Canad. Math. Bull. Vol. 32 (1), 1989

CAUCHY COMPLETION CATEGORIES

BY

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ABSTRACT. A general procedure is introduced for identifying categories of Cauchy spaces which have completions which possess a certain convergence property P, and for constructing completion functors on such categories. These results are applied to obtain a characterization of Cauchy spaces which allow T_3 completions, and to construct T_3 completions for categories of Cauchy groups and Cauchy lattices.

Introduction. In an earlier paper [3], R. Fric and one of the authors introduced a method for constructing Cauchy completion categories and Cauchy space completion functors preserving a certain property P by composing a modification functor derived from P with the Wyler completion functor. A similar procedure is employed here, but with improvements that lead to more general and useful results. The property P employed here is a convergence rather than a Cauchy property, and is subjected to fewer restrictions. Our present theory is applicable to a broader class of Cauchy space categories, including Cauchy groups and lattices, and the completion categories on which the completion functors are constructed are more explicitly characterized.

The first section gives the general theory for determining the completion subcategory $\mathbf{S}_P$ of a given Cauchy category $\mathbf{S}$ relative to a certain property P, and simultaneously gives the construction for the associated completion functor. In the second section, the only property considered for P is regularity, while we allow $\mathbf{S}$ to be the categories Cauchy spaces, Cauchy groups, Cauchy lattices, and Cauchy l-groups, with the appropriate morphisms in each case. First we explicitly characterize the Cauchy spaces and Cauchy groups that allow T_3 completions within their own categories by simply applying the results of Section 1 in these special cases. Then, for Cauchy groups, we go further to show that a Cauchy group has a T_3 Cauchy group completion whenever the associated Cauchy space has a T_3 Cauchy space completion. The final theorem asserts that this result is also valid if "Cauchy group" is replaced by "Cauchy lattice" or "Cauchy l-group."

1. **Completion Categories and Functors.** We define a *Cauchy space* in accordance with the axioms of H. H. Keller [5]; these may be found in almost any of

Received by the editors March 9, 1987.

Key Words and Phrases Cauchy space, Cauchy group, Cauchy lattice, completion functor, Cauchy category.

AMS Subject Classification (1980): 54A20, 54E15, 28A35

https://doi.org/10.4153/CMB-1989-011-9 Published online by Cambridge University Press

our references. We consider a *convergence space* to be synonymous with a *complete Cauchy space*, that is, one in which every Cauchy filter converges.

If (X,C) is a T_2 Cauchy space, recall that two Cauchy filters F,G are *equivalent* iff $F\cap G\in C$. Let $[F]=\{G\in C : F\cap G\in C\}$ be the Cauchy equivalence class containing a given Cauchy filter F, let $X^*=\{[F] : F\in C\}$ and let $j_X : X\longrightarrow X^*$ be the natural mapping given by $j_X(x)=[\dot{x}]$, for all $x\in X$, where $\dot{x}$ designates the fixed ultrafilter generated by x. A *completion* $((Y,D),\phi)$ of a Cauchy space (X,C) consists of a T_2 complete Cauchy space (Y,D), and a Cauchy-embedding $\phi : (X,C)\longrightarrow(Y,D)$ such that $\phi(X)$ is dense in Y. If $((Y,D),\phi)$ is a completion of (X,C) such that $Y=X^*, j_X=\phi$, and $jF\cap[\dot{F}]\in C$ whenever $F\in C$, then the completion is said to be in *standard form*. Reed [9], has shown that every Cauchy completion is equivalent to one in standard form.

For any T_2 Cauchy space (X,C), the *Wyler completion* $((X^*,C^*),j_X)$ is constructed (in standard form) by letting C^* be all filters on X^* finer than a filter of the form $j_X(F)\cap[\dot{F}]$ for some $F\in C$, where $[\dot{F}]$ is the fixed ultrafilter on X^* generated by $[F]$.

The term *Cauchy category* will be used to describe any category **S** in which the objects are Cauchy spaces and the morphisms are Cauchy-continuous maps. The category CHY of all Cauchy spaces and all Cauchy continuous maps is the most obvious example of a Cauchy category; in the next section we shall discuss Cauchy categories in which the objects are Cauchy groups (or Cauchy lattices) and the morphisms are Cauchy continuous group (or lattice) homomorphisms. For any Cauchy category, T_2**S** will denote the full subcategory of all T_2 objects in **S**, and **S*** will denote the full subcategory of all complete objects in **S**. In particular, T_2**S*** is the full subcategory of all T_2, complete objects in **S**.

A Cauchy category **S** is defined to be a *Cauchy completion category* if all objects in **S** are T_2, and there is a reflector $\mathcal{F} : \mathbf{S}\longrightarrow\mathbf{S}^*$ such that, for each $(X,C)\in\mathbf{S}, (\mathcal{F}(X,C),\phi_{(X,C)})$ is a completion of (X,C), where the Cauchy embedding map $\phi_{(X,C)} : (X,C)\longrightarrow\mathcal{F}(X,C)$ is a morphism in **S**. The reflector $\mathcal{F}$ is called the *completion functor* associated with **S**, and is unique up to equivalence. A completion functor is in *standard form* if $(\mathcal{F}(X,C),\phi_{(X,C)})$ is a completion of (X,C) in standard form for each $(X,C)\in\mathbf{S}$. The Wyler completion functor (see [3],[10]), defined by $\mathcal{F}(X,C)=(X^*,C^*)$, is the unique completion functor in standard form on the Cauchy completion category T_2CHY.

Next, let **S** be a Cauchy category. A convergence space property P is defined to be **S**-*admissible* if the following conditions are satisfied:

(1) For each $(X,C)\in\mathbf{S}^*$, there is a finest complete Cauchy structure C_p on X coarser than C such that (X,C_p) is in **S*** and has property P;

(2) If $f : (X,C)\longrightarrow(Y,D)$ is a morphism in **S***, then $f : (X,C_p)\longrightarrow(Y,D_p)$ is also a morphism in **S***.

If P is an **S**-admissible property, then $(X,C)\in\mathbf{S}$ is defined to be a $P_{\mathbf{S}}$-*Cauchy space* if (X,C) is T_2 and, for each filter $F\notin C$, there is a $(Y,D)\in T_2\mathbf{S}^*$ which has property P and an **S**-morphism $f : (X,C)\longrightarrow(Y,D)$ such that $f(F)\notin D$.

https://doi.org/10.4153/CMB-1989-011-9 Published online by Cambridge University Press

LEMMA 1.1. *Let* **S** *be a Cauchy category and P an* **S***-admissible Cauchy property. Then* $(X,C) \in \mathbf{S}^*$ *is a* $P_{\mathbf{S}}$*-Cauchy space iff* (X,C) *is* T_2 *and has property P.*

PROOF. It is obvious that a complete, T_2 space in **S** with property P is a $P_{\mathbf{S}}$-Cauchy space. To prove the converse, we need only show that a complete $P_{\mathbf{S}}$-Cauchy space has property P. If (X,C) does not have property P, then $C \neq C_p$, and so there is an $F \in C_p$ such that $F \notin C_p$ and a space (Y,D) in $\mathbf{S}^*$ with property P and an **S**-morphism $f : (X,C) \longrightarrow (Y,D)$ such that $f(F) \notin D$. But $D_p = D$, and by the definition of **S**-admissibility, $f : (X,C_p) \longrightarrow (Y,D)$ is an **S**-morphism. But $F \in C_p$ and $f(F) \in D$ contradicts the Cauchy-continuity of f. □

Let **S** be a Cauchy category, and let P again be an **S**-admissible property. From the definition of **S**-admissibility, it follows that $\mathcal{P} : \mathbf{S}^* \longrightarrow \mathbf{S}^*$, defined for objects by $\mathcal{P}(X,C) = (X,C_p)$ and for morphisms by $\mathcal{P}(f) = f$, is a functor; indeed, $\mathcal{P}$ is a reflector of $\mathbf{S}^*$ onto the full subcategory consisting of complete spaces with property P.

We shall now make the further assumption that $T_2\mathbf{S}$ is a Cauchy completion category, with completion functor $\mathcal{F} : T_2\mathbf{S} \longrightarrow T_2\mathbf{S}^*$ in standard form. Let $\mathbf{S}_p$ be the full subcategory of **S** (indeed, of $T_2\mathbf{S}$) consisting of all $P_{\mathbf{S}}$-Cauchy spaces. We shall examine the composite functor $\mathcal{P} \circ \mathcal{F}$, restricted to $\mathbf{S}_p$.

LEMMA 1.2. *Under the assumptions of the preceding paragraph, if* $(X,C) \in \mathbf{S}_p$, *then* $\mathcal{P} \circ \mathcal{F}(X,C) \in \mathbf{S}_p^*$.

PROOF. It is clear from our construction that $\mathcal{P} \circ \mathcal{F}(X,C) \in \mathbf{S}$ and has property P, so it remains only to show that $\mathcal{P} \circ \mathcal{F}(X,C)$ is T_2. Let $\mathcal{F}(X,C) = (X^*,\tilde{C})$, and let $[F], [G]$ be distinct elements in X^*. Since $F \cap G \notin C$ and $(X,C) \in \mathbf{S}_p$, there is (Y,D) in $\mathbf{S}_p^*$ and an **S**-morphism $f : (X,C) \longrightarrow (Y,D)$ such that $f(F \cap G) = f(F) \cap f(G) \notin D$. Thus $\mathcal{F}(f) : (X^*,\tilde{C}) \longrightarrow (Y^*,\tilde{D}) = (Y,D)$ is an **S**-morphism. Since $(Y,D) \in \mathbf{S}_p^*$, $\mathcal{F}(f) : (X^*,(\tilde{C})_p) \longrightarrow (Y,D_p) = (Y,D)$ is also an **S**-morphism. Thus $\mathcal{F}(f)(j_X(F))$ and $\mathcal{F}(f)(j_X(G))$ converge to distinct elements in (Y,D), and, consequently, $\mathcal{F}(f)([F]) \neq \mathcal{F}(f)([G])$ in Y. □

THEOREM 1.3. *Let* **S** *be a Cauchy category such that* $T_2\mathbf{S}$ *is a Cauchy completion category with completion functor* $\mathcal{F}$ *in standard form. If P is an* **S***-admissible property, then* $\mathbf{S}_p$ *is a Cauchy completion category with completion functor* $\mathcal{P} \circ \mathcal{F}$.

PROOF. We shall use the notation of the proof of Lemma 1.2. It remains only to show that $(X^*,(\tilde{C})_p)$ is a completion of a given object $(X,C) \in \mathbf{S}_p$. By assumption, $j_X : (X,C) \longrightarrow (X^*,\tilde{C})$ is a Cauchy embedding, and $\mathcal{P}(j_X) = j_X : (X,C) \longrightarrow (X^*,(\tilde{C})_p)$ is Cauchy-continuous. To show that $\mathcal{P}(j_X)^{-1} = j_X^{-1}$ is also Cauchy-continuous, let F be a filter on X such that $j_X(F) \in (\tilde{C})_p$. We shall show that $F \in C$.

If $F \notin C$, we can use the fact that $(X,C) \in \mathbf{S}_p$ to obtain $(Y,D) \in \mathbf{S}_p^*$ and an S-morphism $f : (X,C) \longrightarrow (Y,D)$ such that $f(F) \notin D$. Then $\mathcal{F}(f)(j_X F) \notin D$, since $f(F) = \mathcal{F}(f)(j_X F)$. But $\mathcal{F}(f) : (X^*,(\tilde{C})_p) \longrightarrow (Y,D)$ is Cauchy-continuous, and

https://doi.org/10.4153/CMB-1989-011-9 Published online by Cambridge University Press

$j_X(F) \notin (\tilde{C})_p$ is a contradiction. We have shown that $j_X : (X,C) \longrightarrow (X^*, (\tilde{C})_p) = \mathcal{P} \circ \mathcal{F}(X,C)$ is a Cauchy-embedding, and the proof is complete. □

In case $\mathbf{S} = CHY$, the P-admissible properties include: topological, pretopological, pseudo-topological, regular, T- regular, w-regular, c-embedded, and uniformizable.

2. **Regular Completions.** In this section, we shall consider the case where P is "regularity." A Cauchy space (X,C) is *regular* if $F \in C$ implies that the closure of F (denoted by clF) is in C. A T_2, regular Cauchy space is said to be T_3. A Cauchy space with a T_3 completion will be called a C_3 Cauchy space (see [8]). From Theorem 1.3, we obtain the following characterization of C_3 Cauchy spaces.

PROPOSITION 2.1. *A T_2 Cauchy space (X,C) is C_3 iff, whenever $F \in C$, there is a complete, T_3 Cauchy space (Y,D) and a Cauchy-continuous mapping $f : (X,C) \longrightarrow (Y,D)$ such that $f(F) \notin D$.*

Regularity is also an admissible property for Cauchy groups. A *Cauchy group* $(X,\cdot,C)$ is a group $(X,\cdot)$ with a Cauchy structure C relative to which the group operations are Cauchy-continuous. The category of all Cauchy groups and all Cauchy-continuous group homomorphisms will be denoted by CHG. In [4] (where the objects in CHG are called "pre-Cauchy groups"), T_2CHG is shown to be a Cauchy completion category. The completion functor $\mathcal{F}_G : T_2CHG \longrightarrow T_2CHG^*$ is defined (in standard form) for objects as follows: $\mathcal{F}_G(X,\cdot,C) = (X^*,\cdot,C')$, where the group operation for X^* is given by $[F]\cdot[G] = [F \cdot G]$, and C' is the Cauchy structure generated by all finite products of Cauchy filters in the Wyler completion (X^*,C^*). Since regularity is admissible for CHG, we obtain the next proposition immediately from Theorem 1.2.

PROPOSITION 2.2. *A T_2 Cauchy group $(X,\cdot,C)$ has a T_3 Cauchy group completion iff, whenever $F \notin C$, there is a complete, T_3 Cauchy group $(Y,\cdot,D)$ and Cauchy-continuous group homomorphism $f : (X,\cdot,C) \longrightarrow (Y,\cdot,D)$ such that $f(F) \notin D$. The full subcategory CHG_R consisting of such objects is a completion subcategory of T_2CHG relative to the completion functor $\mathcal{R} \circ \mathcal{F}_G$, where $\mathcal{R}$ is the regular modification functor for convergence spaces.*

A simpler characterization for Cauchy groups with T_3 Cauchy group completions is obtained in Theorem 2.5. In order to prove the latter result, two lemmas are needed.

LEMMA 2.3. *If $(X,\cdot,C)$ is a Cauchy group and C_R is the finest regular Cauchy structure on X coarser than C then $(X,\cdot,C_R)$ is a Cauchy group. If C is complete, then C_R is also complete.*

PROOF. This proof will be presented in outline form, since the details are rather lengthy. In [7], it is shown that the regular modification C_R of C can be expressed as the terminal element in a descending, well-ordered chain C_{R_α} of Cauchy structures on X; this chain $\{C_{R_\alpha}\}$ is called the "regularity series" of C. Beginning with the assumption that $(X,\cdot,C)$ is a Cauchy group and using transfinite induction, one can show that $(X, \cdot, C_{R_\alpha})$ is a Cauchy group for each ordinal α, and consequently that $(X, \cdot, C_R)$ is a Cauchy group. The last assertion of the lemma is Proposition 1.3 of [7]. □

LEMMA 2.4. *Let $(X, \cdot, C)$ be a T_2 Cauchy group. Suppose that $\mathcal{F}_G$ is the completion functor on T_2CHG described above, and let $\mathcal{F}_G(X, \cdot, C) = (X^*, \cdot, C')$. Let $((X^*, C^*), j_X)$ be the Wyler completion of (X, C). Then $(C')_R = (C^*)_R$.*

PROOF. Let cl^* be the closure operator on X^* for the convergence structure C^*. Every filter in C' is finer than a finite product of filters of the form $A = (j_X(F_1) \cap [\dot{F}_1]) \cdot (j_X(F_2) \cap [\dot{F}_2]) \dots \cdot (j_X(F_k) \cap [\dot{F}_k])$. It is easy to verify that $cl^* j_X(F_1 \cdot F_2 \dots \cdot F_k) \leqq A$. Let "$cl_R$" denote the closure operator on X^* for the regular modification $(C^*)_R$ of C^*. Then, since $(C^*)_R \leqq C^*, cl_R j_X(F_1 \cdot F_2 \cdots \cdot F_k) \leqq A$, and consequently $A \in (C^*)_R$. Thus $(C^*)_R \leqq C'$, and, since $C' \leqq C^*$, it follows that $(C^*)_R = (C')_R$. □

THEOREM 2.5. *For a Cauchy group $(X, \cdot, C)$, the following conditions are equivalent.*
(1) $(X, \cdot, C)$ has a T_3 Cauchy group completion.
(2) (X, C) has a T_3 Cauchy space completion.
(3) (X, C) is a C_3 Cauchy space.

PROOF. It is obvious that (1) ⇒ (2), and (2) and (3) are equivalent by definition. Assuming (2), it follows that $((X^*, (C^*)_R), j_X)$ is a T_3 completion of (X, C). Since $(X, \cdot, C)$ is a T_2 Cauchy group, $((X^*, \cdot, (C')_R), j_X)$ is a T_2 Cauchy group completion by Theorem 2.5 of [4]. Thus $j_X : (X, \cdot, C) \longrightarrow (X^*, \cdot, (C')_R)$ is a homomorphism. Also, by Lemma 2.3, $(X^*, \cdot, (C')_R)$ is a regular Cauchy group, which is, in addition, T_2 by the assumption of (2). Thus $((X^*, \cdot, (C')_R), j_X)$ is a T_3 Cauchy group completion of $(X, \cdot, C)$. □

Our results concerning T_3 completions of Cauchy groups can be essentially duplicated for Cauchy lattices and l-groups. In view of the successful application of Cauchy structures in the study of completions of lattices and l-groups (see, for instance, [1] and [2]), it is appropriate to extend our results to such categories. Because the arguments needed to establish these new results are almost duplications of those already given, our presentation will be brief and non-detailed.

A *Cauchy lattice* $(X, \vee, \wedge, C)$ is a lattice $(X, \vee, \wedge)$ with a Cauchy structure C relative to which the lattice operations are Cauchy-continuous. A *lattice ordered group* (or *l-group*) $(X, \cdot, \vee, \wedge)$ is simultaneously a group and a lattice, subject to the following compatibility condition: If $x, y, z \in X$ and $x \leq y$, then $xz \leq yz$ and $zx \leq zy$. An l-group equipped with a Cauchy structure relative to which both group and lattice operations are Cauchy continuous is called a *Cauchy l-group*. An *l-group homomorphism* is a mapping between l-groups which is simultaneously a lattice and group homomorphism.

Let *CHL* denote the category of all Cauchy lattices and all Cauchy-continuous lattice homomorphisms; let *CHLG* be the category of all Cauchy l-groups and all Cauchy-continuous l-group homomorphisms. One can show that regularity is an admissible property for both of these categories. Furthermore, a completion functor $\mathcal{F}_{LG}$ is constructed for the category T_2CHLG in [6].

A completion functor $\mathcal{F}_L$ on T_2CHL is not available for easy reference in the literature, but its construction is easy. Beginning with a T_2 Cauchy lattice $(X, \vee, \wedge, C)$, let $((X^*, C^*), j_X)$ be the Wyler completion of (X, C). The lattice operations can be extended to X^* in the obvious way. Then, let $\hat{C}$ be the complete Cauchy structure on X^* generated by taking all finite meets and joins of filters in C^*. One can show that $(X^*, \vee, \wedge, \hat{C})$ is a T_2-Cauchy lattice completion of $(X, \vee, \wedge, C)$. Let $\mathcal{F}_L(X, \vee, \wedge, C) = (X^*, \vee, \wedge, \hat{C})$ and, for any morphism f in T_2CHL, let $\mathcal{F}_L(f)$ be the extension of f associated with the Wyler completion. One can show that $\mathcal{F}_L(f)$ is a CHL-morphism, and we conclude that $\mathcal{F}_L$ is a completion functor on T_2CHL.

Next, we remark that Lemmas 2.3 and 2.4 remain valid when translated from statements about Cauchy groups into corresponding statements about either Cauchy lattices or Cauchy l-groups. Therefore, the proof of each part of the next theorem is virtually the same as that of Theorem 2.5.

THEOREM 2.6. *A Cauchy lattice $(X, \vee, \wedge, C)$ has a T_3 Cauchy lattice completion iff (X, C) is a C_3 Cauchy space. A Cauchy l-group $(X, \cdot, \vee, \wedge, C)$ has a T_3 Cauchy l-group completion iff (X, C) is again a C_3 Cauchy space.*

Let $\mathcal{R}$ again denote the regular modification functor for convergence spaces, and let CHL_R and $CHLG_R$ denote the full subcategories of CHL and CHLG, respectively, whose objects are equipped with C_3 Cauchy structures.

COROLLARY 2.7. *CHL_R is a Cauchy completion category relative to the completion functor $\mathcal{F}_L$, and $CHLG_R$ is a Cauchy completion category relative to the completion functor $\mathcal{F}_{LG}$.*

Do the results of Theorems 2.5 and 2.6 generalize readily to other Cauchy categories S with completion functor $\mathcal{F}$ on T_2S and to other S-admissible properties P? Such a generalization would require the imposition of conditions on S and P relative to which generalized versions of Lemmas 2.4 and 2.5 would hold. For instance in the case of Lemma 2.4, one might require that the P-modification of a space (X, C) in S^* should necessarily agree with the P-modification of a space (X, C) in CHY^*. If P is the topological property and $S = CHG$, then our generalized version of Lemma 2.4 would fail in this case. This example suggests to us that the potential fruitfulness of the proposed generalization would not justify the effort involved.

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