N}(z - z_i(\lambda_n)) = a_d \prod_{i=1}^{N}(z - z_i) \quad \text{for all } z \in \mathbb{C}.$$

In case (a), we have

$$|z_1(\lambda_n)| < \widehat{\gamma}, \dots, |z_{N(\lambda)+1}(\lambda_n)| < \widehat{\gamma}, \quad \text{for all } n \geq 0.$$

When $n \to +\infty$, this gives, using Remark 3.3,

$$|z_1| < \widehat{\gamma}, \dots, |z_{N(\lambda)+1}| < \widehat{\gamma}.$$

Thus, at least $N(\lambda) + 1$ roots of $E_\lambda(\cdot)$ (counted with their multiplicity) are of modulus strictly less than $\widehat{\gamma}$. This contradicts the definition of $N(\lambda)$.

In case (b), we have

$$|z_{N(\lambda)}(\lambda_n)| > \widehat{\gamma}, \qquad |z_{N(\lambda)+1}(\lambda_n)| > \widehat{\gamma}, \dots, |z_N(\lambda_n)| > \widehat{\gamma}, \quad \text{for all } n \geq 0$$

and this similarly gives, when $n \to +\infty$,

$$|z_{N(\lambda)}| > \widehat{\gamma}, \qquad |z_{N(\lambda)+1}| > \widehat{\gamma}, \dots, |z_N| > \widehat{\gamma}.$$

Thus, at least $N - N(\lambda) + 1$ roots of $E_\lambda(\cdot)$ (counted with their multiplicity) are of modulus strictly larger than $\widehat{\gamma}$. This contradicts the definition of $N(\lambda)$.

Proof of Proposition 3.3. From Lemma 3.2 and (3.6), for all $\lambda \in \Lambda$, we obtain $N(\lambda) = \eta$ for some $\eta \geq 1$. Now we prove the implication (a) $\Rightarrow$ (b). Let $\lambda \in \Lambda$ be any eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$, and let $f := \{f(n)\}_{n\in\mathbb{N}}$ be a nonzero sequence in $\mathcal{B}_{\widehat{\gamma}}$ satisfying $Pf = \lambda f$. In particular, f satisfies the equalities

$$\lambda f(i) = \sum_{j=i-g}^{i+g} a_{j-i} f(j) \quad \text{for all } i \geq g. \tag{3.7}$$

Since the characteristic polynomial associated with these recursive formulae is $E_\lambda(\cdot)$, there exists $\{\alpha_{\lambda,z,k}\}_{z\in\mathcal{E}_\lambda,\, 1\leq k\leq m_z} \in \mathbb{C}^\eta$ such that

$$f = \sum_{z\in\mathcal{E}_\lambda} \sum_{k=1}^{m_z} \alpha_{\lambda,z,k} z^{(k)} \in \mathbb{C}^{\mathbb{N}},$$

where m_z denotes the multiplicity of $z \in \mathcal{E}_\lambda$. Next, since $|f| \leq C V_{\widehat{\gamma}}$ for some $C > 0$ (i.e. $f \in \mathcal{B}_{\widehat{\gamma}}$), it can be easily seen that $\alpha_{\lambda,z,k} = 0$ for every $z \in \mathcal{E}_\lambda$ such that $|z| > \widehat{\gamma}$ and, for every $k = 1, \dots, m_z$: first delete α_{λ,z,m_z} for z of maximum modulus and for m_z maximal if there are several z of maximal modulus (to that effect, divide f by $n(n-1)\cdots(n-m_z+2)\, z^{n-m_z+1}$

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and use $|f| \leq CV_{\widehat{\gamma}}$). Therefore, f is of the form (3.5), and it satisfies the boundary equations in (b) since $Pf = \lambda f$ by hypothesis.

To prove the implication (b) $\Rightarrow$ (a), note that any $f := \{f(n)\}_{n\in\mathbb{N}}$ of the form (3.5) belongs to $\mathcal{B}_{\widehat{\gamma}}$ and satisfies (3.7) since $\mathcal{E}_\lambda^- \subset \mathcal{E}_\lambda$. If, moreover, f is nonzero and satisfies the boundary equations, then $Pf = \lambda f$. This gives (a).

We conclude this study with an additional refinement of Proposition 3.3. For any $\lambda \in \Lambda$, let us define the set $\mathcal{E}_{\lambda,\tau}^-$ as follows:

$$\mathcal{E}_{\lambda,\tau}^- := \{z \in \mathbb{C}\colon E_\lambda(z) = 0,\ |z| < \widehat{\gamma}^\tau\} \quad \text{with} \quad \tau \equiv \tau(\lambda) := \frac{\ln|\lambda|}{\ln\widehat{\delta}}.$$

Moreover, define the associated function $N'(\cdot)$ by

$$N'(\lambda) := \sum_{z\in\mathcal{E}_{\lambda,\tau}^-} m_z,$$

where m_z is the multiplicity of z as root of $E_\lambda(\cdot)$ (with the convention that $N'(\lambda) = 0$ if $\mathcal{E}_{\lambda,\tau}^- = \varnothing$).

Lemma 3.3. *Assume that $P := (P(i,j))_{(i,j)\in\mathbb{N}^2}$ satisfies conditions (3.4a)–(3.4c) and (NERI). Moreover, assume that*

$$\phi(t) < t^{\ln\widehat{\delta}/\ln\widehat{\gamma}} \quad \textit{for all } t \in (1,\widehat{\gamma}). \tag{3.8}$$

Then the function $N'(\cdot)$ is constant on Λ: there exists $\eta' \geq 1$ such that $N'(\lambda) = \eta'$ for all $\lambda \in \Lambda$.

From Lemma 3.3, all the assertions of Proposition 3.3 are still valid when η and $\mathcal{E}_\lambda^-$ are replaced with η' and $\mathcal{E}_{\lambda,\tau}^-$, respectively. That $\mathcal{E}_\lambda^-$ may be replaced with $\mathcal{E}_{\lambda,\tau}^-$ in (3.5) follows from Proposition 2.2. Consequently, under the additional condition $\eta' \leq g$, the elimination procedure of Section 4 may be adapted by using Lemma 3.3. Since $\eta' \leq \eta$, the resulting procedure is computationally interesting when g or d are large.

Remark 3.5. Condition (3.8) is the additional assumption in Lemma 3.3 with respect to Lemma 3.2. Since ϕ is decreasing on $(1,\widehat{\gamma})$ under condition (NERI), condition (3.8) is equivalent to

$$z < \widehat{\gamma}^{\ln\phi(z)/\ln\widehat{\delta}} \quad \text{for all } z \in (1,\widehat{\gamma}). \tag{3.9}$$

Indeed, for every $t \in (1,\widehat{\gamma})$, we have $u := t^{\ln\widehat{\delta}/\ln\widehat{\gamma}} \in (\widehat{\delta},1)$ and $z := \phi^{-1}(u) \in (1,\widehat{\gamma})$. Hence,

$$(3.8) \iff \phi(\widehat{\gamma}^{\ln u/\ln\widehat{\delta}}) < u \quad \text{for all } u \in (\widehat{\delta},1) \iff (3.9). \tag{3.10}$$

Therefore, under condition (3.8), for any $\lambda \in (\widehat{\delta},1)$, we have $\mathcal{E}_{\lambda,\tau}^- \neq \varnothing$ since $z = \phi^{-1}(\lambda)$ satisfies $z < \widehat{\gamma}^{\tau(\lambda)}$ from (3.9).

Proof of Lemma 3.3. The proof is similar to that of Lemma 3.2. Under condition (3.8), Remark 3.3 extends as follows:

$$\mathcal{E}_\lambda = \mathcal{E}_{\lambda,\tau}^- \sqcup (\mathcal{E}_\lambda \cap \{z \in \mathbb{C}\colon |z| > \widehat{\gamma}^\tau\}). \tag{3.11}$$

Indeed, consider $\lambda \in \Lambda$ and $z \in \mathcal{E}_\lambda$ such that $|z| = \widehat{\gamma}^\tau$. Since $\lambda = \phi(z)$, we have $|\lambda| \leq \phi(|z|)$; thus, $|\lambda| \leq \phi(\widehat{\gamma}^\tau)$. This inequality contradicts condition (3.8) (use the definition of τ and the second equivalence in (3.10) with $u := |\lambda|$). Next, using (3.11) and the continuity of $\tau(\cdot)$, Lemma 3.2 easily extends to the function $N'(\cdot)$.

4. Convergence rate for RWs with i.d. bounded increments

Let us recall that any RW with i.d. bounded increments defined by (3.4a)–(3.4c) and satisfying (NERI) has an invariant probability measure π on $\mathbb{N}$ such that $\pi(V_{\widehat{\gamma}}) < \infty$, where $V_{\widehat{\gamma}} := \{\widehat{\gamma}^n\}_{n\in\mathbb{N}}$ and $\widehat{\gamma}$ is defined in Lemma 3.1. Indeed, $\widehat{\delta} := \phi(\widehat{\gamma}) < 1$ so that condition (WD) holds with $V_{\widehat{\gamma}}$ from Proposition 3.2. The expected conclusions on π can be deduced from the first statement of [2, Corollary 5]. Note that, from Lemma 3.1, the previous fact is valid for any $\gamma \in (1, \gamma_0)$ in place of $\widehat{\gamma}$.

The $V_{\widehat{\gamma}}$-geometric ergodicity of the RW may be studied using Proposition 2.1. Next we can derive from Proposition 3.3 an effective procedure to compute the rate of convergence with respect to $\mathcal{B}_{\widehat{\gamma}}$ (see (1.2)), denoted by $\widehat{\rho}(P)$. The most favorable case for initializing the procedure (see (4.3) and (4.5)) is to assume that, for some (any) $\lambda \in \Lambda$,

$$\eta := N(\lambda) \leq g. \tag{4.1}$$

- *First step: checking condition (4.1).* From Lemma 3.2, computing η and testing $\eta \leq g$ of assumption (4.1) can be done by analyzing the roots of $E_\lambda(\cdot)$ for some (any) $\lambda \in \Lambda$.
- *Second step: linear and polynomial eliminations.* This second step consists in applying some linear and (successive) polynomial eliminations in order to find a finite set $\mathcal{Z} \subset \Lambda$ containing all the eigenvalues of P on $\mathcal{B}_{\widehat{\gamma}}$ in Λ. Conversely, the elements of $\mathcal{Z}$ providing eigenvalues of P on $\mathcal{B}_{\widehat{\gamma}}$ can be identified using condition (b) of Proposition 3.3. Note that the explicit computation of the roots of $E_\lambda(\cdot)$ is only required for the elements λ of the finite set $\mathcal{Z}$. This is detailed in Corollary 4.1.

Under the assumptions of Proposition 3.3, we define the set

$$\mathcal{M} := \left\{(m_1, \ldots, m_s) \in \{1, \ldots, s\}^s : s \in \{1, \ldots, \eta\},\ m_1 \leq \ldots \leq m_s, \text{ and } \sum_{i=1}^{s} m_i = \eta\right\}.$$

Note that $\mathcal{M}$ is a finite set and that, for every $\lambda \in \Lambda$, there exists a unique $\mu \in \mathcal{M}$ such that the set $\mathcal{E}_\lambda^-$ is composed of s distinct roots of $E_\lambda(\cdot)$ with multiplicity $m_1, \ldots, m_s$, respectively.

Corollary 4.1. *Assume that assumptions (3.4a)–(3.4c) and (NERI) hold. Set $\ell := \binom{g}{\eta}$. Then there exists a family of polynomial functions $\{\mathcal{R}_{\mu,k},\ \mu \in \mathcal{M},\ 1 \leq k \leq \ell\}$, with coefficients depending only on μ and on the transition probabilities $P(i, j)$, such that the following assertions hold for any $\mu \in \mathcal{M}$.*

(a) *Let $\lambda \in \Lambda$ be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ such that, for some $s \in \{1, \ldots, \eta\}$, the set $\mathcal{E}_\lambda^-$ is composed of s roots of $E_\lambda(\cdot)$ with multiplicity $m_1, \ldots, m_s$, respectively. Then*

$$\mathcal{R}_{\mu,1}(\lambda) = 0, \ldots, \mathcal{R}_{\mu,\ell}(\lambda) = 0. \tag{4.2}$$

(b) *Conversely, let $\lambda \in \Lambda$ satisfy (4.2) such that, for some $s \in \{1, \ldots, \eta\}$, the set $\mathcal{E}_\lambda^-$ is composed of s roots of $E_\lambda(\cdot)$ with multiplicity $m_1, \ldots, m_s$, respectively. Then a necessary and sufficient condition for λ to be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ is that λ satisfies condition (b) of Proposition 3.3.*

Proof. Assertion (b) follows from Proposition 3.3. To prove (a), first assume for convenience that $\eta = g$ and that $\lambda \in \Lambda$ is an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ such that the associated set $\mathcal{E}_\lambda^-$ contains

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η distinct roots $z_1, \ldots, z_\eta$ of $E_\lambda(\cdot)$ with multiplicity 1. We know from Proposition 3.3 that there exists $f := \{f(n)\}_{n\in\mathbb{N}} \neq 0$ of the form

$$f = \sum_{i=1}^{\eta} \alpha_i z_i^{(1)},$$

which satisfies the $g = \eta$ boundary equations: $\lambda f(i) = (Pf)(i)$ for all $i = 0, \ldots, \eta - 1$. In other words, the linear system provided by these η equations has a nonzero solution $(\alpha_i)_{1\le i\le\eta} \in \mathbb{C}^\eta$. Therefore, the associated determinant is 0; this leads to a polynomial equation of the form

$$P_{0,1}(\lambda, z_1, \ldots, z_\eta) = 0. \tag{4.3}$$

Since this polynomial is divisible by $\prod_{i\neq j}(z_i - z_j)$, (4.3) is equivalent to

$$P_0(\lambda, z_1, \ldots, z_\eta) = 0 \quad \text{with} \quad P_0(\lambda, z_1, \ldots, z_\eta) = \frac{P_{0,1}(\lambda, z_1, \ldots, z_\eta)}{\prod_{i\neq j}(z_i - z_j)}. \tag{4.4}$$

Note that the coefficients of P_0 depend only on the $P(i, j)$.

Next, z_η is a common root of the polynomials $P_0(\lambda, z_1, \ldots, z_{\eta-1}, z)$ and $E_\lambda(z)$ with respect to the variable z; this leads to the necessary condition

$$P_1(\lambda, z_1, \ldots, z_{\eta-1}) := \mathrm{Res}_{z_\eta}(P_0, E_\lambda) = 0,$$

where $\mathrm{Res}_{z_\eta}(P_0, E_\lambda)$ denotes the resultant of the two polynomials P_0 and E_λ corresponding to the elimination of the variable z_η. Again, the coefficients of P_1 depend only on the $P(i, j)$. Next, considering the common root $z_{\eta-1}$ of the polynomials $P_1(\lambda, z_1, \ldots, z_{\eta-2}, z)$ and $E_\lambda(z)$ leads to the elimination of the variable $z_{\eta-1}$:

$$P_2(\lambda, z_1, \ldots, z_{\eta-2}) := \mathrm{Res}_{z_{\eta-1}}(P_1, E_\lambda) = 0.$$

Repeating this method, we find that a necessary condition for λ to be an eigenvalue of P is $\mathcal{R}(\lambda) = 0$, where $\mathcal{R}$ is some polynomial with coefficients depending only on the $P(i, j)$.

Now let us consider the case when $\eta < g$, $s \in \{1, \ldots, \eta\}$, and $\lambda \in \Lambda$ is assumed to be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ such that the associated set $\mathcal{E}_\lambda^-$ contains s distinct roots of $E_\lambda(\cdot)$ with respective multiplicity $m_1, \ldots, m_s$ satisfying $\sum_{i=1}^{s} m_i = \eta$. Then the elimination (by using determinants) of $(\alpha_{\lambda,z,\ell}) \in \mathbb{C}^\eta$ provided by the linear system of Proposition 3.3 leads to $\ell := \binom{g}{\eta}$ polynomial equations

$$P_{0,\mu,1}(\lambda, z_1, \ldots, z_\eta) = 0, \ldots, P_{0,\mu,\ell}(\lambda, z_1, \ldots, z_\eta) = 0. \tag{4.5}$$

As in the case $\eta = g$, these polynomials are replaced in the sequel by the polynomials obtained by division of the $P_{0,\mu,k}$ by $\prod_{i\neq j}(z_i - z_j)^{n_{i,j}}$, where $n_{i,j} := \min(m_i, m_j)$.

The successive polynomial eliminations of $z_\eta, \ldots, z_1$ can be derived as above from each polynomial equation $P_{0,\mu,k}(\lambda, z_1, \ldots, z_\eta) = 0$. This gives ℓ polynomial equations

$$\mathcal{R}_{\mu,1}(\lambda) = 0\,, \ldots,\; \mathcal{R}_{\mu,\ell}(\lambda) = 0.$$

Satisfying this set of polynomial equations is a necessary condition for λ to be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$. Finally, the polynomial functions $\mathcal{R}_{\mu,1}, \ldots, \mathcal{R}_{\mu,\ell}$ depend on the $P(i, j)$ and also on $(m_1, \ldots, m_s)$, since the linear system used to eliminate $(\alpha_{\lambda,k,\ell}) \in \mathbb{C}^\eta$ involves the coefficients $i(i-1)\cdots(i-k+1)$ for some finitely many integers i and $k = 1, \ldots, m_i$ $(i = 1, \ldots, s)$.

To compute $\widehat{\rho}(P)$, we define the following (finite and possibly empty) sets:

$$\Lambda_\mu := \{\lambda \in \Lambda\colon \mathcal{R}_{\mu,1}(\lambda) = 0, \dots, \mathcal{R}_{\mu,\ell}(\lambda) = 0\} \quad \text{for all } \mu \in \mathcal{M}.$$

Let us denote by $\mathcal{Z}$ the (finite and possibly empty) set composed of all the complex numbers $\lambda \in \bigcup_{\mu\in\mathcal{M}} \Lambda_\mu$ such that condition (b) of Proposition 3.3 holds.

Corollary 4.2. *Assume that assumptions (3.4a)–(3.4c) and (NERI) hold, and that P is irreducible and aperiodic. Then*

$$\widehat{\rho}(P) = \max(\widehat{\delta}, \max\{|\lambda|,\ \lambda \in \mathcal{Z}\}), \quad \textit{where} \quad \widehat{\delta} := \phi(\widehat{\gamma}).$$

Proof. Under the assumptions on P, we know from Proposition 2.1 that the RW is $V_{\widehat{\gamma}}$-geometrically ergodic. Since $\widehat{r}_{\mathrm{ess}}(P) = \widehat{\delta}$ from Proposition 3.2, the corollary follows from Corollary 4.1 and from Proposition 2.1 applied either with any r_0 such that $\widehat{\delta} < r_0 < \min\{|\lambda|, \lambda \in \mathcal{Z}\}$ if $\mathcal{Z} \neq \varnothing$, or with any r_0 such that $\widehat{\delta} < r_0 < 1$ if $\mathcal{Z} = \varnothing$.

Remark 4.1. When $\eta \geq 2$ and $\mu := (m_1, \dots, m_s)$ with $s < \eta$, the set Λ_μ used in Corollary 4.2 may be reduced. For the sake of simplicity, this fact has been omitted in Corollary 4.2, but it is relevant in practice. Actually, when $s < \eta$, Corollary 4.1(b) can be specified since it requires that $E_\lambda(\cdot)$ admits roots of multiplicity greater than or equal to 2. This involves some additional necessary conditions on λ derived from some polynomial eliminations with respect to the derivatives of $E_\lambda(\cdot)$.

For instance, in the case $g = 2$, $\eta = 2$, and $s = 1$ (thus, $\mu := (2)$), a necessary condition on λ for $E_\lambda(\cdot)$ to have a double root is that $E_\lambda(\cdot)$ and $E'_\lambda(\cdot)$ admit a common root. This leads to

$$Q(\lambda) := \mathrm{Res}_z(E_\lambda, E'_\lambda) = 0.$$

Consequently, if $g = 2$ and $\eta = 2$ (thus, $\ell := 1$), then condition (b) of Proposition 3.3 can be tested in the case $s = 1$ by using the following finite set:

$$\Lambda'_\mu := \Lambda_\mu \cap \{\lambda \in \Lambda\colon Q(\lambda) = 0\}.$$

In general, Λ'_μ is strictly contained in Λ_μ. Even Λ'_μ may be empty while Λ_μ is not (see Subsection 4.2).

Proposition 3.3 and the above elimination procedure obviously extend to any $\gamma \in (1, \gamma_0)$ in place of $\widehat{\gamma}$, where γ_0 is given in Lemma 3.1. Of course, $\widehat{\delta} = \phi(\widehat{\gamma})$ is then replaced by $\delta = \phi(\gamma)$.

4.1. RWs with $g = d := 1$: birth-and-death Markov chains

Let $p, q, r \in [0, 1]$ be such that $p + r + q = 1$, and let P be defined by

$$\begin{gathered} P(0,0) \in (0,1), \quad P(0,1) = 1 - P(0,0) \\ P(n,n-1) := p, \quad P(n,n) := r, \quad P(n,n+1) := q, \quad \text{for } n \geq 1 \text{ with } 0 < q < p. \end{gathered} \tag{4.6}$$

Note that $a_{-1} := p, a_1 := q > 0$ and (NERI) holds. We have $\gamma_0 = p/q \in (1, +\infty)$ and $\widehat{\gamma} := \sqrt{p/q} \in (1, +\infty)$ such that $\widehat{\delta} := \min_{\gamma>1} \phi(\gamma) = \phi(\widehat{\gamma}) < 1$ (see Lemma 3.1). Let $V_{\widehat{\gamma}} := \{\widehat{\gamma}^n\}_{n\in\mathbb{N}}$ and let $\mathcal{B}_{\widehat{\gamma}}$ be the associated weighted-supremum space (as defined in Section 3). Here we have

$$\widehat{r}_{\mathrm{ess}}(P) = \widehat{\delta} = r + 2\sqrt{pq}.$$

Proposition 4.1. *Let P be defined by conditions (4.6). The boundary transition probabilities are denoted by $P(0,0) := a$ and $P(0,1) := 1-a$ for some $a \in (0,1)$. Then P is $V_{\widehat{\gamma}}$-geometrically ergodic. Furthermore, defining $a_0 := 1-q-\sqrt{pq}$, the convergence rate $\widehat{\rho}(P)$ of P with respect to $\mathcal{B}_{\widehat{\gamma}}$ is as follows.*

- *When $a \in (a_0, 1)$,*
$$\widehat{\rho}(P) = r + 2\sqrt{pq}. \tag{4.7}$$
- *When $a \in (0, a_0]$,*
  (a) *in the $2p \le (1-q+\sqrt{pq})^2$ case*
$$\widehat{\rho}(P) = r + 2\sqrt{pq}, \tag{4.8}$$
  (b) *in the $2p > (1-q+\sqrt{pq})^2$ case, setting $a_1 := p - \sqrt{pq} - \sqrt{r(r+2\sqrt{pq})}$,*
$$\widehat{\rho}(P) = \left| a + \frac{p(1-a)}{a-1+q} \right| \quad \text{when } a \in (0, a_1], \tag{4.9a}$$
$$\widehat{\rho}(P) = r + 2\sqrt{pq} \quad \text{when } a \in [a_1, a_0). \tag{4.9b}$$

When $r := 0$, such results have been obtained in [1], [10], and [14] using various methods involving conditions on a (see the end of the introduction). Let us specify the above formulae in the case $r := 0$. We have $a_0 = a_1 = p - \sqrt{pq} = (p-q)/(1+\sqrt{q/p})$, and it can be easily checked that $2p > (1-q+\sqrt{pq})^2$. Then properties (4.7), (4.9a), and (4.9b) can be rewritten as $\widehat{\rho}(P) = (pq + (a-p)^2)/|a-p|$ when $a \in (0, a_0]$, and $\widehat{\rho}(P) = 2\sqrt{pq}$ when $a \in (a_0, 1)$.

Proof of Proposition 4.1. We apply the elimination procedure of Section 4. Then $\Lambda := \{\lambda \in \mathbb{C} \colon \widehat{\delta} < |\lambda| < 1\}$ with $\widehat{\delta} := r + 2\sqrt{pq}$. The characteristic polynomial $E_\lambda(\cdot)$ is
$$E_\lambda(z) := qz^2 + (r-\lambda)z + p.$$
A simple study of the graph of $\phi(t) := p/t + r + qt$ on $\mathbb{R} \setminus \{0\}$ shows that, for any $\lambda \in (\widehat{\delta}, 1)$, the equation $\phi(z) = \lambda$ (i.e. $E_\lambda(z) = 0$) admits a solution in $(1, \widehat{\gamma})$ and another solution in $(\widehat{\gamma}, +\infty)$, so $N(\lambda) = 1$. It follows from Proposition 3.3 that $\eta = 1$. Thus, the linear elimination used in Corollary 4.1 is here trivial. Indeed, a necessary condition for $f := \{z^n\}_{n\in\mathbb{N}}$ to satisfy $Pf = \lambda f$ is obtained by eliminating the variable z with respect to the boundary equation $(Pf)(0) = \lambda f(0)$, namely, $P_0(\lambda, z) := a + (1-a)z = \lambda$, and the equation $E_\lambda(z) = 0$. This leads to
$$P_1(\lambda, z) := \mathrm{Res}_z(P_0, E_\lambda) = (1-\lambda)[(\lambda-a)(1-a-q) + p(1-a)]. \tag{4.10}$$
In the special case $a = 1-q$, the only solution of (4.10) is $\lambda = 1$. Corollary 4.2 then gives $\widehat{\rho}(P) = r + 2\sqrt{pq}$.

Now assume that $a \neq 1-q$. Then $\lambda = 1$ is a solution of (4.10) and the other solution of (4.10), say $\lambda(a)$, and the associated complex number, say $z(a)$, are given by the following formulae (use $a + (1-a)z = \lambda$ to obtain $z(a)$):
$$\lambda(a) := a + \frac{p(1-a)}{a-1+q} \in \mathbb{R} \quad \text{and} \quad z(a) := \frac{p}{a+q-1} \in \mathbb{R}.$$
To apply Corollary 4.2, we must find the values $a \in (0,1)$ for which both conditions $\widehat{\delta} < |\lambda(a)| < 1$ and $|z(a)| \le \widehat{\gamma}$ hold. Observe that
$$|z(a)| \le \widehat{\gamma} \iff |a-1+q| \ge \sqrt{pq}.$$
Hence, if $a \in (a_0, 1)$ (recall that $a_0 := 1-q-\sqrt{pq}$) then $|z(a)| > \widehat{\gamma}$. This gives (4.7).

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Now let $a \in (0, a_0]$. Then $|z(a)| \le \widehat{\gamma}$. Let us study $\lambda(a)$. We have $\lambda'(a) = 1 - pq/(a - 1 + q)^2$, so $a \mapsto \lambda(a)$ is increasing on $(-\infty, a_0]$ from $-\infty$ to $\lambda(a_0) = r - 2\sqrt{pq}$. Thus,

$$\lambda(a) \le r - 2\sqrt{pq} < r + 2\sqrt{pq} \quad \text{for all } a \in (0, a_0],$$

and the equation $\lambda(a) = -(r + 2\sqrt{pq})$ has a unique solution $a_1 \in (-\infty, a_0)$. Note that $a_1 < a_0$ and $\lambda(a_1) = -(r + 2\sqrt{pq})$, that $\lambda(0) = p/(q-1) \in [-1, 0)$, and finally that

$$\lambda(0) - \lambda(a_1) = \frac{p}{q-1} + r + 2\sqrt{pq} = \frac{(q - \sqrt{pq} - 1)^2 - 2p}{1-q}.$$

When $2p \le (1 - q + \sqrt{pq})^2$, we obtain (4.8). Indeed, $|\lambda(a)| < r + 2\sqrt{pq}$ since

$$-(r + 2\sqrt{pq}) = \lambda(a_1) \le \lambda(0) < \lambda(a) < r + 2\sqrt{pq} \quad \text{for all } a \in (0, a_0].$$

When $2p > (1 - q + \sqrt{pq})^2$, we have $a_1 \in (0, a_0]$ and the following statements hold.

- If $a \in (0, a_1)$ then (4.9a) holds. Indeed, $r + 2\sqrt{pq} < |\lambda(a)| < 1$ since

$$-1 \le \lambda(0) < \lambda(a) < \lambda(a_1) = -(r + 2\sqrt{pq}) \quad \text{for all } a \in (0, a_1].$$

- If $a \in [a_1, a_0]$ then (4.9b) holds. Indeed, $|\lambda(a)| < r + 2\sqrt{pq}$ since

$$-(r + 2\sqrt{pq}) = \lambda(a_1) \le \lambda(a) < r + 2\sqrt{pq}.$$

Remark 4.2. (*Discussion on the $\ell^2(\pi)$-spectral gap and the decay parameter.*) As mentioned in the introduction, we are not concerned with the usual $\ell^2(\pi)$-spectral gap $\rho_2(P)$ for birth-and-death Markov chains (BDMCs). In particular, we cannot compare our results with those of [16]. To give a comprehensive discussion on [16], let P be a kernel of a BDMC defined by (4.6) with invariant probability measure π. Here P is reversible with respect to π. It can be proved that the decay parameter of P, denoted by γ in [16] but by γ_{DS} here to avoid confusion with our parameter γ, is also the rate of convergence $\rho_2(P)$:

$$\gamma_{DS} = \rho_2(P) := \lim_n \|P^n - \Pi\|_2{}^{1/n}.$$

Here $\Pi f := \pi(f)\, \mathbf{1}_{\mathbb{X}}$ and $\|\cdot\|_2$ denotes the operator norm on $\ell^2(\pi)$. When P is assumed to be $V_{\widehat{\gamma}}$-geometrically ergodic with $V := \{\widehat{\gamma}^n\}_{n\in\mathbb{N}}$, it follows from [1, Theorem 6.1] that

$$\gamma_{SD} \le \widehat{\rho}(P).$$

Consequently, the bounds of the decay parameter γ_{DS} given in [16] cannot provide bounds for $\widehat{\rho}(P)$ since the converse inequality $\widehat{\rho}(P) \le \gamma_{DS}$ is not known to the best of our knowledge. Moreover, even if the equality $\gamma_{DS} = \widehat{\rho}(P)$ was true, the bounds obtained in our Proposition 4.1 could be derived from [16] only for some specific values of $P(0, 0)$. Indeed, the difficulty in [16, pp. 139–140] with covering all the values $P(0, 0) \in (0, 1)$ is that the spectral measure associated with Karlin and McGregor polynomials cannot be easily computed, except for some specific values of $P(0, 0)$ (see [9] for a recent contribution).

4.2. A nonreversible case: RWs with $g = 2$ and $d = 1$

Let $P := (P(i, j))_{(i,j)\in\mathbb{N}^2}$ be defined by

$$\begin{aligned} P(0,0) &= a \in (0,1), \qquad & P(0,1) &= 1-a, \\ P(1,0) &= b \in (0,1), \qquad & P(1,2) &= 1-b, \end{aligned} \tag{4.11}$$

and for all $n \geq 2$ by

$$P(n, n-2) = a_{-2} > 0, \qquad P(n, n-1) = a_{-1},$$
$$P(n, n) = a_0, \qquad P(n, n+1) = a_1 > 0.$$

The form of boundary probabilities in (4.11) is chosen for convenience. Other (finitely many) boundary probabilities could be considered provided that P is irreducible and aperiodic. To illustrate the procedure proposed in Section 4 for this class of RWs, we also specify the numerical values

$$a_{-2} := \tfrac{1}{2}, \qquad a_{-1} := \tfrac{1}{3}, \qquad a_0 = 0, \qquad a_1 := \tfrac{1}{6}.$$

The procedure could be developed in the same way for any other values of $(a_{-2}, a_{-1}, a_0, a_1)$ satisfying $a_{-2}, a_1 > 0$ and condition (NERI) i.e. $a_1 < 2a_{-2} + a_{-1}$. Here we have

$$\phi(t) := \frac{1}{2t^2} + \frac{1}{3t} + \frac{t}{6} = 1 + \frac{1}{6t^2}(t-1)(t^2 - 5t - 3).$$

The function $\phi(\cdot)$ has a minimum over $(1, +\infty)$ at $\widehat{\gamma} \approx 2.18$, with $\widehat{\delta} := \phi(\widehat{\gamma}) \approx 0.621$. Let $V_{\widehat{\gamma}} := \{\widehat{\gamma}^n\}_{n\in\mathbb{N}}$, and let $\mathcal{B}_{\widehat{\gamma}}$ be the associated weighted space. We know from Proposition 3.2 and from irreducibility and aperiodicity properties that $\widehat{r}_{\mathrm{ess}}(P) = \widehat{\delta}$ and P is $V_{\widehat{\gamma}}$-geometrically ergodic (see Proposition 2.1). The polynomial $E_\lambda(\cdot)$ is given by

$$E_\lambda(z) := \frac{z^3}{6} - \lambda z^2 + \frac{z}{3} + \frac{1}{2} \quad \text{for all } z \in \mathbb{C}.$$

A simple examination of the graph of $\phi(\cdot)$ shows that $\eta = 2$. Thus, the set $\mathcal{M}$ of Corollary 4.2 is $\mathcal{M} := \{(1, 1), (2)\}$. Next, the constructive proof of Corollary 4.1 provides the following procedure to compute $\widehat{\rho}(P)$ (see also Remark 4.1 in the second case). Recall that $\Lambda := \{\lambda \in \mathbb{C} \colon \widehat{\delta} < |\lambda| < 1\}$.

First case: $\mu = (1, 1)$. (a) When $\lambda \in \Lambda$ is such that $\mathcal{E}_\lambda^-$ is composed of two simple roots of $E_\lambda(\cdot)$, a necessary condition for λ to be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ is that

$$R_1(\lambda) := \mathrm{Res}_{z_1}(P_1, E_\lambda) = 0,$$

where

$$P_1(\lambda, z_1) := \mathrm{Res}_{z_2}(P_0, E_\lambda) = \begin{vmatrix} \frac{1}{6} & 0 & A(\lambda, z_1) & 0 & 0 \\ -\lambda & \frac{1}{6} & B(\lambda, z_1) & A(\lambda, z_1) & 0 \\ \frac{1}{3} & -\lambda & C(\lambda, z_1) & B(\lambda, z_1) & A(\lambda, z_1) \\ \frac{1}{2} & \frac{1}{3} & 0 & C(\lambda, z_1) & B(\lambda, z_1) \\ 0 & \frac{1}{2} & 0 & 0 & C(\lambda, z_1) \end{vmatrix}$$

and $P_0(\lambda, z_1, z_2) := A(\lambda, z_1){z_2}^2 + B(\lambda, z_1)z_2 + C(\lambda, z_1)$ is given by

$$P_0(\lambda, z_1, z_2) := \begin{vmatrix} (1-a) & a + (1-a)z_2 - \lambda \\ (1-b)(z_1 + z_2) - \lambda & b + (1-b)z_2^2 - \lambda z_2 \end{vmatrix}. \tag{4.12}$$

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We derive $P_0(\lambda, z_1, z_2)$ from (4.4) with

$$
\begin{aligned}
P_{0,1}(\lambda, z_1, z_2) &:= \begin{vmatrix} a+(1-a)z_1-\lambda & a+(1-a)z_2-\lambda \\ b+(1-b)z_1^2-\lambda z_1 & b+(1-b)z_2^2-\lambda z_2 \end{vmatrix} \\
&= (z_1-z_2)P_0(\lambda, z_1, z_2).
\end{aligned}
$$

(b) *Sufficient part.* Consider

$$\Lambda_{(1,1)} = \mathrm{Root}(R_1) \cap \Lambda = \mathrm{Root}(R_1) \cap \{\lambda \in \mathbb{C}\colon 0.621 \approx \widehat{\delta} < |\lambda| < 1\}.$$

For every $\lambda \in \Lambda_{(1,1)}$,

(i) check that $E_\lambda(z) = 0$ has two simple roots z_1 and z_2 such that $|z_i| < \widehat{\gamma} \approx 2.18$,

(ii) if (i) is satisfied, then test if $P_0(\lambda, z_1, z_2) = 0$ with P_0 given in (4.12).

If (i) and (ii) are satisfied, then λ is an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$.

Second case: $\mu = (2)$. (a) When $\lambda \in \Lambda$ is such that $\mathcal{E}_\lambda^-$ is composed of a double root of $E_\lambda(\cdot)$, a necessary condition for λ to be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ is that (see Remark 4.1)

$$Q(\lambda) = 0 \quad \text{and} \quad R_2(\lambda) := \mathrm{Res}_{z_1}(P_1, E_\lambda) = 0,$$

where

$$Q(\lambda) := \begin{vmatrix} \frac{1}{6} & 0 & \frac{1}{2} & 0 & 0 \\ -\lambda & \frac{1}{6} & -2\lambda & \frac{1}{2} & 0 \\ \frac{1}{3} & -\lambda & \frac{1}{3} & -2\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{3} & -2\lambda \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{3} \end{vmatrix}$$

and

$$P_1(\lambda) := \mathrm{Res}_{z_1}(P_0, E_\lambda) = \begin{vmatrix} \frac{1}{6} & 0 & A(\lambda) & 0 & 0 \\ -\lambda & \frac{1}{6} & B(\lambda) & A(\lambda) & 0 \\ \frac{1}{3} & -\lambda & C(\lambda) & B(\lambda) & A(\lambda) \\ \frac{1}{2} & \frac{1}{3} & 0 & C(\lambda) & B(\lambda) \\ 0 & \frac{1}{2} & 0 & 0 & C(\lambda) \end{vmatrix},$$

where $P_0(\lambda, z_1) := A(\lambda)z_1^2 + B(\lambda)z_1 + C(\lambda)$ is given by

$$P_0(\lambda, z_1) := \begin{vmatrix} a+(1-a)z_1-\lambda & 1-a \\ b+(1-b)z_1^2-\lambda z_1 & 2(1-b)z_1-\lambda \end{vmatrix}. \tag{4.13}$$

(b) *Sufficient part.* Consider

$$\Lambda'_{(2)} = \mathrm{Root}(Q) \cap \Lambda_{(2)} = \mathrm{Root}(Q) \cap \mathrm{Root}(R_2) \cap \{\lambda \in \mathbb{C}\colon 0.621 \approx \widehat{\delta} < |\lambda| < 1\}.$$

For every $\lambda \in \Lambda'_{(2)}$,

(i) check that equation $E_\lambda(z) = 0$ has a double root z_1 such that $|z_1| < \widehat{\gamma} \approx 2.18$,

(ii) if (i) is satisfied, then test if $P_0(\lambda, z_1) = 0$ with P_0 given in (4.13).

If (i) and (ii) are satisfied, then λ is an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$.

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TABLE 1: Convergence rates with different values of the boundary transition probabilities (a, b).

(a, b)	$\Lambda_{(1,1)}$	$\mathcal{Z}_{(1,1)}$	$\Lambda'_{(2)}$	$\mathcal{Z}_{(2)}$	$\widehat{\delta}$	$\widehat{\rho}(P)$
$(\frac{1}{2}, \frac{1}{2})$	$-0.625 \pm 0.466i$	$\varnothing$	$\varnothing$	$\varnothing$	0.621	0.621
	$-0.798, 0.804$	$\varnothing$	$\varnothing$	$\varnothing$	0.621	0.621
$(\frac{1}{10}, \frac{1}{10})$	$-0.681 \pm 0.610i$	$-0.466 \pm -0.506i$	$\varnothing$	$\varnothing$	0.621	0.688
	$-0.466 \pm 0.506i$	$-0.466 \pm -0.506i$	$\varnothing$	$\varnothing$	0.621	0.688
	$-0.384 \pm 0.555i$	$-0.466 \pm -0.506i$	$\varnothing$	$\varnothing$	0.621	0.688
$(\frac{1}{50}, \frac{1}{50})$	$-0.598 \pm 0.614i$	$-0.493 \pm 0.574i$	$\varnothing$	$\varnothing$	0.621	0.757
	$-0.383 \pm 0.542i$	$-0.493 \pm 0.574i$	$\varnothing$	$\varnothing$	0.621	0.757
	$-0.493 \pm 0.574i$	$-0.493 \pm 0.574i$	$\varnothing$	$\varnothing$	0.621	0.757
	$-0.477 \pm 0.584i$	$-0.493 \pm 0.574i$	$\varnothing$	$\varnothing$	0.621	0.757
	0.994	$-0.493 \pm 0.574i$	$\varnothing$	$\varnothing$	0.621	0.757

Final results. Define $\mathcal{Z}_{(1,1)}$ as the set of all the $\lambda \in \Lambda_{(1,1)}$ satisfying (i)–(ii) in the first case, and define $\mathcal{Z}_{(2)}$ as the set of all the $\lambda \in \Lambda'_{(2)}$ satisfying (i)–(ii) in the second case. Finally, set $\mathcal{Z} := \mathcal{Z}_{(1,1)} \cup \mathcal{Z}_{(2)}$. Then

$$\widehat{\rho}(P) = \max(\widehat{\delta}, \max\{|\lambda|,\ \lambda \in \mathcal{Z}\}).$$

The results (obtained using MAPLE®) for different values of the boundary transition probabilities are reported in Table 1. In these specific examples, note that $\Lambda'_{(2)}$ is always the empty set. As expected, we obtain $\rho_{\widehat{\gamma}}(P) \nearrow 1$ when $(a, b) \to (0, 0)$.

5. Convergence rate for RWs with unbounded increments

In this subsection we propose two instances of RWs on $\mathbb{X} := \mathbb{N}$ with unbounded increments for which estimates of the convergence rate with respect to some weighted-supremum space $\mathcal{B}_V$ can be obtained using Proposition 3.1 and Proposition 2.1. The first example is from [11]. The second example is a reversible transition kernel P inspired from the 'infinite star' example in [15]. Note that, using a result of [1] (see Remark 4.2), estimates of $\rho_V(P)$ with respect to $\mathcal{B}_V$ may be useful to obtain estimates on the usual spectral gap $\rho_2(P)$ with respect to the Lebesgue space $\ell^2(\pi)$. Recall that the converse is not true in general.

5.1. A nonreversible RW with unbounded increments [11]

Let P be defined for $n \geq 1$ by

$$P(0, n) := q_n, \qquad P(n, 0) := p, \qquad P(n, n+1) := q = 1 - p,$$

with $p \in (0, 1)$ and $q_n \in [0, 1]$ such that $\sum_{n\geq 1} q_n = 1$.

Proposition 5.1. *Assume that $\gamma \in (1, 1/q)$ such that $\sum_{n\geq 1} q_n\gamma^n < \infty$. Then $r_{\text{ess}}(P) \leq q\gamma$. Moreover, P is V_γ-geometrically ergodic with convergence rate $\rho_{V_\gamma}(P) \leq \max(q\gamma, p)$.*

Proof. We have

$$(PV_\gamma)(n) = q\gamma^{n+1} + p$$

for all $n \geq 1$. Thus, if $\gamma \in (1, 1/q)$ and

$$\sum_{n\geq 1} q_n\gamma^n < \infty,$$

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then condition (WD) holds with V_γ, and we have $\delta_{V_\gamma}(P) \le q\gamma$. Therefore, it follows from Proposition 3.1 that $r_{\mathrm{ess}}(P) \le q\gamma$. Now Proposition 2.1 is applied with any $r_0 > \max(q\gamma, p)$. Let $\lambda \in \mathbb{C}$ be such that $\max(q\gamma, p) < |\lambda| \le 1$, and let $f \in \mathcal{B}_\gamma$, $f \neq 0$, be such that $Pf = \lambda f$. We obtain

$$f(n) = (\lambda/q)f(n-1) - pf(0)/q$$

for any $n \ge 2$, so that

$$f(n) = \left(\frac{\lambda}{q}\right)^{n-1}\left(f(1) - \frac{pf(0)}{\lambda - q}\right) + \frac{pf(0)}{\lambda - q} \quad \text{for all } n \ge 2.$$

Since $f \in \mathcal{B}_{V_\gamma}$ and $|\lambda|/q > \gamma$, we obtain

$$f(1) = pf(0)/(\lambda - q),$$

and, consequently,

$$f(n) = pf(0)/(\lambda - q)$$

for all $n \ge 1$. Next the equality

$$\lambda f(0) = (Pf)(0) = \sum_{n \ge 1 q_n f(n)}$$

gives:

$$\lambda f(0) = pf(0)/(\lambda - q)$$

since

$$\sum_{n \ge 1} q_n = 1.$$

We have $f(0) \neq 0$ since we look for $f \neq 0$. Thus, λ satisfies $\lambda^2 - q\lambda - p = 0$, that is, $\lambda = 1$ or $\lambda = -p$. The case $\lambda = -p$ has not to be considered since $|\lambda| > \max(q\gamma, p)$. If $\lambda = 1$ then $f(n) = f(0)$ for any $n \in \mathbb{N}$, so $\lambda = 1$ is a simple eigenvalue of P on $\mathcal{B}_\gamma$ and is the only eigenvalue such that $\max(q\gamma, p) < |\lambda| \le 1$. Then Proposition 2.1 gives the second conclusion of Proposition 5.1.

Note that p cannot be dropped in the inequality $\rho_{V_\gamma}(P) \le \max(q\gamma, p)$ since $\lambda = -p$ is an eigenvalue of P on $\mathcal{B}_\gamma$ with corresponding eigenvector $f_p := (1, -p, -p, \ldots)$.

5.2. A reversible RW inspired from [15]

Let $\{\pi_n\}_{n\in\mathbb{N}}$ be a probability distribution (with $\pi_n > 0$ for every $n \in \mathbb{N}$), and let P be defined by

$$P(0, n) = \pi_n \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad P(n, 0) = \pi_0, \quad P(n, n) = 1 - \pi_0, \quad \text{for all } n \ge 1.$$

It is easily checked that P is reversible with respect to $\{\pi_n\}_{n\in\mathbb{N}}$, so $\{\pi_n\}_{n\in\mathbb{N}}$ is an invariant probability distribution of P.

Proposition 5.2. *Assume that there exists $V \in [1, +\infty)^{\mathbb{N}}$ such that $V(0) = 1$, $V(n) \to +\infty$ as $n \to +\infty$, and $\pi(V) := \sum_{n\ge 0} \pi_n V(n) < \infty$. Then P is V-geometrically ergodic with $\rho_V(P) \le 1 - \pi_0$.*

It can be checked that P is not stochastically monotone, so the estimate $\rho_V \leq 1 - \pi_0$ cannot be directly deduced from [10].

Proof of Proposition 5.2. From $(PV)(0) = \pi(V)$ and $(PV)(n) = \pi_0 V(0) + (1-\pi_0)V(n)$ for all $n \geq 1$, it follows that

$$PV \leq (1-\pi_0)V + (\pi(V) + \pi_0)\,\mathbf{1}_{\mathbb{X}}\,.$$

That is, condition (WD) holds with $N := 1$, $\delta := 1 - \pi_0$, and $d := \pi(V) + \pi_0$. The inequality $r_{\mathrm{ess}}(P) \leq 1 - \pi_0$ is deduced from Proposition 3.1.

Let $\lambda \in \mathbb{C}$ be an eigenvalue of P, and let $f := \{f(n)\}_{n\in\mathbb{N}}$ be a nontrivial associated eigenvector. Then

$$\lambda f(0) = \sum_{n=0}^{+\infty} \pi_n f(n) \quad \text{and} \quad \lambda f(n) = \pi_0 f(0) + (1-\pi_0) f(n) \quad \text{for all } n \geq 1. \tag{5.1}$$

This gives $f(n) = f(0)\pi_0/(\lambda - 1 + \pi_0)$ for all $n \geq 1$. Since $f \neq 0$, it follows from the first equality in (5.1) that

$$\lambda = \pi_0 + \frac{\pi_0}{\lambda - 1 + \pi_0}(1 - \pi_0),$$

which is equivalent to $\lambda^2 - \lambda = 0$. Thus, $\lambda = 1$ or 0. That 1 is a simple eigenvalue is standard from the irreducibility of P. The result follows from Proposition 2.1.

A specific instance of this model is considered in [15, p. 68]. Let $\{w_n\}_{n\geq 1}$ be a sequence of positive scalars such that $\sum_{n\geq 1} w_n = \frac{1}{2}$. Then P is given by

$$P(n,n) = \tfrac{1}{2} \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad P(0,n) = w_n, \quad P(n,0) = \tfrac{1}{2}, \quad \text{for all } n \geq 1,$$

which is reversible with respect to its invariant probability distribution π defined by $\pi_0 := \frac{1}{2}$ and $\pi_n := w_n$ for $n \geq 1$. It has been proved in [15, p. 68] that, for any $X_0 \sim \alpha \in \ell^2(1/\pi)$, there exists a constant $C_{\alpha,\pi} > 0$ such that

$$\|\alpha P^n - \pi\|_{\mathrm{TV}} \leq C_{\alpha,\pi}\left(\tfrac{3}{4}\right)^n, \tag{5.2}$$

where $\|\cdot\|_{\mathrm{TV}}$ is the total variation distance. Since we know that $\rho_2(P) \leq \rho_V(P)$ from [1] and $\rho_V(P) \leq \frac{1}{2}$ from Proposition 5.2, the rate of convergence in (5.2) is improved.

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