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On Motivic Realizations of the Canonical Hermitian Variations of Hodge Structure of Calabi–Yau Type over type $D^{\mathbb{H}}$ Domains

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Abstract. Let $\mathcal{D}$ be the irreducible Hermitian symmetric domain of type $D_{2n}^{\mathbb{H}}$. There exists a canonical Hermitian variation of real Hodge structure $\mathcal{V}_{\mathbb{R}}$ of Calabi–Yau type over $\mathcal{D}$. This short note concerns the problem of giving motivic realizations for $\mathcal{V}_{\mathbb{R}}$. Namely, we specify a descent of $\mathcal{V}_{\mathbb{R}}$ from $\mathbb{R}$ to $\mathbb{Q}$ and ask whether the $\mathbb{Q}$-descent of $\mathcal{V}_{\mathbb{R}}$ can be realized as sub-variation of rational Hodge structure of those coming from families of algebraic varieties. When $n = 2$, we give a motivic realization for $\mathcal{V}_{\mathbb{R}}$. When $n \geq 3$, we show that the unique irreducible factor of Calabi–Yau type in $\mathrm{Sym}^2 \mathcal{V}_{\mathbb{R}}$ can be realized motivically.

Introduction

Let **D** be a period domain, that is, a classifying space for polarized Hodge structures of weight n with Hodge numbers $\{h^{p,q}\}$ for $p + q = n$. It has been known since Griffiths' pioneering work that any variation of Hodge structure coming from a family of algebraic varieties is contained in a horizontal subvariety of **D** (*i.e.*, an integral manifold of the differential system corresponding to Griffiths transversality). Moreover, if a closed horizontal subvariety is semialgebraic (*cf.* [FL13, Definitions 1.1–1.2]), then it is an unconstrained Mumford–Tate domain (and hence a Hermitian symmetric domain) whose embedding into **D** is equivariant, holomorphic, and horizontal (see [FL13, Theorem 1.4]). It is thus of interest to study the following horizontal subvarieties of **D**.

Definition ([FL13, Definition 2.1]) We say a horizontal subvariety $\mathcal{D} \hookrightarrow \mathbf{D}$ is *of Hermitian type* if $\mathcal{D}$ is a Hermitian symmetric domain embedded into **D** via an equivariant, holomorphic, horizontal embedding. When $\mathcal{D} \subset \mathbf{D}$ is of Hermitian type, the induced variation of Hodge structure $\mathcal{V}$ on $\mathcal{D}$ is called a *Hermitian variation of Hodge structure*.

The Hermitian variations of Hodge structure are those parameterized by Hermitian symmetric domains considered by Deligne [Del79]. Also, when **D** is irreducible, a subvariety $\mathcal{D} \subset \mathbf{D}$ of Hermitian type is the same thing as a Mumford–Tate subdomain that is unconstrained (see [GGK12, p. 12]).

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Remark The irreducible Hermitian symmetric domains are classified by pairs (R, α_s), where R is a connected Dynkin diagram and α_s is a special node of R (see [Del79, 1.2.6]). We use these pairs to denote the isomorphism classes of irreducible Hermitian symmetric domains.

We shall be especially interested in Hermitian variations of Hodge structure of the following two special types: abelian variety type and Calabi–Yau type. Hermitian variations of Hodge structure *of abelian variety type* give families of abelian varieties over the corresponding Hermitian symmetric domains. They have been classified by Satake [Sat65] and Deligne [Del79] (see also [Mil13, Chapter 10]). Following [FL13], we define Hodge structures of Calabi–Yau type as follows.

Definition ([FL13, Definition 2.3]) A *Hodge structure V of Calabi–Yau (CY) type* is an effective weight n Hodge structure such that $V^{n,0}$ is 1-dimensional. If $n = 2$, we say that V is *of K3 type*.

Let $\mathbf{D}$ be a classifying space of certain polarized Hodge structures of CY type. The horizontal subvarieties $\mathcal{D} \subset \mathbf{D}$ of Hermitian type induce Hermitian variations of Hodge structure of CY type over $\mathcal{D}$. Examples of Hermitian variations of Hodge structure of CY type were constructed by by Gross [Gro94] (over tube domains) and Sheng–Zuo [SZ10] (over non-tube domains). Based on these, Friedman and Laza [FL13] classified Hermitian CY variations of real Hodge structure. In this note we mainly consider the tube domain cases. As discussed in [Gro94, Sections 1, 2, and 8] and [FL13, Section 2], there are six types of irreducible Hermitian symmetric domains of tube type:

$$(A_{2n-1}, \alpha_n), (B_n, \alpha_1), (C_n, \alpha_n), (D_n^{\mathbb{R}}, \alpha_1), (D_{2n}^{\mathbb{H}}, \alpha_{2n}), (E_7, \alpha_7).$$

Over every irreducible tube domain $\mathcal{D}$ there exists a canonical $\mathbb{R}$-variation of Hodge structure $\mathcal{V}_{\mathbb{R}}$ of CY type (which descends to a $\mathbb{Q}$-variation of Hodge structure up to some choices). Any other irreducible $\mathbb{R}$-variation of Hodge structure of CY type over $\mathcal{D}$ can be obtained from the canonical $\mathcal{V}_{\mathbb{R}}$ by taking the unique irreducible factor of $\mathrm{Sym}^{\bullet} \mathcal{V}_{\mathbb{R}}$ of CY type. Note that the canonical $\mathbb{R}$-variations of Hodge structure over type B and $D^{\mathbb{R}}$ domains all have weight 2 (*i.e.*, they are of $K3$ type) and are less interesting to us.

Hermitian symmetric domains are universal coverings of connected Shimura varieties that parameterize certain abelian varieties. It is a natural problem to investigate the possibility of constructing Hermitian variations of Hodge structure of CY type from families of abelian varieties. The case for the (C_n, α_n) domains is classical and well known. One can simply take the middle cohomology of abelian n-fold, which will contain a Hodge structure of CY type. At the other extreme, Satake and Deligne showed that there is no variation of Hodge structure of abelian variety type over (E_7, α_7). Thus, the canonical variation of Hodge structure of CY type over (E_7, α_7) cannot come from variations of Hodge structure of abelian variety type. The case when the domain is of type A has been discussed in [Zha15]. Specifically, certain $\mathbb{Q}$-descents of the canonical CY variations over type A tube domains can be realized as sub-variations of Hodge structure of certain $\mathbb{Q}$-variations of Hodge structure that are

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naturally associated with families of abelian varieties of Weil type (or generalized Weil type in the presence of nontrivial real multiplication).

The goal of this short note is to prove the following theorem concerning the remaining $D^{\mathbb{H}}$ case.

Main Theorem *Let $\mathcal{D}$ be the irreducible Hermitian symmetric domain (D_{2n}, α_{2n}) that has real rank n.*

(i) *When $n = 2$, there exist two families of abelian 8-folds $\pi_1 : \mathcal{A}_1 \to \mathcal{D}$ and $\pi_2 : \mathcal{A}_2 \to \mathcal{D}$ such that $R^1\pi_{1*}\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi_{2*}\mathbb{Q}$ contains a Hermitian $\mathbb{Q}$-variation of Hodge structure $\mathcal{V}$ of K3 type. Moreover, $\mathcal{V} \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to the canonical CY variation of real Hodge structure (which has weight 2) over $\mathcal{D}$.*

(ii) *When $n \geq 2$, there exists a family of abelian $4n$-folds $\pi : \mathcal{A} \to \mathcal{D}$ over $\mathcal{D}$ such that $R^{2n}\pi_*\mathbb{Q}$ contains an irreducible Hermitian $\mathbb{Q}$-variation of Hodge structure $\mathcal{V}'$ of CY type. Moreover, $\mathcal{V}' \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to the unique irreducible factor of CY type in $\mathrm{Sym}^2 \mathcal{V}_{\mathbb{R}}$ where $\mathcal{V}_{\mathbb{R}}$ is the the canonical $\mathbb{R}$-variation of Hodge structure of CY type over $\mathcal{D}$ (*n.b. *$\mathcal{V}_{\mathbb{R}}$ *is of weight n).*

Remark Let $\mathcal{V}$ be a Hermitian $\mathbb{Q}$-variation of Hodge structure of CY threefold type over (D_6, α_6) with $\mathcal{V} \otimes_{\mathbb{Q}} \mathbb{R}$ the canonical CY $\mathbb{R}$-variation of Hodge structure. By [FL13, Corollary 3.8], the generic endomorphism algebra of $\mathcal{V}$ is $\mathbb{Q}$. One could consider the more general situation when the generic endomorphism algebra is an arbitrary totally real field. The issue is $\mathcal{V}$ will then be over Hermitian symmetric domains of mixed type D (*cf.* [FL13, Theorem 3.18]) over which there is no variation of Hodge structure of abelian variety type (*cf.* [Mil13, p. 532]).

Remark Over Hermitian symmetric domains of type $D^{\mathbb{H}}$, one important reason why the rank 2 case is distinguished from the higher rank cases is that there are two different symplectic nodes for the rank 2 case, while there is only one for the higher rank cases (*cf.* [Mil13, pp. 529–530]). This fact was also noted and used by Abdulali to solve a quite different problem (*cf.* [Abd02]).

Also, in the higher rank cases, one has to quotient out the kernel of ϖ_1 (viewed as a character) from the simply connected groups of $D^{\mathbb{H}}$ type to obtain faithful symplectic representations (*i.e.,* Hodge representations giving variations of Hodge structure of abelian variety type, see [Mil13, p. 530 and Theorem 10.21]). More specifically, we should view these faithful representations as representations of the groups SO^* (*cf.* [Mil94, Remark 1.22]).

Remark In part (ii) of the Main Theorem, we only realize the $\mathbb{Q}$-descents of $\mathrm{Sym}^2 \mathcal{V}_{\mathbb{R}}$ (not the canonical $\mathcal{V}_{\mathbb{R}}$) when the rank of $\mathcal{D}$ is bigger than or equal to 3. For some representation-theoretic reasons, this is the best our constructions can do. See Remark 3.5.

After reviewing some background materials on Hermitian symmetric domains, Hodge representations, and the groups Spin^* and SO^* in Section 1, we prove the Main Theorem for the rank 2 case and higher rank cases in Section 2 and Section 3, respectively. The constructions for these two cases are different, but the ideas of the proof

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are quite similar and were also used in [Zha15]. Specifically, to give a Hermitian variation of Hodge structure it suffices to give a Hodge representation. In this way one reduces the problem of constructing a sub-variation of Hodge structure to the problem of constructing a subrepresentation. Another key step is to prove the rationality of certain representations (*e.g.,* the half-spin representations) using representation theory and the ideas from [FL14]. We hope that our motivic realizations give a hint as to how to construct families of Calabi–Yau varieties over Hermitian symmetric domains (geometric realizations). For example, the families of abelian varieties we construct can also be obtained (up to isogeny) as certain Prym varieties associated with quaternionic covers of some algebraic curves (*cf.* [vGV03]). We wonder if it is possible to construct Calabi–Yau varieties out of these quaternionic covers.

1 Preliminaries

1.1 Hermitian Variations of Hodge Structure and Hodge Representations

In this subsection, we collect some basic facts on Hermitian variations of Hodge structure. The emphasis will be on Hermitian symmetric domain of type $D^{\mathbb{H}}$ and Hermitian variations of Hodge structure of abelian variety type and of CY type. The general references include [Mil13, GGK12, Ker14].

Let $\mathcal{D} = G(\mathbb{R})/K$ be an irreducible Hermitian symmetric domain (where G is the almost simple and simply connected $\mathbb{R}$-algebraic group associated with $\mathcal{D}$ and K is a maximal compact subgroup of $G(\mathbb{R})$). Recall that irreducible Hermitian symmetric domains are classified by the root system R of $G(\mathbb{C})$ together with one of its special roots α_s. In particular, an irreducible Hermitian symmetric domain of type $D^{\mathbb{H}}_{2n}$ (*n.b.* it has real rank n) corresponds to the pair (D_{2n}, α_{2n}) and the associated simply connected algebraic group is $\mathrm{Spin}^*(4n)$ (*cf.* [Gro94, Section 1]). After choosing a suitable arithmetic subgroup of $\mathrm{Hol}(\mathcal{D})$, we may also assume that the associated algebraic group G is defined over $\mathbb{Q}$ (*cf.* [Mil13, Theorem 3.13]).

To give a Hermitian $\mathbb{Q}$-variation of Hodge structure over $\mathcal{D}$, one needs to construct a representation $\rho\colon G \to GL(V)$ defined over $\mathbb{Q}$ and a compatible polarization Q on V such that $\rho(V) \subseteq \mathrm{Aut}(V, Q)$. As explained in [GGK12, Step 4 of (IV.A)], a compatible polarization typically exists and is unique. Also, without loss of generality, one can assume that ρ is irreducible over $\mathbb{Q}$.

We recall the following theorem of Deligne (see also [FL13, Section 2.1.1]). The necessary and sufficient conditions for $\rho\colon G \to GL(V)$ together with a reference point $\varphi\colon U(1) \to \overline{G}$ (where $\overline{G} = G/Z(G)$ is the adjoint group) to give a Hermitian variation of Hodge structure are as follows: there exists a reductive algebraic group $M \subseteq GL(V)$ defined over $\mathbb{Q}$ (the generic Mumford–Tate group of the variation of Hodge structure) and a morphism of algebraic groups $h\colon \mathbb{S} \to M_{\mathbb{R}} \subseteq GL(V_{\mathbb{R}})$ ($\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$) such that

(a) the homomorphism h defines a Hodge structure on V;
(b) the representation ρ factors through M and $\rho(G) = M_{\mathrm{der}}$;
(c) the induced map $\overline{h}\colon \mathbb{S}/\mathbb{G}_m \to M_{\mathrm{ad},\mathbb{R}} = \overline{G}$ is conjugate to $\varphi\colon U(1) \to \overline{G}$.

Remark (i) Following [GGK12], we call ρ a Hodge representation.

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(ii) As mentioned in [FL13, Section 2.1.1], for variations of Hodge structure of pure weight it suffices to consider the restrictions $H = M \cap \mathrm{SL}(V)$ (thought of as the generic special Mumford–Tate group or the generic Hodge group) and $h|_{U(1)}: U(1) \to H_{\mathbb{R}}$.

(iii) Subrepresentations of V correspond to sub-Hermitian variations of Hodge structure and operations on representations correspond to the same operations on Hermitian variations of Hodge structure.

Satake [Sat65] and Deligne [Del79] (especially Table 1.3.9) classified Hodge representations of abelian variety type (see also [Mil13, Chapter 10]). Based on the earlier work of Gross [Gro94] and Sheng and Zuo [SZ10], Friedman and Laza [FL13] classify Hermitian $\mathbb{R}$-variations of Hodge structure (or Hermitian $\mathbb{Q}$-variations of Hodge structure that remain irreducible over $\mathbb{R}$) of CY type. Over every irreducible Hermitian symmetric domain $\mathcal{D}$, there exists a canonical $\mathbb{R}$-variation of Hodge structure $\mathcal{V}_{\mathbb{R}}$ of CY type; any other irreducible CY Hermitian $\mathbb{R}$-variation of Hodge structure on $\mathcal{D}$ can be obtained from the canonical $\mathcal{V}_{\mathbb{R}}$ by taking the unique irreducible CY factor of $\mathrm{Sym}^{\bullet}\mathcal{V}_{\mathbb{R}}$ or, for non-tube domains, $\mathrm{Sym}^{\bullet}\mathcal{V}_{\mathbb{R}}\{-\frac{a}{2}\}$ ($a \in \mathbb{Z}$, $\{\cdot\}$ denotes the half twist; see [FL13, Section 2.1.3]). To describe the canonical variation of Hodge structure of CY type, we set (R, α_s) to be the pair determined by the domain $\mathcal{D}$, and let G be the associated algebraic group. Also let $\rho: G \to \mathrm{GL}(V)$ be a representation defined over $\mathbb{Q}$ such that $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$ is still an irreducible representation. For tube domains, if the representation $V_{\mathbb{C}} := V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ of $G(\mathbb{C})$ is irreducible (in other words, the representation $V_{\mathbb{R}}$ is of real type, see, for example, [GGK12, p. 88]) and has highest weight ϖ_s which is the fundamental weight corresponding to the special root α_s, then ρ gives rise to a Hermitian $\mathbb{Q}$-variation of Hodge structure of CY type whose scalar extension to $\mathbb{R}$ is the canonical Hermitian $\mathbb{R}$-variation of Hodge structure of CY type. The weight ϖ_s will be called the fundamental *cominuscule weight* associated with the domain $\mathcal{D}$. We refer the reader to [FL13] for the description of the canonical CY variation over non-tube domains.

In particular, the canonical $\mathbb{R}$-variation of Hodge structure of CY type $\mathcal{V}_{\mathbb{R}}$ over (D_{2n}, α_{2n}) is given by a $\mathbb{R}$-representation $S^{+}_{0,\mathbb{R}}$ of $G(\mathbb{R}) = \mathrm{Spin}^{*}(4n)$ with the property that $S^{+}_{0,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is the half-spin representation with highest weight ϖ_{2n} (which is the fundamental cominuscule weight associated with the domain (D_{2n}, α_{2n})). The weight of $\mathcal{V}_{\mathbb{R}}$ equals n, the real rank of (D_{2n}, α_{2n}).

1.2 The Groups Spin^{*} and SO^{*}

We construct a form H of the real algebraic group $\mathrm{SO}^{*}(2m)$ over $\mathbb{Q}$ following [FL14] in this subsection. Then the spin double cover G of H, which is simply connected and of type $D^{\mathbb{H}}_{m}$, gives a $\mathbb{Q}$-form of $\mathrm{Spin}^{*}(2m)$. This specifies the descents of the canonical CY variations from $\mathbb{R}$ to $\mathbb{Q}$.

Let $E = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic extension of $\mathbb{Q}$. Set W to be an E-vector space of dimension $2m$ with an E-basis $e_1, \ldots, e_{2m}$. We write $z = \sum_{i=1}^{2m} z_i e_i$, and similarly for $w \in W$. Suppose that $Q(\cdot, \cdot)$ is a nondegenerate E-bilinear form on W, written in the standard form

$$Q(z, w) = \sum_{i=1}^{2m} (z_i w_{m+i} + z_{m+i} w_i).$$

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Also, let h be the standard $(E, \mathbb{Q})$-Hermitian form of signature (m, m) on W, given by

$$h(z, w) = \sum_{i=1}^{m} z_i \overline{w_i} - \sum_{i=1}^{m} z_{m+i} \overline{w}_{m+i}.$$

Now we define H to be the group of E-linear isomorphisms of W that have determinant 1 and preserve Q and h. The group H is defined over $\mathbb{Q}$, since it is the intersection of $\mathrm{Res}_{E/\mathbb{Q}}\mathrm{SO}(W, Q)$ with $\mathrm{SU}(W, h)$. Recall that the real group $\mathrm{SO}^*(2m)$ is defined to be the isometry group of a skew-Hermitian form on $\mathbb{H}^m$. By [GW09, Exercise 1.1.5(12)], one has $H \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{SO}^*(2m)$.

Let G be the neutral component of the preimage of H in $\mathrm{Res}_{E/\mathbb{Q}}\mathrm{Spin}(W, Q)$ under the spin double covering map. Clearly, G is a $\mathbb{Q}$-form of $\mathrm{Spin}^*(2m)$.

To conclude this subsection, let us note that there are some natural representations of G. The first one is the standard representation $G \to H \to \mathrm{GL}(W)$. Moreover, G also admits two half-spin representations. To construct them, let W_1 (resp. W_2) be the Q-isotropic E-vector subspace of W spanned by $e_1, \ldots, e_m$ (resp. $e_{m+1}, \ldots, e_{2m}$). The half-spin representations are then given by the direct sum of even and odd exterior powers of W_1:

$$S^+ = \bigwedge_E^{\text{even}} W_1; \quad S^- = \bigwedge_E^{\text{odd}} W_1.$$

2 Proof of the Main Theorem for the Rank 2 Case

We shall prove part (i) of the Main Theorem in this section. The notation remains the same as in Subsection 1.2 (with $m = 4$). In particular, $\mathcal{D}$ is an irreducible Hermitian symmetric domain of type (D_4, α_4) and G is the simply connected $\mathbb{Q}$-algebraic group associated with $\mathcal{D}$ (*n.b.* G is a $\mathbb{Q}$-form of $\mathrm{Spin}^*(8)$). First, we construct two families of abelian varieties over $\mathcal{D}$.

Proposition 2.1 *The standard representation $G \to \mathrm{GL}(W)$ and the half-spin representation $G \to \mathrm{GL}(S^-)$ are both Hodge representations giving Hermitian $\mathbb{Q}$-variation of Hodge structure of abelian variety type over $\mathcal{D}$.*

Moreover, there exist two families of abelian 8-folds, $\pi_1 : \mathcal{A}_1 \to \mathcal{D}$ and $\pi_2 : \mathcal{A}_2 \to \mathcal{D}$, such that the associated variation of Hodge structure $R^1\pi_{1*}\mathbb{Q}$ (resp. $R^1\pi_{2*}\mathbb{Q}$) corresponds to the Hodge representation $\mathrm{Res}_{E/\mathbb{Q}}W$ (resp. $\mathrm{Res}_{E/\mathbb{Q}}S^-$).

Proof The representations $G \to \mathrm{GL}(W)$ and $G \to \mathrm{GL}(S^-)$ are both defined over $\mathbb{Q}$. According to [Mil13, Summary 10.11], there are two symplectic nodes associated with the domain $D_4^{\mathbb{H}}$, namely, α_1 and α_3. By the standard representation theory (*e.g.*, [FH91, Chapters 19 and 20]), the irreducible factors of the representations $(\mathrm{Res}_{E/\mathbb{Q}}W) \otimes_{\mathbb{Q}} \mathbb{C}$ and $(\mathrm{Res}_{E/\mathbb{Q}}S^-) \otimes_{\mathbb{Q}} \mathbb{C}$ have highest weight ϖ_1 and ϖ_3, respectively. So they give two Hermitian $\mathbb{Q}$-variation of Hodge structure of abelian variety type. After choosing the underlying integral structures we get two families of abelian 8-folds, π_1 and π_2 (see also [Mil13, Theorem 11.8]). ∎

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Remark 2.2 Recall that there is a classification of irreducible polarized $\mathbb{Q}$-Hodge structures of weight 1 (or the corresponding abelian varieties) according to their endomorphism algebras (see for example [Moo99, (1.19)–(1.20)]). Specifically, there are the following four types: real multiplication (type I), totally indefinite quaternion multiplication (type II), totally definite quaternion multiplication (type III), and complex multiplication (type IV). In our case, [GGK12, Theorem IV.E.4] implies that the generic fiber of π_1 and π_2 are both of type III; therefore, the generic special Mumford–Tate group (a.k.a. Hodge group) of the Hermitian variations of Hodge structures $R^1\pi_{1*}\mathbb{Q}$ and $R^1\pi_{2*}\mathbb{Q}$ are both semisimple (*cf.* [Moo99, Proposition (1.24)]). We also note that a general fiber of the family of abelian varieties $\mathcal{A}_1$ is isogenous to a certain Prym variety associated with a quaternionic cover of a genus three curve (*cf.* [vGV03, Section 3]).

Next we show that $\mathrm{Res}_{E/\mathbb{Q}}S^+$ is a G-subrepresentation of $\mathrm{Res}_{E/\mathbb{Q}}W \otimes_{\mathbb{Q}} \mathrm{Res}_{E/\mathbb{Q}}S^-$.

Lemma 2.3 (i) *S^+ is a subrepresentation of $W \otimes_E S^-$.*

(ii) *There is a natural inclusion $\mathrm{Res}_{E/\mathbb{Q}}(W \otimes_E S^-) \subseteq (\mathrm{Res}_{E/\mathbb{Q}}W) \otimes_{\mathbb{Q}} (\mathrm{Res}_{E/\mathbb{Q}}S^-)$ that also commutes with the G-action.*

Proof (i) Let $\mathfrak{g} = \mathrm{Lie}(G)$. Every representation will be viewed as representation of $\mathfrak{g}$ in this proof. Thanks to the complete reducibility, it suffices to construct a surjection $p\colon W \otimes_E S^- \twoheadrightarrow S^+$ compatible with the action of $\mathfrak{g}$. To define p, we use the inclusion $W \subseteq C(W,Q) \cong \mathrm{End}(S^+ \oplus S^-)$ (where $C(W,Q)$ is the Clifford algebra for Q). In other words, there is an action of W on $S^+ \oplus S^-$. By [FH91, Lemma 20.9], the action of W exchanges S^- and S^+. In other words, we have $W \times S^- \to S^+$, $(w,\xi) \mapsto w(\xi)$, which is clearly E-bilinear and hence can be used to define p. It is not difficult to check that p is surjective.

Next we check that p is compatible with the action of $\mathfrak{g}$, that is, $p(g\cdot(v\otimes\xi)) = g\cdot p(v\otimes\xi)$ for every $g \in \mathfrak{g}$, $v \in W$ and $\xi \in S^-$. To do this, recall that we have $(\mathfrak{g} \subseteq)\mathfrak{so}(W,Q) \cong \bigwedge_E^2 W \hookrightarrow C(W,Q) \cong \mathrm{End}(S^+ \oplus S^-)$, where the first two maps are morphisms of Lie algebras and the last one is an algebra isomorphism (*cf.* [FH91, Lemma 20.7]). Without loss of generality we assume that $g = a \wedge b$ for $a, b \in W$. Let us also recall that the multiplication in the Clifford algebra $C(W,Q)$ is defined by $ab + ba = 2Q(a,b)$. Now we have

$$\begin{aligned}
p(g\cdot(v\otimes\xi)) &= p\big((g\cdot v)\otimes\xi + v\otimes(g\cdot\xi)\big) = (g\cdot v)(\xi) + v(g\cdot\xi)\\
&= 2Q(b,v)a(\xi) - 2Q(a,v)b(\xi) + v\big(ab(\xi)\big) - Q(a,b)v(\xi)\\
&\qquad \big(\text{by [FH91, (20.4) and (20.6)]}\big)\\
&= 2Q(b,v)a(\xi) - 2Q(a,v)b(\xi) + (vab)(\xi) - Q(a,b)v(\xi)\\
&= (abv)(\xi) - Q(a,b)v(\xi)\\
&\qquad \big(\text{by the definition of Clifford algebra}\big)\\
&= (ab)v(\xi) - Q(a,b)v(\xi) = g\cdot p(v\otimes\xi).
\end{aligned}$$

(ii) The proof is essentially the same as that of [Zha15, Lemmas 3.1 and 3.2]. Replacing wedge product by tensor product causes no essential changes. Let us denote

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$\mathrm{Res}_{E/\mathbb{Q}}$ by Res and the E-dual vector space using $\star$. First observe that there is a natural surjection

$$\mathrm{Res}(W^*) \otimes_{\mathbb{Q}} \mathrm{Res}(S^{-*}) \twoheadrightarrow \mathrm{Res}(W^* \otimes_E S^{-*})$$

which gives, by duality, an injection

$$\mathrm{Hom}_{\mathbb{Q}}\left(\mathrm{Res}(W^* \otimes_E S^{-*}), \mathbb{Q}\right) \hookrightarrow \mathrm{Hom}_{\mathbb{Q}}\left(\mathrm{Res}(W^*) \otimes_{\mathbb{Q}} \mathrm{Res}(S^{-*}\right), \mathbb{Q}).$$

Also, for any E-vector space M there is a natural isomorphism

$$\mathrm{Res}\,\mathrm{Hom}_E(M, E) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Res}M, \mathbb{Q}), \quad f \longmapsto \mathrm{Tr} \circ f$$

as in *op. cit*. The natural inclusion can be defined as follows.

$$\begin{aligned}
\mathrm{Res}(W \otimes_E S^-) &\cong \mathrm{Res}\big(\mathrm{Hom}_E(W^*, E) \otimes_E \mathrm{Hom}_E(S^{-*}, E)\big) \\
&\cong \mathrm{Res}\,\mathrm{Hom}_E(W^* \otimes_E S^{-*}, E) \\
&\cong \mathrm{Hom}_{\mathbb{Q}}\left(\mathrm{Res}(W^* \otimes_E S^{-*}), \mathbb{Q}\right) \\
&\subseteq \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Res}(W^*) \otimes_{\mathbb{Q}} \mathrm{Res}(S^{-*}), \mathbb{Q}) \\
&\cong \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Res}W^*, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Res}S^{-*}, \mathbb{Q}) \\
&\cong \mathrm{Res}(W^{**}) \otimes_{\mathbb{Q}} \mathrm{Res}(S^-)^{**} \cong \mathrm{Res}W \otimes_{\mathbb{Q}} \mathrm{Res}S^-.
\end{aligned}$$

Finally, to check that this map is G-equivariant (also after scalar extensions by arbitrary $\mathbb{Q}$-algebras) is straightforward and quite similar to what we did in *op. cit.*. ∎

Now we show that the half-spin representation S^+ is defined over $\mathbb{Q}$.

Lemma 2.4 *There exists a G-subrepresentation on a $\mathbb{Q}$-vector space $S_0^+ \subseteq \mathrm{Res}_{E/\mathbb{Q}}S^+$ such that $S_0^+ \otimes_{\mathbb{Q}} E \cong S^+$.*

Proof As is well known, it suffices to construct an E-conjugate linear operator $\star\colon \mathrm{Res}_{E/\mathbb{Q}}S^+ \to \mathrm{Res}_{E/\mathbb{Q}}S^+$ that is compatible with the G-action and satisfy $\star \circ \star = \mathrm{id}$. Let us consider the Hodge star operator $\star$ associated with the Hermitian form $h|W_1$ and the volume form $e_1 \wedge \cdots \wedge e_4$ defined in [FL13, Section 3.5] (see also Lemma 3.3). One can easily show that $\star$ is E-conjugate linear and maps $\bigwedge_E^{2+2k} W_1$ to $\bigwedge_E^{2-2k} W_1$ for $k = -1, 0, 1$. The more difficult part is to verify that $\star$ is a morphism of G-representations. But this has been done in [FL14, Section 3]. ∎

Remark Over an arbitrary totally real field, $\star$ may not commute with the corresponding group action. To fix this, one should use the "twisted Hodge star operator", which is defined in [FL14, Definition 3.10].

We also need the following lemmas to prove the Main Theorem. Let us denote the special Mumford–Tate group (a.k.a. Hodge group) of a $\mathbb{Q}$-Hodge structure V by $\mathrm{Hg}(V)$.

Lemma 2.5 *Let V be a $\mathbb{Q}$-Hodge structure. Let $W \subseteq V$ be a sub-Hodge structure.*

(i) *There exists a surjective homomorphism* $\mathrm{Hg}(V) \twoheadrightarrow \mathrm{Hg}(W)$.
(ii) *If* $\mathrm{Hg}(V)$ *is semisimple, then* $\mathrm{Hg}(W)$ *is also semisimple.*

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Proof Part (i) follows from [GGK12, (I.B.7)]. Since any quotient of a semisimple algebraic group is semisimple, part (ii) is clear from part (i). ∎

Lemma 2.6 *Let V be a $\mathbb{Q}$-Hodge structure. If $\mathrm{Hg}(V)$ is semisimple, then $\mathrm{Hg}(\bigwedge^k_{\mathbb{Q}} V)$ is semisimple (with k a non-negative integer).*

Proof Note that $\bigwedge^k_{\mathbb{Q}} V$ is a sub-Hodge structure of $\bigotimes^k_{\mathbb{Q}} V$. By Lemma 2.5 it suffices to show that $\mathrm{Hg}(\bigotimes^k_{\mathbb{Q}} V)$ is semisimple. According to [Moo99, (1.8)], we have $\mathrm{Hg}(\bigotimes^k_{\mathbb{Q}} V) = r(\mathrm{Hg}(V))$ where $r\colon \mathrm{GL}(V) \to \mathrm{GL}(\bigotimes^k_{\mathbb{Q}} V)$ is the natural homomorphism. In other words, we have a surjective homomorphism $\mathrm{Hg}(V) \twoheadrightarrow \mathrm{Hg}(\bigotimes^k_{\mathbb{Q}} V)$. Because $\mathrm{Hg}(V)$ is semisimple, $\mathrm{Hg}(\bigotimes^k_{\mathbb{Q}} V)$ is also semisimple. ∎

Now let us prove part (i) of the Main Theorem.

Proof By Lemma 2.3 and Lemma 2.4, we have $S_0^+ \subseteq \mathrm{Res}_{E/\mathbb{Q}} S^+ \subseteq \mathrm{Res}_{E/\mathbb{Q}}(W \otimes_E S^-) \subseteq (\mathrm{Res}_{E/\mathbb{Q}} W) \otimes_{\mathbb{Q}} (\mathrm{Res}_{E/\mathbb{Q}} S^-)$ as representations of G. Also, $\mathrm{Res}_{E/\mathbb{Q}} W$ (resp. $\mathrm{Res}_{E/\mathbb{Q}} S^+$) corresponds to a family of abelian 8-folds $\pi_1\colon \mathcal{A}_1 \to \mathcal{D}$ (resp. $\pi_2\colon \mathcal{A}_2 \to \mathcal{D}$) (*cf.* Proposition 2.1). Let A_i be the generic fiber of π_i ($i = 1, 2$), which is a simple abelian variety. Using [GGK12, Theorem IV.E.4], it is easy to see that A_1 and A_2 are both of type III. By [Moo99, Proposition (1.24)], the special Mumford–Tate group of $H^1(A_1 \times A_2, \mathbb{Q})$ is semisimple. The special Mumford–Tate group of $H^2(A_1 \times A_2, \mathbb{Q})$ is also semisimple because $H^2(A_1 \times A_2, \mathbb{Q}) \cong \bigwedge^2_{\mathbb{Q}} H^1(A_1 \times A_2, \mathbb{Q})$ (*cf.* Lemma 2.6). Since $H^1(A_1, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(A_2, \mathbb{Q})$ is a sub-Hodge structure of $H^2(A_1 \times A_2, \mathbb{Q})$, $H^1(A_1, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(A_2, \mathbb{Q})$ has a semisimple special Mumford–Tate group as well (Lemma 2.5).

As a result, the special Mumford–Tate group of the Hermitian variation of Hodge structure $R^1\pi_{1*}\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi_{2*}\mathbb{Q}$ (which corresponds to the Hodge representation $(\mathrm{Res}_{E/\mathbb{Q}} W) \otimes_{\mathbb{Q}} (\mathrm{Res}_{E/\mathbb{Q}} S^-)$) is semisimple. Let us denote it by Hg. By Deligne's theorem in Subsection 1.1 (especially condition (ii)), Hg is the image of G in $\mathrm{SL}((\mathrm{Res}_{E/\mathbb{Q}} W) \otimes_{\mathbb{Q}} (\mathrm{Res}_{E/\mathbb{Q}} S^-))$ (see also [Roh09, Corollary 1.3.19] and [Mil15, Corollary 22.123]). It follows that S_0^+ is a Hg-subrepresentation of $(\mathrm{Res}_{E/\mathbb{Q}} W) \otimes_{\mathbb{Q}} (\mathrm{Res}_{E/\mathbb{Q}} S^-)$ (*n.b.* Hg is the generic special Mumford–Tate group) and hence gives a variation of sub-Hodge structure $\mathcal{V}$ (*cf.* [Moo99, (1.12)]).

Now it suffices to show that $S_0^+ \otimes_{\mathbb{Q}} \mathbb{R}$ (or equivalently, $\mathcal{V} \otimes_{\mathbb{Q}} \mathbb{R}$) gives the canonical $\mathbb{R}$-variation of Hodge structure of $K3$ type. Note that $S_0^+ \otimes_{\mathbb{Q}} \mathbb{C} \cong S_0^+ \otimes_{\mathbb{Q}} E \otimes_{\mathbb{Q}} \mathbb{R} \cong S^+ \otimes_{\mathbb{Q}} \mathbb{R}$. Since $S^+ \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\bigwedge^{\mathrm{even}}_{\mathbb{C}}(W_1 \otimes_{\mathbb{Q}} \mathbb{R})$ (by the construction), $S_0^+ \otimes_{\mathbb{Q}} \mathbb{R}$ is the half-spin representation of $G(\mathbb{C}) \cong \mathrm{Spin}(8, \mathbb{C})$ with highest weight ϖ_4. Because ϖ_4 is the fundamental cominuscule weight associated with the domain $\mathcal{D}$, the theorem follows from [Gro94, Section 3] or [FL13, Theorem 2.22]. ∎

3 Proof of the Main Theorem for the Higher Rank Cases

Let $\mathcal{D}$ be the irreducible Hermitian symmetric domain of type (D_{2n}, α_{2n}) with $n \geq 2$. We will prove part (ii) of the Main Theorem for $\mathcal{D}$ in this section. Recall that G is the simply connected $\mathbb{Q}$-algebraic group associated with $\mathcal{D}$ and H is a $\mathbb{Q}$-form of $\mathrm{SO}^*(2m)$. There is a spin double cover $G \to H$. We also use the other notation in Subsection 1.2 (in particular, note that $m = 2n$).

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Proposition 3.1 *The standard representation $H \to \mathrm{GL}(W)$ is a faithful Hodge representation corresponding to a Hermitian $\mathbb{Q}$-variation of Hodge structure of abelian variety type over $\mathcal{D}$. Furthermore, there exists a family of abelian $4n$-folds $\pi\colon \mathcal{A} \to \mathcal{D}$ such that the associated variation of Hodge structure $R^1\pi_*\mathbb{Q}$ is given by the Hodge representation* $\mathrm{Res}_{E/\mathbb{Q}} W$.

Proof By [Mil13, Summary 10.11], the only symplectic node of $D_{2n}^{\mathbb{H}}$ ($n \geq 3$) is α_1. The rest is the same as the proof of Proposition 2.1. ∎

Remark As discussed in Remark 2.2, the generic fiber of π is of type III and has a semisimple special Mumford–Tate group.

From [Zha15, Lemmas 3.1 and 3.2], we deduce the following lemma.

Lemma 3.2 $\mathrm{Res}_{E/\mathbb{Q}}(\bigwedge_E^m W)$ *is naturally an H-subrepresentation of* $\bigwedge_{\mathbb{Q}}^m(\mathrm{Res}_{E/\mathbb{Q}} W)$.

We construct elements L and $\star$ of the endomorphism algebra $\mathrm{End}_{\mathbb{Q}[H]}(\bigwedge_E^m W)$, which will be used to decompose $\bigwedge_E^m W$. The operator $\star$ is defined in the same way as in [FL13, Section 3.5]. Specifically, there are two natural pairings on $\bigwedge_E^m W$: the wedge product

$$\wedge\colon \bigwedge_E^m W \times \bigwedge_E^m W \longrightarrow \bigwedge_E^{2m} W \cong E$$

and

$$\wedge^m h\colon \bigwedge_E^m W \times \bigwedge_E^m W \longrightarrow E, \quad (\wedge^m h)(w_1 \wedge \cdots \wedge w_m, u_1 \wedge \cdots \wedge u_m) := \det\big(h(w_i, u_j)\big).$$

They give an E-linear isomorphism $\varphi\colon \bigwedge_E^m W \to \bigwedge_E^m W^*$ and an E-conjugate-linear isomorphism $\rho\colon \bigwedge_E^m W \to \bigwedge_E^m W^*$ respectively. The operator $\star$ is then defined by $\star = \varphi^{-1} \circ \rho$. Note that there is another E-linear isomorphism $\tau\colon \bigwedge_E^m W \to \bigwedge_E^m W^*$ given by the pairing

$$\wedge^m Q\colon \bigwedge_E^m W \times \bigwedge_E^m W \to E, \quad (\wedge^m Q)(w_1 \wedge \cdots \wedge w_m, u_1 \wedge \cdots \wedge u_m) := \det(Q(w_i, u_j)).$$

We define L by $L = \varphi^{-1} \circ \tau$.

Concerning $\star$ and L, they satisfy the following properties.

Lemma 3.3

(i) *The E-linear operator L commutes with the H-action and $L \circ L = \mathrm{id}$.*
(ii) *The E-conjugate-linear operator $\star$ commutes with the H-action and $\star \circ \star = \mathrm{id}$.*

Proof For (i), the action of H preserves the pairing

$$\wedge\colon \bigwedge_E^m W \times \bigwedge_E^m W \longrightarrow \bigwedge_E^{2m} W \cong E$$

and the symmetric bilinear form $\wedge^m Q$, and hence commutes with L. By [FH91, Theorem 19.2(iii)], $L \circ L = \mathrm{id}$. Part (ii) follows from [FL13, Lemma 3.21]. ∎

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Lemma 3.4 *The operators L and $\star$ commute (*i.e., *$L\circ\star = \star\circ L$) in* $\mathrm{End}_{\mathbb{Q}[H]}(\bigwedge_E^m W)$.

Proof We first set up some notation. Let $\{e_1,\dots,e_{2m}\}$ be a basis of W such that the symmetric bilinear form Q and the Hermitian form h can be expressed in the same form as in Subsection 1.2. Also denote the corresponding dual basis by $\{e_1^*,\dots,e_{2m}^*\}$. Now define $B\colon W\to W^*$ by $B(v)(w) = Q(v,w)$, and $F\colon W\to W^*$ by $F(v)(w) = h(w,v)$. It is clear that $B(e_i) = e_{m+i}^*$, $B(e_{m+i}) = e_i^*$ and that $F(e_i) = e_i^*$, $F(e_{m+i}) = -e_{m+i}^*$ for $1\le i\le m$. Using these we can make the operators τ and ρ more explicit. Specifically, we have $\tau(e_{l_1}\wedge e_{l_2}\wedge\cdots\wedge e_{l_m}) = B(e_{l_1})\wedge B(e_{l_2})\wedge\cdots\wedge B(e_{l_m})$ and $\rho(e_{l_1}\wedge e_{l_2}\wedge\cdots\wedge e_{l_m}) = F(e_{l_1})\wedge F(e_{l_2})\wedge\cdots\wedge F(e_{l_m})$.

Now we consider φ. Let $I=\{i_1,i_2,\dots,i_m\}$ with $1\le i_1<i_2<\cdots<i_m\le 2m$. Set $J = \{1,2,\dots,2m\}\smallsetminus I = \{j_1,j_2,\dots,j_m\}$ with $j_1<j_2<\cdots<j_m$. Then it is not difficult to see that $\varphi(e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_m}) = \epsilon_{I,J}\cdot e_{j_1}^*\wedge e_{j_2}^*\wedge\cdots\wedge e_{j_m}^* = (-1)^{n+i_1+i_2+\cdots+i_m} e_{j_1}^*\wedge e_{j_2}^*\wedge\cdots\wedge e_{j_m}^*$ (recall that $n=\frac{m}{2}$). So $\varphi^{-1}(e_{j_1}^*\wedge e_{j_2}^*\wedge\cdots\wedge e_{j_m}^*) = (-1)^{n+j_1+j_2+\cdots+j_m} e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_m}$.

Let us prove the lemma. Let I and J be the ordered set as above. Clearly, it suffices to verify that $\tau\circ\varphi^{-1}\circ\rho = \rho\circ\varphi^{-1}\circ\tau$ for $e_I = e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_m}$ ($i_1<i_2<\cdots<i_m$). We first determine which e_j^*'s appear for the left-hand-side (*i.e.,* $(\tau\circ\varphi^{-1}\circ\rho)(e_I)$) and the right-hand side (*i.e.,* $(\rho\circ\varphi^{-1}\circ\tau)(e_I)$). To do this, we define $s(i)\in\{1,2,\dots,2m\}$ (for every $1\le i\le 2m$) by $s(i) = m+i$ if $i\le m$ and $s(i) = i-m$ if $i>m$. If $i\in I$ while $s(i)\notin I$, then e_i^* will appear for both the left-hand-side and the right-hand side; if $i\in I$ and $s(i)\in I$, then neither e_i^* nor $e_{s(i)}^*$ will be contained in $(\tau\circ\varphi^{-1}\circ\rho)(e_I)$ or $(\rho\circ\varphi^{-1}\circ\tau)(e_I)$; if $i\notin I$ and $s(i)\notin I$, then e_i^* and $e_{s(i)}^*$ will be contained in both the left-hand-side and the right-hand side. So $(\tau\circ\varphi^{-1}\circ\rho)(e_I)$ and $(\rho\circ\varphi^{-1}\circ\tau)(e_I)$ consist of the same e_j^*'s. It follows that both $(\tau\circ\varphi^{-1}\circ\rho)(e_I)$ and $(\rho\circ\varphi^{-1}\circ\tau)(e_I)$ can be expressed uniquely, up to a sign, as $e_{l_1}^*\wedge e_{l_2}^*\wedge\cdots\wedge e_{l_m}^*$ with the same sub-indices $l_1<l_2<\cdots<l_m$. Now let us verify that the signs are the same. Let $k = \mathrm{Card}(I\cap\{m+1,\dots,2m\})$. Then it is straightforward to check that the sign of $(\tau\circ\varphi^{-1}\circ\rho)(e_I)$ is $(-1)^{k+n+i_1+\cdots+i_m+k}$, and the sign for $(\rho\circ\varphi^{-1}\circ\tau)(e_I)$ is $(-1)^{k+n+s(i_1)+\cdots+s(i_m)+(m-k)}$. Since m is an even number, we get $i\equiv s(i)\pmod 2$ for every i, which implies that the two signs are the same. ∎

We now prove part (ii) of the Main Theorem. Note that $m = 2n$.

Proof Let $S = \ker(L-\mathrm{id})$. Then S is an H-subrepresentation of $\bigwedge_E^m W$. By Lemma 3.4, we have $L\circ\star = \star\circ L$ and hence $L(\star(s)) = \star(L(s)) = \star(s)$ for any $s\in S$. So the restriction of $\star$ to S is well defined. Let $S_0 = \ker(\star\,|_S - \mathrm{id})\subseteq S$. Since $\star$ is E-conjugate-linear, S_0 is a $\mathbb{Q}$-subrepresentation of $\mathrm{Res}_{E/\mathbb{Q}}S$.

Note that we have $S_0\subseteq \mathrm{Res}_{E/\mathbb{Q}}S\subseteq \mathrm{Res}_{E/\mathbb{Q}}(\bigwedge_E^m W)\subseteq\bigwedge_{\mathbb{Q}}^m(\mathrm{Res}_{E/\mathbb{Q}}W)$ as representations of H (*cf.* also Lemma 3.2). Recall that the Hodge representation $\mathrm{Res}_{E/\mathbb{Q}}W$ corresponds to a family of abelian $4n$-folds $\pi\colon\mathcal{A}\to\mathcal{D}$ constructed in Proposition 3.1. By [GGK12, Theorem IV.E.4] and [Moo99, Proposition (1.24)], the special Mumford–Tate group of the Hermitian variation of Hodge structure $R^1\pi_*\mathbb{Q}$ (which corresponds to the Hodge representation $\mathrm{Res}_{E/\mathbb{Q}}W$) is semisimple. It follows that the special Mumford–Tate group of the variation of Hodge structure $R^m\pi_*\mathbb{Q}$ given by the Hodge representation $\bigwedge_{\mathbb{Q}}^m(\mathrm{Res}_{E/\mathbb{Q}}W)$ is semisimple (Lemma 2.6). Let us denote it by Hg.

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The next part of the proof is quite similar to the rank 2 case. Specifically, by the result of Deligne (see Subsection 1.1, especially Condition (ii)) Hg is the image of H in $\mathrm{SL}(\bigwedge^m_{\mathbb{Q}}(\mathrm{Res}_{E/\mathbb{Q}}W))$. As a result, $S_0 \subseteq \bigwedge^m_{\mathbb{Q}}(\mathrm{Res}_{E/\mathbb{Q}}W)$ is invariant under the action of the generic special Mumford–Tate group Hg and hence corresponds to a variation of sub-Hodge structure $\mathcal{V}'$.

It remains to prove that S_0 is the Hodge representation of CY type and compare $\mathcal{V}'$ to the canonical CY variation. Let us consider $S_{0,\mathbb{R}} := S_0 \otimes_{\mathbb{Q}} \mathbb{R}$. Because $S_0 \otimes_{\mathbb{Q}} E \cong S$, we get $S_{0,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong S_0 \otimes_{\mathbb{Q}} \mathbb{C} \cong S_0 \otimes_{\mathbb{Q}} E \otimes_{\mathbb{Q}} \mathbb{R} \cong S \otimes_{\mathbb{Q}} \mathbb{R}$. According to [FH91, Theorem 19.2(ii) and (iii)], $S \otimes_{\mathbb{Q}} \mathbb{R} \subseteq \bigwedge^m_{\mathbb{C}} W_{\mathbb{R}}$ is the irreducible representation of $H(\mathbb{R}) \cong \mathrm{SO}^*(2m)$ with highest weight $2\varpi_m$. Since ϖ_m is the fundamental cominuscule weight associated with the domain $\mathcal{D}$, $\mathcal{V}'$ is of CY type [FL13, Theorem 2.22]. Consider the half-spin representation S^+. Note that S^+ is defined over $\mathbb{Q}$ (that is, there exists a G-subrepresentation on a $\mathbb{Q}$-vector space S_0^+ such that $S_0^+ \otimes_{\mathbb{Q}} E \cong S^+$). By [Gro94] or [FL13], $S_0^+ \otimes \mathbb{R}$ gives the canonical CY variation $\mathcal{V}_{\mathbb{R}}$ over $\mathcal{D}$. Using the highest weight theory, it is easy to see that $S_{0,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong S \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to an irreducible summand of $\mathrm{Sym}^2(S_0^+ \otimes_{\mathbb{Q}} \mathbb{C})$. The theorem then follows. ∎

Remark 3.5 Note that in part (ii) of the Main theorem we, only realize $\mathrm{Sym}^2 \mathcal{V}_{\mathbb{R}}$ (not the canonical CY variation $\mathcal{V}_{\mathbb{R}}$). This is the best our constructions can do when the rank of the domain is bigger or equal to 3. One important reason is that the half-spin representation with highest weight ϖ_m is not a representation of the orthogonal group $H(\mathbb{C}) \cong \mathrm{SO}(2m, \mathbb{C})$ (*cf.* [FH91, Proposition 23.13]). As a result, $\mathcal{V}_{\mathbb{R}}$ is not contained in any tensor construction of the cohomology of the universal family of abelian varieties $\pi: \mathcal{A} \to \mathcal{D}$. Specifically, let $W_{\mathbb{R}}$ be the standard representation corresponding to a Hermitian variation of Hodge structure of abelian variety type, and set $S^+_{0,\mathbb{R}}$ to be the Hodge representation corresponding to the canonical $\mathbb{R}$-variation of Hodge structure of CY type. By [GGK12, Theorem (IV.E.4)], $W_{\mathbb{R}}$ is of quaternion type (*i.e.,* $W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = U \oplus U^*$ with $U \cong U^*$ and $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} U = W_{\mathbb{R}}$). Suppose we have $S^+_{0,\mathbb{R}} \subseteq \bigotimes^l_{\mathbb{R}} W_{\mathbb{R}}$ as representations of $G(\mathbb{R}) \cong \mathrm{Spin}^*(2m)$. Then $S^+_{0,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is a subrepresentation of $(\bigotimes^l_{\mathbb{R}} W_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C} \cong \bigotimes^l_{\mathbb{C}}(U \oplus U^*)$ (as representations of $G(\mathbb{C}) \cong \mathrm{Spin}(2m, \mathbb{C})$) which factors through $H(\mathbb{C}) \cong \mathrm{SO}(2m, \mathbb{C})$. Since $\mathrm{SO}(2m, \mathbb{C}) = \mathrm{Spin}(2m, \mathbb{C})/\{\pm 1\}$, (-1) also acts trivially on the half-spin representation $S^+_{0,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ which is a contradiction. This argument also works for other tensor constructions $(\bigotimes^{l_1}_{\mathbb{R}} W_{\mathbb{R}}) \otimes (\bigotimes^{l_2}_{\mathbb{R}} W^*_{\mathbb{R}})$.

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