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FINITELY GENERATED PSEUDOCOMPLEMENTED DISTRIBUTIVE LATTICES

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If L is a pseudocomplemented distributive lattice which is generated by a finite set X, then we will show that there exists a subset G of L which is associated with X in a natural way such that $|G| \leqq |X| + 2^{|X|}$ and whose structure as a partially ordered set characterizes the structure of L to a great extent. We first prove in Section 2 as a basic fact that each element of L can be obtained by forming sums (joins) and products (meets) of elements of G only. Thus, L considered as a distributive lattice with 0, 1 (the operation of pseudocomplementation deleted), is generated by G. We apply this to characterize for example, the maximal homomorphic images of L in each of the equational subclasses of the class B_ω of pseudocomplemented distributive lattices, and also to find the conditions which have to be satisfied by G in order that X freely generates L.

In Section 3 we investigate the pseudocomplemented meet semilattice $\bar{G}$ which is generated by G for the case that L is freely generated by X. It is shown that $\bar{G} \sim \{0\}$ is exactly the set of join-irreducibles of L (Urquhart (to appear)). Furthermore we show that $\bar{G}$ is the pseudocomplemented meet-semilattice which is freely generated by X (cf. Balbes (1973)) and that L is isomorphic to the algebra freely generated by $\bar{G}$ over the class of distributive lattices, where $\bar{G}$ is considered as a partial lattice.

It follows from the basic result in Section 2 mentioned above, that L considered as a distributive lattice with 0, 1, is a lattice homomorphic image of the distributive lattice with 0, 1 which is freely generated by a set of cardinality $|G|$. It is a natural question to ask whether $|G|$ is minimal with this property. This question is answered in Section 4 in the affirmative.

In Section 5 we generalize some of the results obtained in the previous sections to the case that L is infinite.

1. Preliminaries

For the notions of *algebra*, *subalgebra*, *partial algebra*, *relative* (*partial*) *algebra, homomorphism between partial algebras, principal congruence relation, maximal homomorphic image*, etc. we refer the reader to Grätzer (1968). We will often denote a (partial) algebra $\langle A, F\rangle$ by the symbol A only. If A and B are (partial) algebras of the same similarity type then $[A, B]$ will denote the set of homomorphisms from A to B. It will often be useful, if we deal with a class V of (partial) algebras of a certain similarity type and if A and $B \in V$, to write $[A, B]_V$ instead of $[A, B]$. If V is an equational class of algebras and $A \in V$, $T \subseteq A$, then $[T]_V$ will denote the *subalgebra* of A *generated by* T. If $T = \{x_1, \cdots, x_n\}$, then we will write $[x_1, \cdots, x_n]_V$ instead of $[\{x_1, \cdots, x_n\}]_V$. If V is an equational class of algebras then $FV(X)$ denotes the *free algebra over V on a free generating set X*. If $|X| = \alpha$, then we also use the symbol $FV(\alpha)$. Again, if V is an equational class and A is a partial algebra of the same similarity type then $FV(A)$ denotes the *algebra freely generated by A over V*. Thus $FV(A) \in V$ and there exists an isomorphism f between A and a relative subalgebra A' of $FV(A)$ such that $[A']_V = FV(A)$ and for each $g \in [A, B]$, there exists an $h \in [FV(A), B]$ with $h \cdot f = g$.

Of particular interest in this paper are the equational classes of algebras:

D: distributive lattices with operations $\cdot$ and $+$.

D_{01}: distributive lattices with 0, 1 and operations $+, \cdot, 0, 1$.

B_ω: pseudocomplemented distributive lattices with operations $+, \cdot, *, 0$.

M: pseudocomplemented (meet) semilattices with operations $\cdot, *, 0$.

The operation $*$ in B_ω and M is defined by $xx^* = 0$ and if $xy = 0$, then $y \leqq x^*$. For the properties of these classes see Grätzer (1968), Frink (1962) and Balbes (1973) Recall that for $L \in B_\omega$ or $L \in M$ we have for $x, y \in L$.

1.1 (i) $x \leqq y$ implies $x^* \geqq y^*$

(ii) $x \leqq x^{**}$

(iii) $x^* = x^{***}$

The two element Boolean algebra is denoted by $\mathbf{2}$ and $\mathbf{2}^m \oplus \mathbf{1}$, $m \geqq 0$ stands for the algebra obtained from $\mathbf{2}^m$ by adjoining another one element. Note $\mathbf{2}^m \oplus \mathbf{1} \in B_\omega$ for $m \geqq 0$. For $L \in B_\omega$, we let $S(L) = \{x^* \mid x \in L\}$. It is well known that $S(L)$ is a Boolean algebra under the partial ordering of L. It is known that besides B_ω the only equational subclasess of B_ω are the classes B_m, $m = -1, 0, 1, \cdots$ and where B_{-1} is the trivial class and where for $m \geqq 0$ B_m is the class generated by $\mathbf{2}^m \oplus \mathbf{1}$ (Lakser (1971), Lee(1970)). If $L \in B_\omega$, then $L \in B_\omega$, $m \geqq 1$, is equivalent to either of the following conditions (Grätzer (1971)).

1.2. For $z_0, z_1, \cdots, z_m \in L$:

(i) $(z_1 z_2 \cdots z_m)^* + (z_1^* z_2 \cdots z_m)^* + \cdots + (z_1 z_2 \cdots z_m^*)^* = 1$

(ii) if $z_i z_j = 0$ for all $i \neq j$, then $z_0^* + z_1^* + \cdots + z_m^* = 1$.

Finally, for notational convenience, if X is a set, $T \subseteq\cdot X$ means T is a finite non-void subset of X.

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2. Lattice theoretic generation of B_ω algebras

It is well known that the congruence relation $\sim$ on $L \in B_\omega$ defined by $x \sim y$ if and only if $x^* = y^*$ is such that $S(L) \cong L/\sim$. In the following lemma we give an alternate characterization of this congruence relation for the case that L is finite. This result will be used to characterize the * operation of L, L finite, in terms of the atoms of L.

2.1 LEMMA. *Let $L \in B_\omega$, L finite. For $x, y \in L$, define $x \equiv y$ if and only if $\{a \in L \mid a$ is an atom of L, $a \leqq x\} = \{a \in L \mid a$ is an atom of L, $a \leqq y\}$. Then $x \equiv y$ if and only if $x^* = y^*$.*

PROOF. Let $x \in L$. Let $y = \Sigma\{a \in L \mid a$ is an atom, $a \nleqq x\}$. Let $z = \Sigma\{w \in L \mid w \equiv y\}$. Then by distributivity $z \equiv y$. Hence $xz = 0$. Moreover, suppose $xu = 0$ for some $u \in L$. It must be that $\{a \in L \mid a$ is an atom, $a \leqq u\} \subseteq \{a \in L \mid a$ is an atom, $a \leqq y\}$. So $u \leqq z$. Hence $z = x^*$. The lemma now follows.

2.2 NOTATION. Let $L \in B_\omega$ with $L = [x_1, \cdots, x_n]_{B_\omega}$. Define $x_i^0 = x_i$ and $x_i^1 = x_i^*$. For $1 \leqq j \leqq 2^n$, let $a_j = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, with $(\varepsilon_1, \cdots, \varepsilon_n) \in \{0, 1\}^n$. Define $b_j = a_j^*$. For $1 \leqq i \leqq n$ and $1 \leqq j \leqq 2^n$ let $(a_j)_i = (b_j)_i = \varepsilon_i$. Also let $X = \{x_1, \cdots, x_n\}$, $A = \{a_1, \cdots, a_{2^n}\}$, $B = \{b_1, \cdots, b_{2^n}\}$ and $G = X \cup B$. In the sequel, the sets G and B will be of particular interest. In this section we will show that the partial order structure of G and B determine the algebraic structure of L.

2.3 LEMMA. *Let $L \in B_\omega$ with $L = [x_1, \cdots, x_n]_{B_\omega}$. Then each $a_i \in A$ is an atom or 0. Moreover, every atom in L is equal to some a_i for exactly one i.*

PROOF. Clearly $x_j \cdot a_i \in \{0, a_i\}$ for all x_j. Let $y, z \in L$ be such that $ya_i \in \{0, a_i\}$ and $za_i \in \{0, a_i\}$. Certainly $(yz)a_i \in \{0, a_i\}$ and by distributivity $(y + z)a_i \in \{0, a_i\}$. If $ya_i = 0$, then $a_i \leqq y^*$. Thus $y^*a_i = a_i$ If $ya_i = a_i$, then $y \geqq a_i$. So by 1.1, $y^* \leqq a_i^*$, so $y^*a_i = 0$. Since $L = [x_1, \cdots, x_n]_{B_\omega}$ this completes the proof of the first claim. Next observe that $s = \Sigma\{a_i \mid 1 \leqq i \leqq 2^n\} = (x_1 + x_1^*) \cdots (x_n + x_n^*)$. Hence $s^* = 1^*$, so by 2.1 every atom is equal to some a_i. If $a_i = a_j$ in L, for $i \neq j$, then there exists k for which $(a_i)_k \neq (a_j)_k$. So $a_i \leqq x_k x_k^* = 0$.

From 2.1 it follows that if a is an atom of L, then a^* is a dual atom of $S(L)$. So from 2.3 it follows that each $b_i \in B$ is either 1 or a dual atom in the Boolean algebra $S(L)$. Moreover, every dual atom of $S(L)$ is equal to exactly one b_i. Thus, $S(L)$ is generated by B under the formation of products. (Note $\Pi\phi = 1$). Indeed, let $z^* \in S(L)$. Form $T = \{b_i \mid a_i \leqq z\}$. It is easily seen that $z^* = \Pi T$.

2.4 THEOREM. *Let $L \in B_\omega$, $L = [X]_{B_\omega}$, X finite. Then $L = [G]_{D_{01}}$.*

PROOF. Since $L = [X]_{B_\omega}$ and $X \subseteq G$, only applications of * need be considered. By the remarks following 2.3, any application of * is equivalent to forming ΠT for some $T \subseteq B \subseteq G$. Indeed, if $z \in L$, then $z = \Pi T_1 + \cdots + \Pi T_r$ for some family of sets $T_i \subseteq G$, $1 \leqq i \leqq r$.

2.6 THEOREM. *Let $L \in B_\omega$, $L = [x_1, \cdots, x_n]_{B_\omega}$. Then $L \in B_\omega$, $m \geqq 1$, if and only if for all $I \subseteq \{1, 2, \cdots, 2^n\}$ such that $|I| \geqq m+1$, the equality $\sum_{i \in I} b_i = 1$ holds.*

PROOF. By 2.3, $a_i a_j = 0$ for $i \neq j$. So by 1.2 $\sum_{i \in I} b_i = 1$. Conversely, suppose $\{y_0, y_1, \cdots, y_m\} \subseteq L$ and $y_i y_j = 0$ for $i \neq j$. Note $y_i^* = \Pi\{b_k | a_k \leqq y_i\} = \Pi T_i$. Hence $T_i \cap T_j = \varnothing$ or $\{1\}$. So $\sum_{i=0}^m y_i^* = \sum_{i=0}^m (\Pi T_i) = (\Sigma Q_1) \cdots (\Sigma Q_r)$ where each Q_i contains the element 1 or $m+1$ b_j. Hence $\Sigma Q_i = 1$ for all i.

2.7 THEOREM. *Let $L = [x_1, \cdots, x_n]_{B_\omega}$. Define $u_m = \Pi\{\Sigma S | S \subseteq B, |S| = m+1\}$. Let $\theta(u_m, 1)$ be the principal B_ω congruence relation generated by $\{u_m, 1\}$. Define $L_m = L/\theta(u_m, 1)$. Then all of the following hold:*

i) $L_m \in B_m$

ii) *L_m is a maximal homomorphic image of L in B_m*

iii) *L_m is isomorphic to the interval $[0, u_m] \subseteq L$.*

PROOF. By 2.6, $L_m \in B_m$. If $L/\theta = L_1 \in B_m$ then again by 2.6 $1 \equiv u_m(\theta)$. Hence, $\theta \geqq \theta(u_m, 1)$, so (ii) holds. Observe that since $\theta(u_m, 1)$ is determined by a principal filter, $x \equiv y\ (\theta(u_m, 1))$ if and only if $xu_m = yu_m$ (Lakser (1973)). So every congruence class of L_m contains exactly one element in $[0, u_m]$. Hence (iii) follows.

We now specialize to the case where L is free in B_ω.

2.8 THEOREM. *Let $G = X \cup B \subseteq FB_\omega'X)$ with $X = \{x_1, \cdots, x_n\}$. For $S, T \subseteq\cdot\ G$, $\Pi S \leqq \Sigma T$ if and only if at least one of the following hold:*

(i) $S \cap T \neq \varnothing$.

(ii) *There exist $1 \leqq j \leqq 2^n$ and $1 \leqq i \leqq n$ with $b_j \in T$ and $(b_j)_i = 1$ for some $x_i \in S$.*

(iii) $B = \{b_j | b_j \in S\} \cup \{b_j | (b_j)_i = 1,\ x_i \in S\}$.

For $FB_m(X)$ the following condition may be added to the list:

(iv) $|T \cap B| > m$.

PROOF. $\Leftarrow$ (i) suffices in any lattice. For (ii) observe that $(b_j)_i = 1$ implies $x_i^* \geqq a_j$. So by 1.1 $x_i \leqq x_i^{**} \leqq b_j \in T$. If (iii) holds then $\Pi S \leqq \Pi B = 0$. In the case of $FB_m(X)$, if $|T \cap B| > m$, then $\Sigma T = 1$ by 2.6. $\Rightarrow$ Suppose in $FB_\omega(X)$ $\Pi S \leqq \Sigma T$ and neither (i), (ii) nor (iii) hold. Note that since the Boolean algebra $\mathbf{2}^{2^n}$ is a B_ω homomorphic image of $FB_\omega(n)$, each of the b_i, $1 \leqq i \leqq 2^n$, are distinct in $FB_\omega(n)$. Let $|T \cap B| = t$. If $t = 0$, by the negation of (iii) a b_j may be adjoined to T for which conditions (i) (ii) nor (iii) will still not hold. So assume $|T \cap B| = t \geqq 1$. There exists $f \in [FB_\omega(X), \mathbf{2}^t]_{B_\omega}$ such that $f(a_i)$ is an atom of $\mathbf{2}^t$ for all i, $b_i \in T$, $f(a_i) = 0$ otherwise. Adjoin a new maximal element $1'$ to $\mathbf{2}^t$ to obtain $L = \mathbf{2}^t \oplus \mathbf{1}'$. Thus $L \in B_\omega$, $0^* = 1'$ and $1'$ is join-irreducible. Assume $x_i \in S$ for

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$1 \leqq i \leqq k$, $x_i \notin S$, $i > k$. Define $\gamma\colon X \to L$ by $\gamma(x_i) = 1'$ for $1 \leqq i \leqq k$, $\gamma(x_i) = f(x_i)$ otherwise. Let $g \in [FB_\omega(n), L]_{B_\omega}$ extend γ. If $x_i \in T$, then by the negation of (i), $g(x_i) < 1'$. If $b_j \in T$, then by the negation of (ii) $g(a_j) \geqq f(a_j)$ so $1' > f(b_j) \geqq g(b_j)$. Thus $g(\Sigma T) < 1'$. If $x_i \in S$ or $b_j \in S$ with $(b_j)_i = 1$ for some $1 \leqq i \leqq k$, then $g(x_i) = g(b_j) = 1'$. So suppose $b_j \in S$, $a_j = x_1 \cdots x_k x_{k+1}^{\varepsilon_{k+1}} \cdots x_n^{\varepsilon_n}$. Let $C = \{a_i \mid (a_j)_r = (a_i)_r \text{ for all } r > k\}$. Then $0 = \Sigma f(C) = f(x_{k+1}^{\varepsilon_{k+1}} \cdots x_n^{\varepsilon_n})$ $g(x_{k+1}^{\varepsilon_{k+1}} \cdots x_n^{\varepsilon_n})$. Thus $g(b_j) = 1'$ also. So $g(\Pi S) = 1'$. This contradicts the assumption $\Pi S \leqq \Sigma T$. For the case of $FB_m(X)$, note that if (iv) does not hold, then $t = |T \cap B| \leqq m$. But $\mathbf{2}^t \oplus \mathbf{1}' \in B_m$ for $t \leqq m$. So the above contradiction can be obtained.

2.9 THEOREM. *The lattice $FB_\omega(n)$ contains for each $m < \omega$ an ideal which is lattice isomorphic to $FB_m(n)$. Moreover, these ideals form a chain when ordered by inclusion.*

PROOF. Since $B_m \subseteq B_\omega$, $FB_m(n)$ is a homomorphic image of $FB_\omega(n)$. Let θ_m be a congruence on $FB_\omega(n)$ such that $FB_\omega(n)/\theta \simeq FB_m(n)$. Apply 2.7 to show $\theta_m = \theta(u_m, 1)$. So by 2.7 (iii) the ideal $[0, u_m]$ is lattice isomorphic to $FB_m(n)$. Finally, note $u_1 \leqq u_2 \leqq \cdots u_m \leqq \cdots$ in $FB_\omega(n)$.

Next, independence conditions are obtained for B_ω and B_m. See Marczewski (1958) of Grätzer (1968) for a general discussion of independence.

2.10 THEOREM. *Let $L \in B_\omega$, $L = [x_1, \cdots, x_n]_{B_\omega}$. $L \simeq FB_\omega(n)$ if and only if whenever $S, T \subseteq\cdot\ G$ and $\Pi S \leqq \Sigma T$, then one or more of* 2.8 (*i*), (*ii*) *or* (*iii*) *hold. Moreover, if $L \in B_m$, then $L \simeq FB_m(n)$ if and only if the additional condition of* 2.8 (*iv*) *is included.*

PROOF. $\Rightarrow$ Use 2.8. $\Leftarrow$ Let $X = \{x_1, \cdots, x_n\}$. Define $\gamma\colon X \to L$ by $\gamma(x_i) = x_i$. Then γ extends to a B_ω homomorphism g from $FB_\omega(n)$ ($FB_m(n)$) onto $L \in B_\omega$ ($L \in B_m$). A standard argument shows that conditions (i) (ii) (iii) (and (iv)) guarantee that g is also one-to-one.

2.11 COROLLARY (Grätzer, Lakser (1971; page 190)), *For $k \geqq 2^n$, $FB_\omega(n) \simeq FB_k(n)$.*

PROOF. For $k \geqq 2^n$, since $|B| \leqq 2^n$, condition 2.8 (iv) cannot hold.

3. The semilattice generated by G

3.1 NOTATION. Consider $L \in B_\omega$, $L = [X]_{B_\omega}$ with X finite. Let G be as in 2.2. Form $\bar{G} = \{\Pi T \mid T \subseteq G\}$. Thus, $\bar{G}$ is the closure of G under the formation of products. Observe $\Pi\phi = 1 \in \bar{G}$ and from the remarks following 2.3, $S(L) \subseteq \bar{G}$. Also, 2.4 implies that the set of join-irreducible elements of L is contained in $\bar{G}$.

3.2 NOTATION. Let $\bar{G}$ be as in 3.1 with $z \in \bar{G}$. Let $\beta(z) = \{b_i \in B \mid b_i \geqq z\}$ and $\chi(z) = \{x_i \in X \mid x_i \geqq z\}$. Then $z = (\Pi\beta(z))(\Pi\chi(z))$.

In the remainder of this section we will be concerned with the set $\bar{G} \subseteq FB_\omega(X)$ or $\bar{G} \subseteq FB_m(X)$ for $|X| = n$, n finite.

3.3 THEOREM. *The join-irreducible elements of $FB_m(n)$ are precisely those $z \in \bar{G}$, $z \neq 0$, for which $2^n > |\beta(z)| \geqq 2^n - m$. In particular, the set of join-irreducibles of $FB_\omega(X)$ is $\bar{G}\backslash\{0\}$.*

PROOF. If $z \in \bar{G}$ and if $|\beta(z)| = 2^n$, then $z = 0$. If $|\beta(z)| < 2^n - m$, then there exist, say $b_0, \cdots, b_m \in B$, with $b_i \ngeqq z$ for $0 \leqq i \leqq m$. Thus in $FB_m(n)$ we have $z = z \cdot 1 = z(b_0 + \cdots + b_m) = zb_0 + \cdots + zb_m$. Finally, suppose $2^n > |\beta(z)| \geqq 2^n - m$. Let $z = \Pi T_1 + \cdots + \Pi T_r$, with $T_i \subseteq G$. If $\Pi T_i \neq z$ for all i, then for each i there exist $t_i \in T_i$ such that $t_i \ngeqq z$.

Thus

$$0 \neq z = (\Pi\beta(z))(\Pi\chi(z)) \leqq t_1 + \cdots + t_r.$$

Since $|\{t_1, \cdots, t_r\} \cap B| \leqq m$, 2.8 gives a contradiction. So z is join-irreducible. The second part of the theorem follows from 2.11.

For an alternate characterization of the join-irreducibles of $FB_m(n)$ and $FB_\omega(n)$ see Urquhart (to appear).

We now determine the number of join-irreducible elements in $FB_m(n)$ and $FB_\omega(n)$. This, of course, gives the lengths of these lattices. Compare Balbes (1973).

3.4 THEOREM. *Define $p(s,t) = \sum_{i=1}^{t} \binom{2^s}{i}$. Then the number of join-irreducible elements of $FB_m(n)$ is $\sum_{k=0}^{n} \binom{n}{k} p(n-k, m)$. In particular, for $FB_\omega(n)$ this is equal to $\sum_{k=0}^{n} \binom{n}{k} (2^{2^k} - 1)$.*

PROOF. If $z = (\Pi\beta(z))(\Pi\chi(z))$ with $|\chi(z)| = k$, then $B\backslash\beta(z)$ is contained in a set of cardinality 2^{n-k}. Since $B\backslash\beta(z) \neq \varnothing$, there are $p(n-k, m)$ choices for $\beta(z)$. Hence, $\binom{n}{k}$ $p(n-k, m)$ possible choices for z. Finally observe that if $m \geqq 2^n$, then $p(n-k, m) = 2^{2^k} - 1$ for all k. Apply 2.11 to complete the proof.

3.5 REMARK. Note that $\bar{G}$ is closed with respect to $\cdot$ and $*$. If $\bar{G}$ is considered as a relative partial B_ω subalgebra of $FB_\omega(n)$, then for $y, z \in \bar{G}$, $y + z$ is defined if and only if $y + z \in \bar{G}$. So by 3.3, $y + z$ is defined if and only if y and z are comparable elements of $FB_\omega(n)$.

3.6 THEOREM. *Let $\bar{G} \subseteq FB_\omega(X)$, $|X|$ finite, be as in 3.1. Consider $\bar{G}$ as a partial lattice where $x + y$ is defined if and only if x and y are comparable. Then $FB_\omega(X)$ is the distributive lattice freely generated over D by the partial lattice $\bar{G}$.*

PROOF. By 3.5 $\bar{G}$ is a relative partial lattice and by 2.4 $[\bar{G}]_D = FB_\omega(X)$. Let $L \in D$ and $f \in [\bar{G}, L]_D$. Since $L_1 = [f(\bar{G})]_D$ is finite, $L_1 \in B_\omega$. So the function f restricted to X has an extension to $g \in [FB_\omega(X), L_1]_{B_\omega}$. But since $\bar{G} \subseteq [X]_D$ and f is a partial homomorphism, g extends f as well. So $g \in [FB_\omega(X), L]_D$.

The next lemma gives an embedding of an arbitrary pseudocomplemented semilattice in a pseudocomplemented lattice. See also Balbes (1969) and Dyson (1965).

3.7 LEMMA. *Let $S \in M$, S arbitrary. There exists $L \in B_\omega$ such that the reduct $L' = \langle L, \cdot, {}^*, 0 \rangle$ of L has a subalgebra S' which is M-isomorphic to S. Moreover, $[S']_D = L$.*

PROOF. For $a \in S$ let $(a] = \{z \in S \mid 0 \leqq z \leqq a\}$. Form $S' = \{(a] \mid a \in S\}$. Let L be the ring of sets generated by S'. It is easily verified that S' is a pseudocomplemented semilattice with zero element $(0]$, with $\cap$ for product and with $(a]^* = (a^*]$. To complete the proof it remains to show $L \in B_\omega$. Let $T \in L$. So there exist $t_1, \cdots, t_p \in S$ such that $T = (t_1] \cup \ldots \cup (t_p]$. Omit all t_i for which $t_i \leqq t_j$ for some $j \neq i$. Then this is a unique representation for T. For, if otherwise, $T = (t_1] \cup \cdots \cup (t_p] = (r_1] \cup \cdots \cup (r_q]$. So for any i, $t_i \leqq r_j$ for some j. Similarly $r_j \leqq t_k$. Thus $t_i \leqq t_k$ so $t_i = r_j = t_k$. Thus $\{t_1, \cdots, t_p\} = \{r_1, \cdots, r_q\}$. Define $T^* = (t_1^*] \cup \cdots \cup (t_p^*]$. By the above, T^* is well defined and easily seen to be the pseudocomplement of T in L.

3.8 COROLLARY. *Let $S \in M$ be an arbitrary pseudocomplemented semilattice. Consider S as a partial B_ω lattice where $x + y$ is defined if and only if x and y are comparable. Then the B_ω lattice L constructed from S in 3.7 is isomorphic to the B_ω lattice freely generated by the partial B_ω algebra S.*

PROOF. It is easily seen that the set $S' \backslash (0]$ consists only of join-irreducibles in L. So S' is a relative partial B_ω subalgebra of L. Using the uniqueness of the representation of elements of L as union of principal ideals in S, the mapping extension property can be verified.

3.9 THEOREM. *Let $\bar{G} \subseteq FB_\omega(n)$, n finite. Then $\bar{G}$ is M-isomorphic to $FM(n)$.*

PROOF. Let $X = \{x_1, \cdots, x_n\}$ and $Y = \{y_1, \cdots, y_n\}$. Consider arbitrary $S \in M$, $S = [Y]_M$. Suppose $FB_\omega(n) = [X]_{B_\omega}$. Construct S' and L as in 3.7. Identify S with $S' \subseteq L$. Observe $[Y]_{B_\omega} = L$. Let $\gamma(x_i) = y_i$ and extend γ to $g \in [FB_\omega(n), L]_{B_\omega}$. Then $g(\bar{G})$ is closed under $\cdot$ and $*$ and contains 0 and Y. Hence $S = g(\bar{G})$. So $g \in [\bar{G}, S]_M$ and g extends γ as desired.

For an alternate characterization of $FM(n)$, see Balbes (1973).

4. Minimality

It follows from 2.4 that $FB_\omega(n)$ is a D_{01} homomorphic image of $FD_{01}(n + 2^n)$. A natural question is whether the number $n + 2^n$ is minimal with this property. Or equivalently, does there exist a subset $S \subseteq FB_\omega(n)$, $|S| < n + 2^n$, for which $FB_\omega(n) = [S]_{D_{01}}$?

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4.1 LEMMA. *Every element of* $B \subseteq FB_\omega(n)$ *is both meet-irreducible and join-irreducible.*

PROOF. Let $b_j \in B$. By 2.8 b_j is join-irreducible. Suppose $b_j = pq$. By 2.4 we may write $p = (\sum S_1)\cdots(\sum S_k)$ and $q = (\sum T_1)\cdots(\sum T_l)$, with $S_i, T_i \subseteq G$. Use 2.8 and an argument similar to that in 3.3 to show that for some i, $b_j \geqq \sum S_i$ or $b_j \geqq \sum T_i$.

4.2 THEOREM. *If* $FB_\omega(n) = [Y]_{D_{01}}$, *then* $|Y| \geqq n + 2^n$.

PROOF. By 4.1 $B \subseteq Y$. Each x_i, $1 \leqq i \leqq n$, in the free generating set for $FB_\omega(n)$ is join-irreducible. So $x_i = \Pi S_i$, $S_i \subseteq Y$. Define $T_i = \{y \in Y \mid y \in S_i, y \notin B\}$. By 2.8, $T_i \neq \emptyset$ for each i. Let $b_1 \in B$ be such that $(b_1)_i = 0$ for all i, $1 \leqq i \leqq n$. If for some i, $T_i \subseteq \bigcap_{j \neq i} T_j$, then $S_i \subseteq (\bigcup_{j \neq i} T_j) \cup (B\backslash\{b_1\})$. Hence $x_i \geqq (\Pi_{j \neq i} x_j)(\Pi(B\backslash\{b_1\}))$. This violates 2.8. So each T_i contains, say, y_i for which $y_i \notin B$ and $y_i \notin T_j$ for $j \neq i$. Thus $|Y| \geqq n + 2^n$.

It is interesting to note that 4.2 is not true for $FB_m(n)$, m arbitrary.

5. The infinite case

In this final section we generalize some of the results of the previous sections to $FB_\omega(X)$, where X is an infinite set of arbitrary cardinality.

5.1 DEFINITION. *For* $Y \subseteq\cdot\ X$, *define* $B(Y) \subseteq FB_\omega(X)$ *by*

$$B(Y) = \{[(\Pi S^*)(\Pi((Y\backslash S))]^* \mid S \subseteq Y\}.$$

Let $B = \cup\{B(Y) \mid Y \subseteq\cdot\ X\}$. *Form* $G = X \cup B$.

5.2 THEOREM. $FB_\omega(X) = [G]_{D_{01}}$. *Moreover, if* α *is any infinite cardinal,* $FB_\omega(\alpha)$ *is a* D_{01} *homomorphic image of* $FD_{01}(\alpha)$.

PROOF. For $z \in FB_\omega(X)$, z may be obtained from a finite subset $Y \subseteq X$ by a finite series of applications of $+$, $\cdot$ and $*$. Apply 2.4 to show $[Y]_{B_\omega} = [Y \cup B(Y)]_{D_{01}}$. The second claim follows from the fact that $|G| = |X|$, whenever X is infinite.

5.3 REMARK. If $z_1, \cdots, z_k \in FB_\omega(X)$, then there exists some set $Y \subseteq X$ such that $z_1, \cdots, z_k \in [Y]_{B_\omega}$. Observe $[Y]_{B_\omega} \cong FB_\omega(Y)$. Thus, the results of sections 2 and 3 apply to $z_1, \cdots, z_k$. In particular, for X infinite and G as in 5.1, define $\bar{G} = \{\Pi T \mid T \subseteq G, T \text{ finite}\}$. It can be seen that the set of join-irreducible elements of $FB_\omega(X)$ is $\bar{G}\backslash\{0\}$. Thus $\bar{G}$ is a relative partial sublattice of $FB_\omega(X)$: $y + z$ is defined only in the case that y and z are comparable. Arguments similar to those in Section 3 give the following:

5.4 THEOREM. $FB_\omega(X)$ *is the distributive lattice freely generated in D by the partial lattice* $\bar{G}$.

5.5 THEOREM. *The pseudocomplemented semilattice $\bar{G}$ is M isomorphic to $FM(X)$.*

Recall that for $L \in D$, a subset $F \subseteq L$ is a prime filter if and only if there exists $h \in [L, \{0, 1\}]_D$, h onto such that $h^{-1}(1) = F$. Observe that every proper filter in the partial lattice $\bar{G}$ is a prime. Apply 5.4 to obtain

5.6 THEOREM. *The partially ordered set of prime filters of $FB_\omega(X)$ is isomorphic to the partially ordered set of proper filters of $\bar{G}$.*

References

R. Balbes (1969), 'A representation theory for prime and implicativde semilattices', *Trans. Amer. Math. Soc.* **136,** 261–267.

R. Balbes (1973), 'On free pseudo-c'omplemented and relatively pseudo-complemented semi-lattices', *Fund. Math.* **8,** 119–131.

V. Dyson (1965), *An embedding theorem for semilattices*, (Hughes Aircraft Technical Report, 1965).

O. Frink (1962), 'Pseudo-complements in semi-lattices', *Duke Math. J.* **29,** 505–514.

G. Grätzer (1968), *Universal Algebra*, (Van Nostrand, 1968).

G. Grätzer (1971), *Lattice Theory*, (W. H. Freeman and Co., 1971).

H. Lakser (1971), 'The structure of pseudo-complemented, distributive lattices I. Subdirect decomposition', *Trans. Amer. Math. Soc.* **156,** 335–342.

H. Lakser (1973), 'Principal congruences of pseudocomplemented distributive lattices', *Proc. Amrel Math. Soc.* **37,** 32–36.

K. B. Lee (1970), 'Equational classes of distributive pseudocomplemented lattices', *Canad. J. Math.* **22** 881–891.

E. Marczewski (1958), 'A general scheme of the notions of independence in mathematics', *Bull. Acad. Polon. Sci.* **6** 731–736.

A. Urquhart (1973), 'Free distributive pseudo-complemented lattices', Algebra Universalis **3,** 13–15.

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