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STRICT TOPOLOGY ON SPACES OF CONTINUOUS VECTOR-VALUED FUNCTIONS

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1. Introduction. In this paper, X denotes a completely regular Hausdorff space, $C_b(X)$ all real-valued bounded continuous functions on X, E a Hausforff locally convex space over reals $\mathbf{R}$, $C_b(X, E)$ all bounded continuous functions from X into E, $C_b(X) \otimes E$ the tensor product of $C_b(X)$ and E. For locally convex spaces E and F, $E \otimes_\epsilon F$ denotes the tensor product with the topology of uniform convergence on sets of the form $S \times T$ where S and T are equicontinuous subsets of E', F', the topological duals of E, F respectively ([**11**], p. 96). For a locally convex space G, G' will denote its topological dual.

If $\mathscr{U}$ is an algebra of subsets of a set Y, E, F Hausdorff locally convex spaces, $L(E, F)$ the set of all linear continuous mappings from E into F, $S(Y, \mathscr{U}, E)$ all E-valued, $\mathscr{U}$-simple functions on Y with the topology of uniform convergence on Y, and μ: $\mathscr{U} \rightarrow L(E, F)$ a finitely additive set function, then μ will be called a measure if the corresponding linear mapping μ: $S(Y, \mathscr{U}, E) \rightarrow F$ is continuous ([**12**], p. 375). Denoting by $B(Y, \mathscr{U}, E)$ the closure of $S(Y, \mathscr{U}, E)$, in the space of all bounded functions from Y into E with the topology of uniform convergence, the measure μ can be uniquely extended to a linear continuous mapping μ: $B(Y, \mathscr{U}, E) \rightarrow \tilde{F}$, $\tilde{F}$ being the completion of F. It is easy to verify that $C_b(X) \otimes E \subset B(X, \mathscr{B}, E)$, $\mathscr{B}$ being the class of all Borel subsets of X. $M_t(X)$ will denote all tight measures on X ([**6**], [**8**], [**14**]) and $M_t(X, E') = \{\mu: \mathscr{B} \rightarrow E' = L(E, R)$, μ is a measure and for every $x \in E$, μ_x: $\mathscr{B} \rightarrow R$, defined by $\mu_x(B) = \langle \mu(B), x \rangle$, is in $M_t(X)\}$.

The strict topology β_0 on $C_b(X, E)$ is defined by the family of seminorms $\| \cdot \|_{h,p}$, as h varies through all real-valued functions on X vanishing at infinity and p ranges over all continuous seminorms on E;

$$\| f \|_{h,p} = \sup_{x \in X} p(h(x) f(x)), f \in C_b(X, E).$$

When E is a normed space, it is proved in [**6**] that $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_0)$, $(C_b(X, E), \beta_0)' = M_t(X, E')$, and β_0 is the finest locally convex topology which coincides with the compact-open topology on norm-bounded subsets of $C_b(X, E)$; also bounded subsets of $(C_b(X, E), \beta_0)$ are norm-bounded. (For $E = R$ this result is proved in [**13**], but it immediately carries over to the case when E is a normed space since $M_t(X, E')$ is a closed subspace of the Banach space $(C_b(X, E), \| \cdot \|)'$.) Considering $M_t(X, E')$ a

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Banach space, with the norm induced by $(C_b(X, E), \|\cdot\|)'$, we have

$$\|\mu\| = |\mu|(X), \forall \mu \in M_t(X, E').$$

(It is simple to verify this; cf. [**8**], p. 315.) Conway [**5**] showed $(C_b(X), \beta_0)$ is strongly Mackey when X is paracompact. If X is a P-space and E is a normed (Banach) space, then $(C_b(X, E), \beta_0)$ is Mackey (strongly Mackey) [**10**].

2. Topological properties. The following are useful results. The proofs are easy, and the same results are proved for $E = \mathbf{R}$ in [**13**] and X being locally compact in [**3**].

THEOREM 2.1. *Let X be a completely regular Hausdorff space and E be a locally convex space. Then*

(1) *$p \leqq k \leqq \beta_0 \leqq u$, where p is the topology of pointwise convergence, k is the compact-open topology and u is the topology of uniform convergence on X.*

(2) *u and β_0 have the same bounded sets.*

(3) *β_0 and k agree on a u-bounded set.*

THEOREM 2.2. *The two topologies on $C_b(X) \otimes E$, $(C_b(X) \otimes E, \beta_0)$ and $((C_b(X))_{\beta_0} \otimes_\epsilon E)$, are identical.*

Proof. Take a net $\{f_\alpha\}$ in $C_b(X) \otimes E$, $f_\alpha \to 0$ in ϵ-topology. Take a continuous seminorm q on E and a scalar-valued function h on X vanishing at infinity. For

$$S = \{f \in C_b(X): \|fh\| \leqq 1\} \quad \text{and} \quad T = \{y \in E: q(y) \leqq 1\},$$

let S^0, T^0 be the polars of S and T in $M_t(X)$ and E' respectively. Since $f_\alpha \to 0$ in ϵ-topology, $f_\alpha \to 0$ uniformly on $S^0 \times T^0$. Fix $\eta > 0$. There exists α_0 such that $|\mu(g \circ f_\alpha)| \leqq \eta \,\forall \alpha \geqq \alpha_0$, $\forall g \in T^0$, and $\forall \mu \in S^0$. Thus $g \circ f_\alpha/\eta \in S^{00} = S$ (note S is pointwise closed and so closed in $(C_b(X), \beta_0)$), $\forall g \in T^0$ and $\forall \alpha \geqq \alpha_0$. This means

$$|(1/\eta)h(x)g \circ f_\alpha| \leqq 1, \forall g \in T^0, \forall x \in X, \quad \text{and} \quad \forall \alpha \geqq \alpha_0,$$

and so $(1/\eta)h(x) f_\alpha(x) \in T^{00} = T$. We get $\sup_{x \in X} q(f_\alpha(x)h(x)) \leqq \eta$, $\forall \alpha \geqq \alpha_0$, which proves that $f_\alpha \to 0$ in β_0.

Conversely, suppose $f_\alpha \to 0$ in $(C_b(X) \otimes E, \beta_0)$. Take absolutely convex equicontinuous subsets P and Q of $M_t(X)$ and E' respectively. Since P^0 is a 0-neighbourhood in $(C_b(X), \beta_0)$, there exists a scalar-valued function h on X, vanishing at infinity, such that

$$P^0 \supset \{f \in C_b(X): \|fh\| \leqq 1\}.$$

Since the seminorm q on E, $q(y) = \sup\{|g(y)|: g \in Q\}$, is continuous, $\sup_{x \in X} q(f_\alpha(x)h(x)) \to 0$. Fix $\eta > 0$. We get α_0 such that

$$|g \circ f_\alpha(x)h(x)| \leqq \eta, \forall \alpha \geqq \alpha_0, \forall x \in X, \quad \text{and} \quad \forall g \in Q.$$

From this it follows that $(1/\eta)g \circ f_\alpha \in P^0$, and so

$$|\mu(g \circ f_\alpha)| \leqq \eta, \forall\, g \in Q, \forall\, \mu \in P, \quad \text{and} \quad \forall\, \alpha \geqq \alpha_0.$$

This proves $f_\alpha \to 0$ in ϵ-topology.

THEOREM 2.3. *Let E be a Banach space. Then the following statements are equivalent:*
(1) *X is compact;*
(2) *$\beta_0 = \|\cdot\|$, where $\|\cdot\|$ is the sup-norm topology;*
(3) *$(C_b(X, E), \beta_0)$ is normable;*
(4) *$(C_b(X, E), \beta_0)$ is metrizable;*
(5) *$(C_b(X, E), \beta_0)$ is bornological;*
(6) *$(C_b(X, E), \beta_0)$ is barreled.*

Proof. (1) implies (2): This is clear from the definition. (2) implies (1): Assume $\beta_0 = \|\cdot\|$ on $C_b(X, E)$ and fix $y_0 \in E$, $y_0 \neq 0$. Then $\beta_0 = \|\cdot\|$ on the closed subspace $C_b(X) \otimes y_0$ of $C_b(X, E)$. Now consider the mapping

$$L: (C_b(X), \beta_0) \to (C_b(X) \otimes y_0, \beta_0)$$

defined by $L(f) = f \otimes y_0$. Then $\beta_0 = \|\cdot\|$ on $C_b(X)$. Therefore X is compact ([**13**], p. 321).
(2) implies (3): Since $\|\cdot\|$ is normable, the result follows.
(3) implies (4): This is trivial as is (4) implies (5).
(5) implies (2): Let

$$I: (C_b(X, E), \beta_0) \to (C_b(X, E), \|\cdot\|)$$

be an identity mapping. Since β_0 and $\|\cdot\|$ have the same bounded set, $I(B)$ is $\|\cdot\|$-bounded in $C_b(X, E)$, for each β_0-bounded set B of $C_b(X, E)$. Hence I is continuous ([**11**], Theorem 8.3, p. 62), which implies that $\|\cdot\| \leqq \beta_0$. This proves that $\|\cdot\| = \beta_0$.

(1) implies (6): If X is compact, then $C_b(X, E)$ is a Banach space and so the result follows.

(6) implies (2): Let $B = \{f \in C_b(X, E): \|f\| \leqq 1\}$. Then B is radial, convex and circled. Let $\{f_\alpha\}_{\alpha \in I}$ be a net in B such that $f_\alpha \to f$ in β_0-topology. Then $f_\alpha \to f$ in p and $\|f\| = \lim \|f_\alpha\| \leqq 1$. Therefore $f \in B$ and B is β_0-closed, and so B is a barrel. This proves that $\|\cdot\| \leqq \beta_0$.

3. P-space and k-space. A completely regular Hausdorff space X is a *P-space* is every zero set in X is open, and it is well known that X is a P-space if and only if every G_δ set in X is open. A topological space X is a k-space if a set $A \subset X$ is closed if and only if $A \cap K$ is closed for all compact subsets K in X. If X is a k-space, then $f: X \to Y$ is continuous if and only if $f|_K$ is continuous for each compact subset K in X, where Y is a topological space. All locally compact spaces are k-spaces ([**9**], p. 131).

THEOREM 3.1. *If X is a P-space and E is a complete locally convex space, then $(C_b(X, E), \beta_0)$ is sequentially complete.*

Proof. For the β_0-Cauchy sequence $\{f_n\}$, let $f(x) = \lim f_n(x)$ for each x in X. Suppose there is a sequence $\{x_m\}$ and a continuous seminorm q on E such that $q(f(x_m)) \geqq 4^m$, $m = 1, 2, \ldots$. Put

$$h = \sum_{m=1}^{\infty} \frac{1}{2^m} \chi_{\{x_m\}}.$$

Then h is a real-valued function on X which vanishes at infinity. Since $\{f_n\}$ is a Cauchy sequence, there is a $n_0 \in \mathbf{N}$, $\mathbf{N}$ the set of natural numbers, such that

$$q(f_n(x_m) - f(x_m)) < 2^m, \forall\, n \geqq n_0, \forall\, m \in \mathbf{N}.$$

Thus $q(f_{n_0}(x_m)) > 4^m - 2^m$, which is impossible. Let U be a neighbourhood of $f(x)$ and V_n be a neighbourhood of x, $\forall\, n$ such that $f_i(V_n) \subset U$, $i = 1, 2, \ldots n$. Then $W = \bigcap_{n=1}^{\infty} V_n$ is open since W is a G_δ set. Hence $f(W) \subset U$ which shows that f is continuous. Now, take a real-valued function h which vanishes at infinity and a continuous seminorm q on E. Put

$$W = \{g \in C_b(X, E): \sup_{x \in X} q(h(x)g(x)) \leqq 1\}.$$

Then W is a β_0 0-neighbourhood which is closed in the pointwise topology, and since $\{f_n\}$ is a Cauchy sequence and $f_n \to f$, there is $n_0 \in N$ such that $f_n - f \in W$, for all $n \geqq n_0$, which gives $f_n \to f$ in β_0.

Remark. If X is a k-space, then a similar argument shows that $(C_b(X, E), \beta_0)$ is complete.

THEOREM 3.2. *Let E be a Banach space. Let $f: X \to E$ be bounded and $f|_K$ be continuous for each compact set K in X and also let $(C_b(X, E), \beta_0)$ be quasi-complete. Then f is continuous.*

Proof. If the conditions hold, then by Aren's extension theorem [**1**] there exists a continuous extension; $f_K: \beta X \to E$ such that $\bar{f}_K(\beta X) \subset \overline{\operatorname{conv}(f_K(X))}$, where $f_K = f|_K$, βX is the Stone–Čech compactification and $\overline{\operatorname{conv}(f_K(X))}$ is the closure of the convex hull of $f_K(X)$. We note that βX is paracompact. Put $g_K = \bar{f}_K|_X$. Then $g_K = f_K$ on K. Order compact subsets of X by inclusion. Then $\{g_K: K \text{ a compact subset of } X\}$ is norm-bounded in $C_b(X, E)$ and is evidently a Cauchy net with the compact-open topology. Hence $\{g_K: K \text{ a compact subset of } X\}$ is a β_0-Cauchy net and so $g_K \to g$ in β_0, for some $g \in C_b(X, E)$. Since f is the only possible limit of $\{g_K\}$, we have $f \in C_b(X, E)$.

LEMMA 3.3. *Let X be a k-space. Then $(C_b(X), \beta_0)$ is nuclear if and only if X is finite.*

Proof. Let $(C_b(X), \beta_0)$ be nuclear. Then every bounded set in $(C_b(X), \beta_0)$ is relatively compact, and hence the unit ball $B = \{f \in C_b(X): \|f\| \leqq 1\}$ is β_0-compact in $C_b(X)$. Now, let $x_0 \in X$ and $\{f_\alpha\}$ be a net in B which converges pointwise to the characteristic function $\chi_{\{x_0\}}$ of $\{x_0\}$. Then any β_0-cluster point of $\{f_\alpha\}$ coincides with $\chi_{\{x_0\}}$ and hence $\chi_{\{x_0\}}$ is continuous. Thus $\{x_0\}$ is open and evidently X is discrete. Since a discrete space is locally compact, by Collins' result ([**4**], p. 364), X is finite.

THEOREM 3.4. *Let X be a k-space. Then $(C_b(X, E), \beta_0)$ is nuclear if and only if X is finite and E is nuclear.*

Proof. Suppose $(C_b(X, E), \beta_0)$ is nuclear and let $y_0 \in E$, $y_0 \neq 0$. Then the subspace $(C_b(X) \otimes y_0, \beta_0)$ is nuclear. Now define a mapping

$$L: (C_b(X), \beta_0) \rightarrow (C_b(X) \otimes y_0, \beta_0)$$

by $L(f) = f \otimes y_0$, for each $f \in C_b(X)$. Then $(C_b(X), \beta_0)$ is nuclear and hence X is finite by the lemma. So we can write $C_b(X, E) = E^n$, where n is the number of points in X. Note that $E \subset E^n$, and evidently E is nuclear.

Conversely, let X be finite and E be nuclear. Then the result follows from $C_b(X, E) = E^n$, with the product topology and n is the number of points in X.

We need the following Husain's definition ([**7**], p. 61).

Definition 3.5. Let E be a locally convex Hausdorff space and E' be its dual. The *ew*-topology* is defined to be the finest topology (not necessarily locally convex) which coincides with weak*-topology on each equicontinuous subset of E'. The topology t_p on E' is defined to be the topology of uniform convergence on precompact subsets of E. The equicontinuous weak*-topology (*ew**) on E' is, in general, finer than t_p ([**4**], p. 364, [**7**]).

LEMMA 3.6. *Let E be a Banach space. Then $H \subset (C_b(X, E), \beta_0)' = M_t(X, E')$ is equicontinuous if and only if H is uniformly bounded and, for a given $\epsilon > 0$, there exists a compact subset K of X such that $|\mu|(X \backslash K) < \epsilon$ for all $\mu \in H$.*

Proof. See [**10**].

THEOREM 3.7. *Let X be a k-space and E be a Banach space. Then X is discrete and E is finite dimensional if and only if the ew*-topology and the norm topology on $M_t(X, E') = (C_b(X, E), \beta_0)'$ are the same.*

Proof. Suppose X is discrete and E is of finite dimension. If

$$B = \{f \in C_b(X, E): \|f\| \leqq 1\},$$

then $B = S^X$ and is compact where S is the closed unit ball of E. Since compact subsets of X are finite and β_0 coincides with k on B, the topology on B induced by β_0 is the one obtained on S^X by the product topology. Thus B is β_0-compact and every bounded subset of $(C_b(X, E), \beta_0)$ is relatively compact. Thus the topology on $M_t(X, E')$ is the topology t_p, and hence $\|\cdot\| \leqq ew^*$. Now, to prove $ew^* \leqq \|\cdot\|$, suppose it is not true; then there exists a sequence $\{\mu_n\}$

in $M_t(X, E')$ such that $\mu_n \to 0$ in $\|\cdot\|$, but $\mu_n \notin V$ for all n, where V is a ew^* 0-neighbourhood. Put $H = \{0, \mu_1, \mu_2, \ldots \mu_n \ldots\}$. Then H is norm compact. Also, given $\epsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $|\mu_n|(X) < \epsilon$, $\forall\, n \geqq n_0$. By regularity there exists a compact subset K of X such that $|\mu_n|(X\backslash K) < \epsilon$, $n = 1, 2, \ldots, n_0 - 1$. For any n, if $n \geqq n_0$, then $|\mu_n|(X\backslash K) \leqq |\mu_n|(X) < \epsilon$ and if $n \geqq n_0$, then $|\mu n|(X^X K) < \epsilon$. Therefore by Lemma 3.6 H is a β_0-equicontinuous subset of $M_t(X, E')$. Since H is norm-compact, weak* $= \|\cdot\|$ on H. Thus $\mu_n \to 0$ in weak* and hence $\mu_n \to 0$ in ew^* which is a contradiction.

Conversely, let $ew^* = \|\cdot\|$ on $M_t(X, E')$. Let B be the closed unit ball of E', the dual of E and $H = \{f\epsilon_x : f \in B\}$, where $x \in X$ is fixed and ϵ_x is the point measure of x. Then $H \subset M_t(X, E')$ and it is equicontinuous and weak*-closed since, for any $\mu \in H$, $|\mu| = \|f\|\epsilon_x$, if we take $K = \{x\}$, then $|\mu|(X\backslash K) = 0 < \epsilon$ for any $\epsilon > 0$. Thus weak* $= ew^* = \|\cdot\|$ on H. Now, define a mapping $L: (B, \|\cdot\|) \to (H, \|\cdot\|)$ by $L(f) = f\epsilon_x$. Then L is one to one, onto and continuous, and also L^{-1} is continuous. Thus $\|\cdot\| =$ weak* on B and hence B is norm-compact, and evidently E is of finite dimension.

Next we want to show that X is discrete. Take an arbitrary point p in X and let $\chi_{\{p\}}$ be the characteristic function of $\{p\}$. Fix y_0 in E, $\|y_0\| = 1$ and define a mapping $L: (C_b(X, E), \beta_0) \to R$ by

$$L(\mu) = \int \chi_{\{p\}} \otimes y_0 d\mu, \forall\, \mu \in M_t(X, E').$$

Then it is obvious that L is linear. We want to show that L is $\sigma(F', F)$-continuous, where $F = (C_b(X, E), \beta_0)$ and $F' = (C_b(X, E), \beta_0)'$. Let H be an equicontinuous subset of $M_t(X, E')$. Since $ew^* = \|\cdot\|$ on $M_t(X, E')$, by Grothendieck's completeness theorem, it is sufficient to show that L is continuous on H with respect to the $\|\cdot\|$-topology. Let $\mu_n \to \mu$ in $\|\cdot\|$ in H. Then

$$\|\mu_n - \mu\| = |\mu_n - \mu|(X) = \sup_{\|y_i\| \leqq 1} |\sum (\mu_n - \mu)(B_i) y_i| \to 0,$$

where the supremum is taken over all partitions of X into a finite number of disjoint Borel sets $\{B_i\}$ and all finite collections of elements $\{y_i\}$ in E with $\|y_i\| \leqq 1$. In particular, $\|\mu_n - \mu\| \to 0$ implies that

$$|(\mu_n - \mu)\{p\} y_0| \to 0.$$

Hence L is a weak*-continuous linear functional and thus $\chi_{\{p\}} \otimes y_0$ is continuous and so is $\chi_{\{p\}}$. Therefore $\{p\}$ is open, and we conclude that X is discrete.

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