10
Time-varying fields

Until this point, we have only examined magnetic effects due to steady currents or magnetic materials in stationary configurations. In this chapter, we will partially relax this constraint by considering phenomena where there are slow variations in current or magnetic flux. By slow, we mean slow enough that we can ignore all effects of electromagnetic radiation. We begin with a discussion of Faraday's law, which presents another connection between electric and magnetic phenomena. This is followed by a more detailed discussion of the energy associated with a magnetic field, including the energy loss from the hysteresis cycle in ferromagnetic materials. We find that Faraday's law leads to the production of eddy currents in some materials, while the skin effect can restrict currents to a layer near the surface. We introduce the displacement current, which finally allows us to give a complete set of Maxwell's equations for stationary media. We conclude the chapter with a brief discussion of magnetic measurements.

10.1 Faraday's law

Michael Faraday discovered that a changing magnetic flux through a wire circuit C induced a voltage in the wire.

$$V \propto \frac{d\Phi_B}{dt} \tag{10.1}$$

The changing flux could be due to changing the current in C itself, changing the current in a second, nearby circuit, moving a second circuit or permanent magnet with respect to C, or changing the shape of C. Here we will mostly consider effects due to explicit changes in the current, in which case we can replace the total time derivative in Equation 10.1 with a partial derivative. According to *Lenz's law*, the voltage induced by the changing flux is such as to induce a current that gives rise to an additional flux that opposes the original change in flux. Thus Faraday's law can be written as

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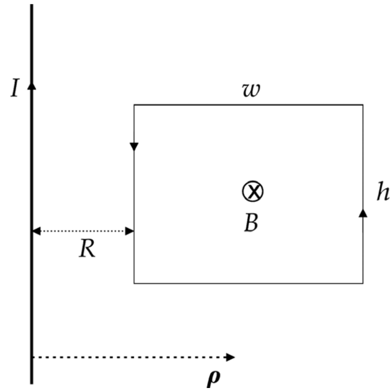

Figure 10.1 Electromotance induced in a rectangular loop.

$$\mathcal{E} = \oint \overrightarrow{E} \cdot \overrightarrow{dl} = -\frac{\partial}{\partial t} \int \overrightarrow{B} \cdot \hat{n} \, dS, \tag{10.2}$$

where E is the *electric field intensity* and the surface S is bounded by the closed circuit. The field E acts on a distance element dl in its rest frame. Because of the tangential boundary condition on E, it follows that C can refer to any closed loop in space, not just a physical circuit.[1] The line integral[1] on the left side of Equation 10.2 is called the *electromotance* ε. If the flux links a coil with N turns, the electromotance must be multiplied by N. Since the contour integral is non-zero, the induced electric field in this case is nonconservative, i.e., work is done on a charge going around the contour.

Example 10.1: ε induced in a current loop
Consider a rectangular loop near a wire with increasing current I, shown in Figure 10.1. The time-dependent field from the wire is

$$\overrightarrow{B}(t) = \frac{\mu_0}{2\pi\rho} I(t) \; \hat{\phi}.$$

The flux through the square loop is

$$\Phi_B = \frac{\mu_0 I(t)}{2\pi} h \int_R^{R+w} \frac{d\rho}{\rho}$$
$$= \frac{\mu_0 I(t)}{2\pi} h \ln\left(\frac{R+w}{R}\right)$$

[1] Historically, ε has also been referred to as an *emf* or electromotive force.

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and the electromotance is

$$\varepsilon = -\frac{\mu_0}{2\pi}\frac{dI}{dt}\,h\ln\left(\frac{R+w}{R}\right).$$

Using Stokes's theorem in Equation 10.2, we find

$$\begin{aligned}\int(\nabla\times\vec{E})\cdot\overrightarrow{dS} &= -\frac{\partial}{\partial t}\int\vec{B}\cdot\overrightarrow{dS}\\ &= -\int\frac{\partial\vec{B}}{\partial t}\cdot\overrightarrow{dS}.\end{aligned}$$

Then since the surface S is arbitrary, we find the differential form of Faraday's law is

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}. \tag{10.3}$$

This equation is valid at any point in space. We can relate this to the vector potential by

$$\begin{aligned}\nabla\times\vec{E} &= -\frac{\partial}{\partial t}(\nabla\times\vec{A})\\ &= -\nabla\times\frac{\partial\vec{A}}{\partial t},\end{aligned}$$

so that

$$\nabla\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right) = 0.$$

Since its curl vanishes, the quantity in parentheses must be the gradient of a scalar function, which we denote V_e.

$$-\nabla V_e = \vec{E}+\frac{\partial\vec{A}}{\partial t}.$$

Thus the electric field

$$\vec{E} = -\nabla V_e - \frac{\partial\vec{A}}{\partial t}. \tag{10.4}$$

can arise from static charge distributions or from time-varying magnetic fields.

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From Equation 1.32, the inductance is related to the flux by

$$L\,I = N\,\Phi_B.$$

Taking the time derivative of both sides, we find that an alternative definition of L is

$$L = -\frac{\varepsilon}{dI/dt}. \tag{10.5}$$

10.2 Energy in the magnetic field

We return now to the subject of the energy associated with a magnetic field. Consider a current element in an isolated loop together with an associated power source. Suppose that we want to increase the current in the loop from 0 up to some value I. For each step in the process of raising the current, the source must produce a voltage change

$$dV_e = -\nabla V_e \cdot \overrightarrow{dl}$$

across the current element and the source must supply the power

$$\begin{aligned} dP &= I\,dV_e \\ &= J\,dA\ \ (-\nabla V_e \cdot \overrightarrow{dl}), \end{aligned}$$

where A is the cross-sectional area of the conductor. Since J and dl are parallel, we can use Equation 10.4 to write

$$dP = \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) \cdot \vec{J}\ d\tau,$$

where, to minimize confusion, we use $d\tau$ in this section to represent the volume element. The total power provided by the source for the full loop is then [2]

$$P = \int \left(\vec{E} \cdot \vec{J} + \frac{\partial \vec{A}}{\partial t} \cdot \vec{J}\right) d\tau.$$

The first term on the right side is the power used to compensate for energy losses from heating in the conductor. The second term is the power used to set up the magnetic field associated with the increasing current, which is the subject of interest here. If we let W represent the energy stored in the magnetic field, then

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$$\frac{dW}{dt} = \int \frac{\partial \vec{A}}{\partial t} \cdot \vec{J}\, d\tau. \tag{10.6}$$

Consider a small volume element of the conductor where J can be considered constant.[2] Then we can write A as the product of J with a factor that only depends on the geometry. Thus we can assume that $\vec{A} = \alpha \vec{J}$, where α is a constant. Then substituting

$$\frac{\partial}{\partial t}(\vec{A} \cdot \vec{J}) = 2\vec{J} \cdot \frac{\partial \vec{A}}{\partial t}$$

into Equation 10.6, we find the energy stored in the magnetic field is

$$W = ½ \int \vec{J} \cdot \vec{A}\ \ d\tau. \tag{10.7}$$

If the current distribution is a current loop, we let $\vec{J}\ d\tau \rightarrow I\ \vec{dl}$ and Equation 10.7 becomes

$$W = ½\, I \int \vec{A} \cdot \vec{dl}.$$

This can be expressed in terms of the magnetic flux by

$$W = ½\ I\ \Phi_B. \tag{10.8}$$

Returning again to Equation 10.7, we can use the curl $H = J$ equation to write the energy as

$$W = ½ \int (\nabla \times \vec{H}) \cdot \vec{A}\ d\tau.$$

Using the vector identity B.4, we find

$$W = ½ \int \left[\vec{H} \cdot (\nabla \times \vec{A}) - \nabla \cdot (\vec{A} \times \vec{H})\right] d\tau.$$

Rewriting the first term in terms of B and using the divergence theorem in the second, we get

[2] J.D. Jackson, Classical Electrodynamics, Wiley, 1962, p. 176.

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$$W = \frac{1}{2} \int \vec{H} \cdot \vec{B}\, d\tau + \frac{1}{2} \int \vec{H} \times \vec{A} \cdot \hat{n}\, dS.$$

Looking at the surface integral, we know that the field from a conductor element falls off like $1/R^2$ and the vector potential falls off like $1/R$, while the surface area only grows like R^2. By evaluating at a sufficiently large distance, the second integral vanishes. Thus the energy stored in the magnetic field from conduction currents is

$$W = \frac{1}{2} \int \vec{B} \cdot \vec{H}\, d\tau \tag{10.9}$$

and the magnetic energy density in the field is

$$w_B = \frac{1}{2}\ \vec{B} \cdot \vec{H}. \tag{10.10}$$

The energy of a permeable body with magnetization M in an applied magnetic field B_a can be expressed as [3]

$$W = \frac{1}{2} \int \vec{M} \cdot \vec{B}\ \ d\tau. \tag{10.11}$$

10.3 Energy loss in hysteresis cycles

Consider a Rowland ring containing a ferromagnetic sample, as discussed in Section 2.5. If we increase the current in a conductor wound around the sample, we get an induced electromotance that opposes the change in current. The extra power expended by the source is

$$\begin{aligned} \frac{dW}{dt} &= NI\ \frac{d\Phi_B}{dt} \\ &= NIA\ \frac{dB}{dt} \\ &= \frac{NI}{l}\ Al\ \frac{dB}{dt}, \end{aligned}$$

where N is the number of conductor turns, A is the cross-sectional area of the sample, l is the mean circumference of the ring, and B is the average flux density inside the sample. Using the Ampère law, this can be written

$$\frac{dW}{dt} = HV\ \frac{dB}{dt},$$

where V is the volume of the sample.

Now consider the hysteresis loop shown in Figure 10.2. The energy supplied by the source in moving from point a to point b along the loop is

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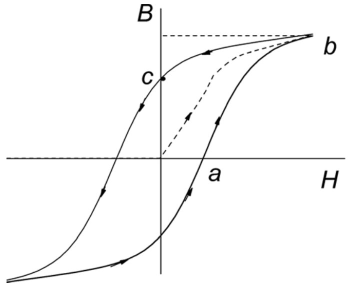

Figure 10.2 Energy loss in a hysteresis loop.

$$W_{ab} = V\int_a^b H\, dB.$$

Since dB is the independent variable, the value of this integral is the area projected on the B (vertical) axis in the figure. Going from point b to point c along the loop, I is in the same direction, but is decreasing. Thus the electromotance changes sign and some energy is returned to the source.

$$W_{bc} = -V\int_b^c H\, dB.$$

The sum of these two integrals is the area inside the hysteresis loop in the first quadrant. If we continue this analysis for a complete cycle, we find that the net energy lost in the ferromagnetic material per cycle is [4]

$$W = V \oint H\, dB. \tag{10.12}$$

This energy loss can be minimized by choosing ferromagnetic materials with a narrow hysteresis loop.

10.4 Eddy currents

Faraday's law shows that time-varying magnetic fields produce a voltage in materials such as conductors, iron, or mechanical supports. If a closed path exists inside the material, this voltage can drive currents, known as *eddy currents*.[5] The eddy currents can in turn create new magnetic fields that are superimposed over the original field. Eddy currents can be used for a number of desirable purposes, including displacement and position measurements, induction heating, magnetic shielding, levitation, and braking. On the other hand, undesirable effects from eddy currents include resistive power losses, Lorentz

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forces, multipole errors in a desired field, and a time lag in reaching an equilibrium field value.

Starting from Faraday's law

$$\nabla \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t},$$

multiplying both sides by the electrical conductivity σ and taking the curl, we find

$$\nabla \times (\nabla \times \sigma \overrightarrow{E}) = -\sigma \frac{\partial}{\partial t}(\nabla \times \overrightarrow{B}).$$

We can write Ohm's law in the form

$$\overrightarrow{J} = \sigma \overrightarrow{E}. \tag{10.13}$$

The range of current densities over which this linear relation holds depends on the material. Thus we have

$$\nabla \times (\nabla \times \overrightarrow{J}_e) = -\sigma \mu \frac{\partial}{\partial t}(\nabla \times \overrightarrow{H}),$$

where J_e is the eddy current density. We can use the vector identity B.7 on the left side of this equation and the curl $H = J$ equation on the right side to get

$$\nabla(\nabla \cdot \overrightarrow{J}_e) - \nabla^2 \overrightarrow{J}_e = -\sigma \mu \frac{\partial \overrightarrow{J}_e}{\partial t}.$$

Since the divergence term on the left side vanishes, we find that [6]

$$\nabla^2 \overrightarrow{J}_e = \sigma \mu \frac{\partial \overrightarrow{J}_e}{\partial t}. \tag{10.14}$$

This is a form of the diffusion equation. The rate of build-up of the eddy currents is controlled by the factor $\sigma\mu$.

If instead, we begin by taking the curl of the Ampère law, we find

$$\begin{aligned} \nabla \times (\nabla \times \overrightarrow{H}) &= \nabla \times \overrightarrow{J}_e \\ &= \sigma \nabla \times \overrightarrow{E}. \end{aligned}$$

Applying Equation B.7 on the left-hand side and Faraday's law on the right, we obtain the equation

$$\nabla^2 \overrightarrow{H} = \sigma \mu \frac{\partial \overrightarrow{H}}{\partial t}. \tag{10.15}$$

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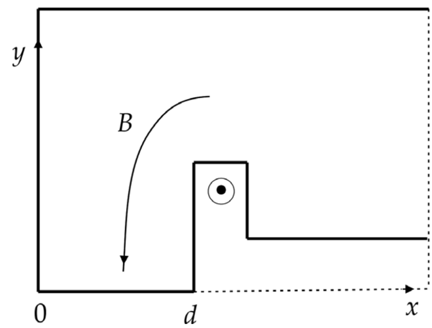

Figure 10.3 One-quarter of a symmetric *H*-dipole.

Thus the magnetic field associated with the eddy currents also satisfies a diffusion equation with the same characteristic constant. If one specifies the time dependence for *H* and the geometry of the configuration, the diffusion equation can be solved for the spatial and time dependence of the magnetic field due to eddy currents.[5] This may lead to a series of terms, each with its own characteristic time dependences.

Example 10.2: time constant for eddy currents in a solid iron core
Consider a long *H*-dipole with a solid iron yoke, as shown in Figure 10.3. For slow time changes, eddy currents can flow throughout the volume of the iron yoke surrounding the coil.[7] The magnetic flux from the eddy currents is not symmetric with the flux from the coils, which causes the iron saturation to vary with transverse position.

Consider a path through the iron yoke at the midplane in the region $0 \le x \le d$. Assuming there is no leakage flux, all of the return flux from the conductor has to pass across this path. Assume the current in the conductor is changing with time. Then the magnetic field in the vicinity of the path is in the y direction, the induced electric field is in the z direction, and on the midplane both fields are only functions of x and t.

$$\begin{aligned} \overrightarrow{B} &= B_y(x,t)\,\hat{y} \\ \overrightarrow{E} &= E_z(x,t)\,\hat{z}. \end{aligned}$$

The eddy currents flow parallel to the z axis until they reach the magnet end, where they reverse direction and flow back at the symmetric (x, y) position on the other side of the magnet. From the Ampère and Ohm's laws, we have

$$\frac{\partial B_y(x,t)}{\partial x} = \sigma\,\mu\,E_z(x,t),$$

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while from Faraday's law

$$\frac{\partial B_y(x,t)}{\partial t} = \frac{\partial E_z(x,t)}{\partial x}.$$

Applying the Laplace transform to the time variable for these two equations,[8] we get

$$\partial_x \mathcal{B}_y(x,p) = \sigma\,\mu\,\varepsilon_z(x,p) \tag{10.16}$$

and

$$p\,\mathcal{B}_y(x,p) = \partial_x\,\varepsilon_z(x,p), \tag{10.17}$$

where p is the variable conjugate to t in the Laplace transform. Taking the derivative of Equation 10.16 with respect to x and substituting Equation 10.17, we get

$$\partial_x^2 \mathcal{B}_y(x,p) = \sigma\,\mu\,p\,\mathcal{B}_y(x,p).$$

Defining $k^2 = \sigma\mu p$, the solution for the magnetic field consistent with the boundary conditions is [7]

$$\mathcal{B}_y(x,p) = \mathcal{B}_0 \cosh kx.$$

The field across the return yoke is asymmetric and is larger on the side nearer the coil. At the edge of the path closest to the conductor, we have

$$\begin{aligned} k\,d &= d\sqrt{\sigma\,\mu\,p} \\ &= \sqrt{\sigma\,\mu\,p\,d^2} \\ &= \sqrt{p\,\tau}, \end{aligned}$$

where [7]

$$\tau = \sigma\mu d^2. \tag{10.18}$$

The variable τ has the dimensions of time. It gives a characteristic time for eddy current effects in this configuration. Note that it depends quadratically on the width d of the return yoke.

Eddy currents can be suppressed by restricting the rate of change of the desired field or by constructing the magnet in such a way that potential eddy current loops are minimized. Magnet yokes are frequently constructed by assembling thin iron laminations for this reason.

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10.5 Skin effect

Consider a current density with the periodic time variation

$$\vec{J} = \vec{J}_0 \, e^{i\omega t},$$

where ω is the angular frequency and J_0 only depends on the spatial dimensions. The diffusion equation for the current density, analogous to Equation 10.14, is

$$\begin{aligned} \nabla^2 \vec{J} &= \sigma \mu \frac{\partial \vec{J}}{\partial t} \\ &= i \; \omega \sigma \mu \, \vec{J} . \end{aligned}$$

Defining $\zeta^2 = i\omega\sigma\mu$, we obtain

$$\nabla^2 \vec{J} - \zeta^2 \vec{J} = 0. \tag{10.19}$$

Now assume that the current is flowing along a conducting slab that occupies the space $y \leq 0$. Then the component of J flowing in the z direction, for example, is

$$J_z = J_{z0} \, e^{i\omega t}.$$

Applying Equation 10.19 to this, we find

$$\frac{d^2 J_{z0}}{dy^2} = \zeta^2 J_{z0}.$$

This differential equation has the solution

$$J_{z0} = J_S \, e^{-\zeta \, | \, y \, |},$$

where J_S is the spatial dependence of the current density on the surface of the slab. Using the relation

$$i = ½ \, (1 + 2 \, i - 1) = ½ \, (1 + i)^2,$$

we can write

$$\zeta = \pm \frac{1}{\sqrt{2}} \, (1 + i) \, \sqrt{\omega\sigma\mu}.$$

The boundary condition for large $| \, y \, |$ eliminates the negative solution for ζ. Thus

$$\zeta = \; (1 + i) \, \sqrt{\frac{\omega\sigma\mu}{2}}.$$

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Defining the *skin depth* as

$$\delta = \sqrt{\frac{2}{\omega\sigma\mu}}, \tag{10.20}$$

the solution for the current density is [9]

$$J_z = J_S \exp\left\{-\frac{|y|}{\delta}\right\} \exp\left\{i\left(\omega t - \frac{|y|}{\delta}\right)\right\}. \tag{10.21}$$

We see that the current density decreases exponentially with distance into the surface. In addition, there is a phase shift of the current flowing inside the material with respect to the current flowing on the surface. These effects scale with the skin depth parameter δ. For a copper conductor with current varying at 1 kHz, the skin depth is ~2.1 mm.

10.6 Displacement current

We have seen in Chapter 1 that $\nabla \cdot \vec{J} = 0$ in magnetostatic problems. However, once we allow for time variations, the conservation of charge requires

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0, \tag{10.22}$$

where ρ is the electric charge density of free (i.e., unbound) charges. Therefore, when time variation is allowed, the divergence of the conduction current density no longer needs to vanish. From electrostatics, we know that [10]

$$\nabla \cdot \vec{D} = \rho, \tag{10.23}$$

The vector D is called the *electric flux density*[3] and is related to the electric field intensity by

$$\vec{D} = \varepsilon \vec{E} \tag{10.24}$$

for linear materials, where ε is the *permittivity*. Taking the time derivative of Equation 10.23, we get

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \frac{\partial \vec{D}}{\partial t}. \tag{10.25}$$

[3] Historically, the vector D is also known as the electric displacement.

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Table 10.1 *Maxwell's equations*

Differential form	Integral form
$\nabla\cdot\overrightarrow{D} = \rho$	$\int \overrightarrow{D}\cdot\overrightarrow{dS} = \int \rho\, dV$
$\nabla\times\overrightarrow{E} = -\frac{\partial\overrightarrow{B}}{\partial t}$	$\oint \overrightarrow{E}\cdot\overrightarrow{dl} = -\int \frac{\partial\overrightarrow{B}}{\partial t}\cdot\overrightarrow{dS}$
$\nabla\cdot\overrightarrow{B} = 0$	$\int \overrightarrow{B}\cdot\overrightarrow{dS} = 0$
$\nabla\times\overrightarrow{H} = \overrightarrow{J} + \frac{\partial\overrightarrow{D}}{\partial t}$	$\oint \overrightarrow{H}\cdot\overrightarrow{dl} = \int \left(\overrightarrow{J} + \frac{\partial\overrightarrow{D}}{\partial t}\right)\cdot\overrightarrow{dS}$

Comparing Equations 10.22 and 10.25, we see that the quantity $\partial\overrightarrow{D}/\partial t$ acts like an additional kind of current. Thus we define the displacement current density as

$$\overrightarrow{J}_d = \frac{\partial\overrightarrow{D}}{\partial t}. \tag{10.26}$$

Taking this into account, the Ampère law must then be modified as [11]

$$\nabla\times\overrightarrow{H} = \overrightarrow{J} + \frac{\partial\overrightarrow{D}}{\partial t}. \tag{10.27}$$

This shows that a magnetic field can also be produced by a time-varying electric field.

At this point, we can summarize the complete set of Maxwell's equations for stationary media in Table 10.1. It is important to keep in mind that writing the equations in this form assumes the validity of the *constitutive relations*

$$\begin{aligned}\overrightarrow{B} &= \mu\,\overrightarrow{H}\\ \overrightarrow{J} &= \sigma\,\overrightarrow{E}\\ \overrightarrow{D} &= \varepsilon\,\overrightarrow{E}.\end{aligned}$$

10.7 Rotating coil measurements

Magnetic fields can be measured using a number of techniques. Nuclear magnetic resonance (NMR) probes are used for high-precision measurements.[12] Hall effect probes are simple to use and are commercially available.[13] Other common methods of measuring the magnetic field are based on electromagnetic induction.

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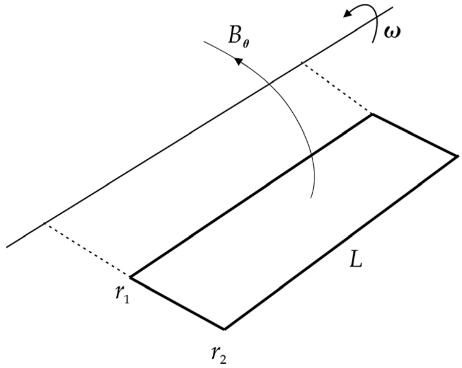

Figure 10.4 Field measurement with a radial coil.

One technique, which relates directly with our previous discussions of the multipole content of fields, involves measurements in long magnets with large aperture using a rotating coil.[14] The azimuthal component of the field can be measured using a radial coil, the principle of which is shown in Figure 10.4. The flux through the wire loop with N turns is

$$\begin{aligned}\Phi_B(\theta) &= N\,L\int_{r_1}^{r_2} B_\theta(r,\theta)\;dr \\ &= N\,L\sum_{n=1}^{\infty}(A_n \sin n\theta + B_n \cos n\theta)\;\int_{r_1}^{r_2} r^{n-1}\;dr \\ &= N\,L\sum_{n=1}^{\infty}(A_n \sin n\theta + B_n \cos n\theta)\;\left[\frac{r_2^n - r_1^n}{n}\right],\end{aligned}$$

where we have used Equation 4.8 to express the azimuthal field in terms of multipole field components. If the coil rotates at a constant rate, we have $\theta = \omega\,t$ and

$$\frac{d\Phi_B}{dt} = \frac{d\Phi_B}{d\theta}\;\frac{d\theta}{dt} = \omega\;\frac{d\Phi_B}{d\theta}.$$

The induced voltage in the coil from Faraday's law is then

$$V(\theta) = -\omega\,N\,L\sum_{n=1}^{\infty}(A_n \cos n\theta - B_n \sin n\theta)\;(r_2^n - r_1^n). \tag{10.28}$$

Performing a Fourier analysis on the voltage signal allows the multipole coefficients to be determined.

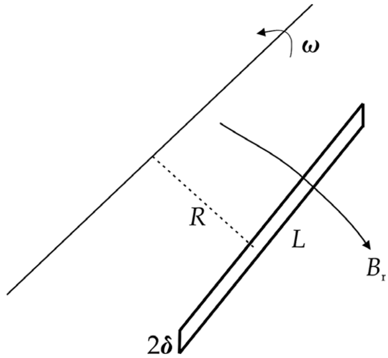

Figure 10.5 Field measurement with a tangential coil.

$$\int_0^{2\pi} V(\theta)\sin m\theta\, d\theta = -\omega NL\sum_{n=1}^{\infty}(r_2^n - r_1^n)\left[A_n\int_0^{2\pi}\cos n\theta\ \sin m\theta\, d\theta - B_n\int_0^{2\pi}\sin n\theta\ \sin m\theta\, d\theta\right] = \omega NL(r_2^n - r_1^n)\, B_n\, \pi\, \delta_{mn}.$$

Thus

$$B_m = \frac{1}{\pi\,\omega N\, L\ (r_2^m - r_1^m)}\int_0^{2\pi} V(\theta)\sin m\theta\, d\theta. \tag{10.29}$$

Similarly we find that

$$A_m = \frac{-1}{\pi\,\omega N\, L\ (r_2^m - r_1^m)}\int_0^{2\pi} V(\theta)\cos m\theta\, d\theta. \tag{10.30}$$

It is possible to do a similar analysis on the radial component of the field B_r using the rotating tangential coil illustrated in Figure 10.5. Using Equation 4.7, we have

$$\begin{aligned}\Phi_B(\theta) &= NL\int_{\theta-\delta}^{\theta+\delta} B_r(R,\theta)\, R\ d\theta \\ &= 2NL\sum_{n=1}^{\infty}\frac{R^n}{n}\sin n\delta\,(-A_n\cos n\theta + B_n\sin n\theta).\end{aligned}$$

The induced voltage in this case is

$$V(\theta) = -2\omega N\, L\sum_{n=1}^{\infty} R^n \sin n\delta\,(A_n\sin n\theta + B_n\cos n\theta). \tag{10.31}$$

Performing a Fourier analysis, we find that

$$B_m = \frac{-1}{2\pi\omega NLR^m \sin m\delta} \int_0^{2\pi} V(\theta) \cos m\theta \, d\theta \tag{10.32}$$

and

$$A_m = \frac{-1}{2\pi\omega NLR^m \sin m\delta} \int_0^{2\pi} V(\theta) \sin m\theta \, d\theta. \tag{10.33}$$

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