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A theorem on homeomorphism groups and products of spaces

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Let $H(C)$ be the group of homeomorphisms of the Cantor set, C, onto itself. Let $p : C \to M$ be a map of C onto a compact metric space M, and let $G(p, M)$ be $\{h \in H(C) \mid \forall x \in C, p(x) = ph(x)\}$. $G(p, M)$ is a group. The map $p : C \to M$ is *standard*, if for each $(x, y) \in C \times C$ such that $p(x) = p(y)$, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset C$ and a sequence $\{h_n\}_{n=1}^{\infty} \subset G(p, M)$ such that $x_n \to x$ and $h_n(x_n) \to y$. Standard maps and their associated groups characterize compact metric spaces in the sense that: Two such spaces, M and N, are homeomorphic if and only if, given p standard from C onto M, there is a standard q from C onto N for which $G(p, M) = h^{-1} G(q, N)h$, for some $h \in H(C)$

The present paper exhibits a structure theorem connecting these characterizing subgroups of $H(C)$ and products of spaces: Let M_1 and M_2 be compact metric spaces. Then there are standard maps $p : C \to M_1 \times M_2$ and $p_i : C \to M_i$, $i = 1, 2$, such that $G(p, M_1 \times M_2) = G(p_1, M_1) \cap G(p_2, M_2)$.

The following definition and results were given in [1]:

Let $H(C)$ be the group of homeomorphisms of the Cantor set, C, onto itself. Let $p : C \to M$ be a map (continuous) of C onto a compact metric

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space M , and let $G(p\ ,\ M) = \{h \in H(C)\ |\ \forall\ x \in C\ ,\ p(x) = ph(x)\}$. $G(p\ ,\ M)$ is a group.

A map, p , of C onto M is a *standard map*, if for each pair of points, x and y , such that $p(x) = p(y)$, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset C$ and a sequence $\{h_n\}_{n=1}^{\infty} \subset G(p\ ,\ M)$ such that $x_n \to x$ and $h_n(x_n) \to y$. Standard maps are very naturally obtained and their groups characterize compact metric spaces in the following sense: Two compact metric spaces, M and N , are homeomorphic if and only if, given p standard from C onto M , there is a standard q from C onto N for which $G(p\ ,\ M) = h^{-1}G(q\ ,\ N)h$, for some $h \in H(C)$.

The following theorem is intended to exhibit a natural connection between these characterizing subgroups of $H(C)$ and products of compact metric spaces.

THEOREM. *Let* M_1 *and* M_2 *be compact metric spaces. Then there are standard maps* $p : C \to M_1 \times M_2$ *and* $p_i : C \to M_i$, $i = 1\ ,\ 2$, *such that* $G(p\ ,\ M_1 \times M_2) = G(p_1\ ,\ M_1) \cap G(p_2\ ,\ M_2)$.

While the following hardly deserves to be called a lemma, it is inserted before the proof of the theorem to simplify subsequent constructions.

LEMMA. *If* $p : C_2 \to M$, *a compact metric space, is a standard map, and* $h : C_1 \to C_2$ *is a homeomorphism, then* $ph : C_1 \to M$ *is a standard map.*

Proof of Lemma. Suppose, for $x\ ,\ y \in C_1$, $ph(x) = ph(y)$. Let $z = h(x)$ and $w = h(y)$, so that $p(z) = p(w)$. Then, from the standardness of p , there exist sequences $\{x_n\}_{n=1}^{\infty} \subset C_2$ and $\{f_n\}_{n=1}^{\infty} \subset G(p\ ,\ M)$ such that $x_n \to z$ and $f_n(x_n) \to w$. If $s_n = h^{-1}(x_n)$, then $h(s_n) \to h(x) = z$ and $s_n \to x$. Defining $\{g_n = h^{-1}f_n\ h\}_{n=1}^{\infty}$, observe that, for $v \in C_1$, $phg_n(v) = ph(h^{-1}f_nh)(v) = p\ f_n(h(v)) = ph(v)$, so that $\{g_n\}_{n=1}^{\infty} \subset G(ph\ ,\ M)$. Also $g_n(s_n) = h^{-1}f_n(x_n) \to h^{-1}(w) = y$.

Proof of Theorem. Let C_1 and C_2 be Cantor sets; let $h : C \to C_1 \times C_2$ be a homeomorphism, and let $q_i : C_i \to M_i$, $i = 1\ ,\ 2$,

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be standard maps. With $\Pi_i : C_1 \times C_2 \to C_i$ defined by $\Pi_i(y_1 , y_2) = y_i$, $i = 1 , 2$, let $p : C \to C_1 \times C_2 \to M_1 \times M_2$ be defined by $p(x) = (q_1\Pi_1h(x) , q_2\Pi_2h(x))$. Clearly, p is continuous onto $M_1 \times M_2$.

Next we see that each $p_i = q_i\Pi_i h$ is a standard map: We first show that $q_i\Pi_i : C_1 \times C_2 \to M_i$ is standard and then remark that, by the Lemma, $q_i\Pi_i h$ is still a standard map. Let $q_i\Pi_i(x_1 , x_2) = q_i\Pi_i(y_1 , y_2)$; then standardness of q_i says there is a sequence $\{z_n = \Pi_i(w_{1,n} , w_{2,n})\}_{n=1}^{\infty} \subset C_i$ and a sequence $\{h'_n\}_{n=1}^{\infty} \subset G(q_i , M_i)$ such that $z_n = w_{i,n} \to x_i$ and $h'_n(w_{i,n}) \to y_i$.

From now on, it will be notationally convenient to work with a particular choice of i, say $i = 1$. Let $w_{2,n} = x_2$ for the ordered pairs above - we are only interested in the projection onto the first coordinate - and let $h_0 : C_2 \to C_2$ be a homeomorphism for which $h_0(x_2) = y_2$. Now let $h_n : C_1 \times C_2 \to C_1 \times C_2$ be defined by $h_n(v_1 , v_2) = (h'_n(v_1) , h_0(v_2))$, so that $(z_n , x_2) \to (x_1 , x_2)$ and $h_n(z_n , x_2) \to (y_1 , y_2)$. We may claim h_n, $n = 1 , \ldots ,$ is in $G(q_1\Pi_1 , M_1)$ because

$$\begin{aligned} q_1\Pi_1(v_1 , v_2) &= q_1(v_1) \\ &= q_1h'_n(v_1) \\ &= q_1\Pi_1(h'_n(v_1) , h_0(v_2)) \\ &= q_1\Pi_1h_n(v_1 , v_2) . \end{aligned}$$

The proof for $q_2\Pi_2$, with $i = 2$, is obviously similar. As noted, $q_i\Pi_ih$, $i = 1 , 2$, is also standard.

Next, we must show that $p : C \to C_1 \times C_2 \to M_1 \times M_2$ is standard: It suffices to show standardness of $q : h(C) = C_1 \times C_2 \to M_1 \times M_2$ defined by $q(v_1 , v_2) = (q_1(v_1) , q_2(v_2))$. Suppose $q(x_1 , x_2) = q(y_1 , y_2)$; this means $q_1(x_1) = q_1(y_1)$ and $q_2(x_2) = q_2(y_2)$. Standardness of q_1 implies there exist sequences $\{z_n\}_{n=1}^{\infty} \subset C_1$ and $\{f_n\}_{n=1}^{\infty} \subset G(q_1 , M_1)$ such that $z_n \to x_1$ and $f_n(z_n) \to y_1$. Likewise, there exist sequences $\{w_n\}_{n=1}^{\infty} \subset C_2$

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and $\{g_n\}_{n=1}^{\infty} \subset G(q_2 , M_2)$ such that $w_n \to x_2$ and $g_n(w_n) \to y_2$.

Consider the sequence $\{(z_n , w_n)\}_{n=1}^{\infty} \subset C_1 \times C_2$; since $z_n \to x_1$ and $w_n \to x_2$, $(z_n , w_n) \to (x_1 , x_2)$. Let $h_n : C_1 \times C_2$ be defined by $h_n(v_1 , v_2) = (f_n(v_1) , g_n(v_2))$. Then $h_n(z_n , w_n) \to (y_1 , y_2)$. We claim h_n , $n = 1 , 2 , \ldots$, is in $G(q , M_1 \times M_2)$ because

$$\begin{aligned} q(v_1 , v_2) &= (q_1\Pi_1(v_1 , v_2) , q_2\Pi_2(v_1 , v_2)) \\ &= (q_1f_n\Pi_1(v_1 , v_2) , q_2g_n\Pi_2(v_1 , v_2)) \\ &= (q_1f_n(v_1) , q_2g_n(v_2)) \\ &= (q\Pi_1(f_n(v_1) , g_n(v_2)) , q_2\Pi_2(f_n(v_1) , g_n(v_2))) \\ &= (q_1\Pi_1h_n(v_1 , v_2) , q_2\Pi_2h_n(v_1 , v_2)) \\ &= qh_n(v_1 , v_2) , \end{aligned}$$

each $(v_1 , v_2) \in C_1 \times C_2$.

Since h is a homeomorphism, $p = qh$ is also standard.

Finally, $G(p , M_1 \times M_2) = G(p_1 , M_1) \cap G(p_2 , M_2)$: First, for $f \in G(p , M_1 \times M_2)$ and each $x \in C$,

$$\begin{aligned} p(x) &= (q_1\Pi_1h(x) , q_2\Pi_2h(x)) \\ &= pf(x) \\ &= (q_1\Pi_1hf(x) , q_2\Pi_2hf(x)) , \end{aligned}$$

which says $q_i\Pi_ih(x) = q_i\Pi_ihf(x)$, $i = 1 , 2$, and $f \in G(p_1 , M_1) \cap G(p_2 , M_2)$. Second, for $f \in G(p_1 , M_1) \cap G(p_2 , M_2)$ and each $x \in C$,

$$\begin{aligned} p(x) &= (q_1\Pi_1h(x) , q_2\Pi_2h(x)) \\ &= (q_1\Pi_1hf(x) , q_2\Pi_2hf(x)) \\ &= pf(x) \end{aligned}$$

and $f \in G(p , M_1 \times M_2)$.

Reference

[1] Arnold R. Vobach, "On subgroups of the homeomorphism group of the Cantor set", *Fund. Math.* 60 (1967), 47-52.

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