J. Austral. Math. Soc. (*Series A*) **28** (1979), 385–397

UNIFORM LABELLED SEMILATTICES

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(Received 17 May 1978)

Communicated by T. E. Hall

Abstract

Let P be a partially-ordered set in which every two elements have a common lower bound. It is proved that there exists a lower semilattice L whose elements are labelled with elements of P in such a way that (i) comparable elements of L are labelled with elements of P in the same strict order relation; (ii) each element of P is used as a label and every two comparable elements of P are labels of comparable elements of L; (iii) for any two elements of L with the same label, there is a label-preserving isomorphism between the corresponding principal ideals. Such a structure is called a full, uniform P-labelled semilattice.

1980 Mathematics subject classification (*Amer. Math. Soc.*): primary 06 A 12, 08 A 99; secondary 20 M 10.

0. Introduction

Let P be a partially-ordered set. A P-labelling of a lower semilattice, L, is an assignment, to each element a of L, of an element $l(a)$ of P such that, if $a, b \in L$ and $a < b$ in the ordering of L, then $l(a) < l(b)$ in the ordering of P. Such a pair (L, l) will be called a P-labelled semilattice, or *P-semilattice*.

A P-semilattice will be called *full* if, whenever $p, q \in P$ and $p \leqslant q$, then there exist $a, b \in L$ with $a \leqslant b$, $l(a) = p$ and $l(b) = q$. (In particular, for each $p \in P$ there exists $a \in L$ with $l(a) = p$.) Clearly, there can only exist a full P-semilattice if P is downward-directed, that is, for any $p, q \in P$ there exists $r \in P$ with $r \leqslant p$ and $r \leqslant q$.

If $\mathscr{L} = (L, l)$ is a P-semilattice, then each $a \in L$ determines a P-semilattice $\mathscr{L}^{<a}$ consisting of those elements c of L with $c < a$. We say that $\mathscr{L}$ is *uniform* if, whenever $a, b \in L$ and $l(a) = l(b)$, then $\mathscr{L}^{<a}$ and $\mathscr{L}^{<b}$ are isomorphic P-semilattices.

In this paper we show that, for each downward-directed partially-ordered set, P, there exists a full, uniform P-semilattice. This answers a question originally suggested by the study of Green's relation $\mathcal{J}$ on semigroups, see Rhodes (1972), Hall (1973), Ash and Hall (1975), and also by the related topic of generalized cardinal systems (Ash (1979c)). The construction we give is quite involved and seems of independent interest, so we will record the other matters elsewhere in Ash (1979a, b, c).

Two cases present themselves, according to whether P has or has not a least element. In the first case, a fairly simple construction is sufficient and this is essentially the method used in Ash and Hall (1975). This case is dealt with in Section 2, giving part of the main theorem, Theorem A.

The body of this paper is concerned with the second case. Here no such straightforward method appears possible and we give a less direct construction which depends heavily on Jonsson's homogeneous-universal structures (Jonsson (1960), Morley and Vaught (1962)). The relevant definitions and theorems are given, without proof, in Section 3, and applied in Section 4 to give the remaining part, Theorem B, of the Main Theorem, which is then stated.

In Section 5 we discuss ways in which the set-theoretic assumptions used in Theorem B may be avoided. We continue by adding a refinement, Corollary C, of the main theorem, concerning the cardinalities of full, uniform P-semilattices. We conclude with some further remarks about the method of proof of Theorem B.

The axiom of choice is used freely throughout.

1. Preliminaries

The numbering refers to the section in which the terminology is first used.

1.0. A *lower semilattice* is a partially ordered set $(L, \leqslant)$ in which every two elements a, b have a greatest lower bound, or *meet*, $a \wedge b$. The ordering is determined by the pair $(L, \wedge)$, since then $a \leqslant b$ if and only if $a = a \wedge b$. We adopt this convention throughout. Also, throughout the paper, *semilattice* means lower semilattice.

An *isomorphism* between semilattices $(L_1, \wedge_1)$ and $(L_2, \wedge_2)$ is a bijection $f: L_1 \to L_2$ such that, for all $a, b \in L_1$, $f(a \wedge_1 b) = f(a) \wedge_2 f(b)$. If L_1, L_2 are semilattices and $\mathcal{L}_1 = (L_1, l_1)$, $\mathcal{L}_2 = (L_2, l_2)$ are P-semilattices, then an isomorphism from $\mathcal{L}_1$ to $\mathcal{L}_2$ is a semilattice isomorphism $f: L_1 \to L_2$ such that, for all $a \in L_1$, $l_1(a) = l_2(f(a))$.

If $L = (L, \wedge)$ is a semilattice and L_0 is a subset of L such that, for all $a, b \in L_0$, $a \wedge b \in L_0$, then $L_0 = (L_0, \wedge)$ is also a semilattice, called a *subsemilattice* of L. If $\mathcal{L} = (L, l)$ is a P-semilattice and L_0 is a subsemilattice of L, then $\mathcal{L}_0 = (L_0, l)$ is

https://doi.org/10.1017/S1446788700012520 Published online by Cambridge University Press

also a P-semilattice, called a *P-subsemilattice* of $\mathscr{L}$. Here and elsewhere, we use the labour-saving device of letting $\wedge$ and l also denote their restrictions to the obvious sets.

1.3. A *structure* consists of a non-empty set together with families of finitary relations and finitary operations on the set. Thus one may define a structure as a triple $\mathscr{A}=(A,\{R_i\}_{i\in I},\{f_j\}_{j\in J})$ where, for each $i\in I$, R_i is a $\sigma(i)$-ary relation on A and, for each $j\in J$, f_j is a $\tau(j)$-ary operation on A. The quadruple (I,J,σ,τ) is referred to as the *type* of the structure. A *monomorphism* $f\colon \mathscr{A}\to\mathscr{B}$, where $\mathscr{A}=(A,\{R_i\}_{i\in I},\{f_j\}_{j\in J})$ and $\mathscr{B}=(B,\{S_i\}_{i\in I},\{g_j\}_{j\in J})$ are structures of the same type, is an injection $f\colon A\to B$ such that for $i\in I$, $\sigma(i)=k$ and $a_1,\dots,a_k\in A$,

$$(a_1,\dots,a_k)\in R_i \text{ if and only if } (f(a_1),\dots,f(a_k))\in S_i$$

and for $j\in J$, $\tau(j)=k$ and $a_1,\dots,a_k\in A$,

$$f(f_j(a_1,\dots,a_k))=g_j(f(a_1),\dots,f(a_k)).$$

$\mathscr{A}$ is a *substructure* of $\mathscr{B}$, denoted by $\mathscr{A}\subseteq\mathscr{B}$, if $A\subseteq B$ and the insertion of A into B is a monomorphism. An *isomorphism* is a monomorphism which is also a surjection. An *automorphism* of $\mathscr{A}$ is an isomorphism from $\mathscr{A}$ to $\mathscr{A}$.

For any structure $\mathscr{A}=(A,\dots)$, the *underlying set* of $\mathscr{A}$ is A, denoted by $|\mathscr{A}|$. The cardinality of a set A is denoted by $c(A)$. The *power* of a structure $\mathscr{A}$ is $c(|\mathscr{A}|)$, also denoted by $c(\mathscr{A})$. The Greek letters κ, λ, μ, ν are used to denote infinite cardinal numbers, without further explanation. If μ is the cardinality of the set S, then 2^μ denotes the cardinality of the set of all subsets of S. $\aleph_0$ is the first infinite cardinal.

A *regular* cardinal, κ, is one which is not a sum of fewer than κ cardinals each of size less than κ.

1.4. A semilattice $(L,\wedge)$ is *free* on a set X of generators if $X\subseteq L$ and if every element of L is the meet of some unique finite subset of X.

The free P-semilattice (L,X,l) defined below in Section 4 may be construed as the structure $(L,X,\{R_p\}_{p\in P},\wedge)$ in the sense of 1.3, where, for $p\in P$, R_p is the 1-ary relation true for $a\in L$ if and only if $l(a)=p$. The definition of monomorphism then agrees with that of 1.3.

1.5. *Zermelo–Fraenkel set theory* is described in, for example, Cohen (1966). κ^+ denotes the least cardinal greater than κ. λ^μ denotes the cardinality of the set of all functions from a set of cardinality μ to a set of cardinality λ.

Constructible sets, sets (relatively) constructible from another set and standard models of ZFC and parts thereof are also described in Cohen (1966).

2. P has a least element

Assume now that P has a least element, 0. Let L be the set of finite sequences $\langle p_1, \ldots, p_n \rangle$, $n \geqslant 1$, of the members of $P-\{0\}$ for which $p_1 > p_2 > \ldots > p_n$, together with a single new object $\diamond$. Define an ordering on $L-\{\diamond\}$ by

$$\langle q_1, \ldots, q_m \rangle \leqslant \langle p_1, \ldots, p_n \rangle$$

if $p_1 = q_1, p_2 = q_2, \ldots, p_n = q_r$ for some $r \leqslant m$. Extend $\leqslant$ to L by making $\diamond$ the least element. Under this ordering, L is clearly a lower semilattice. Let l be defined by $l(\langle p_1, \ldots, p_n \rangle) = p_n$ and $l(\diamond) = 0$. It is easy to see that $\mathscr{L} = (L, l)$ is a full P-semilattice. Moreover, if $x = \langle p_1, \ldots, p_n \rangle$, $y = \langle q_1, \ldots, q_m \rangle$ and $l(x) = l(y)$ then $p_n = q_m \neq 0$ and one may define an order isomorphism (and therefore a semilattice ismorphism) from $\mathscr{L}^{<x}$ to $\mathscr{L}^{<y}$ by

$$\diamond \mapsto \diamond \quad \text{and} \quad \langle p_1, \ldots, p_n, r_1, \ldots, r_k \rangle \mapsto \langle q_1, \ldots, q_m, r_1, \ldots, r_k \rangle.$$

Since this mapping also respects l, we have thus proved the following

THEOREM A. *If P is a partially ordered set with least element then there exists a full, uniform P-semilattice.*

COMMENT. A *finite* downward-directed partially ordered set must have a least element and so we have, in Theorem A, dealt with all cases where P is finite.

3. Homogeneous-universal structures

We now summarize those parts of the theory of homogeneous-universal structures which we shall need. The results are essentially as stated in Jonsson (1960). Further results and extensive proofs are given in Morley and Vaught (1962).

Jonsson's constructions apply to arbitrary classes, $\mathscr{M}$, of structures of the same type satisfying the following conditions.

JONSSON'S CONDITIONS.

I. $\mathscr{M}$ is closed under isomorphism.

II. $\mathscr{M}$ contains members of arbitrarily large powers.

III. (The joint embedding property.) For all $\mathscr{A}_1, \mathscr{A}_2 \in \mathscr{M}$ there exist $\mathscr{A}_3 \in \mathscr{M}$ and monomorphisms

$$g_1 : \mathscr{A}_1 \to \mathscr{A}_3 \quad \text{and} \quad g_2 : \mathscr{A}_2 \to \mathscr{A}_3.$$

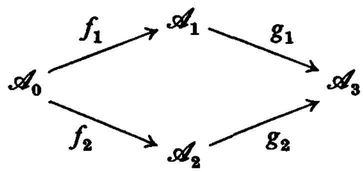
IV. (The amalgamation property.) For all $\mathscr{A}_0$, $\mathscr{A}_1$, $\mathscr{A}_2 \in \mathscr{M}$ and monomorphisms

$$f_1: \mathscr{A}_0 \to \mathscr{A}_1 \quad \text{and} \quad f_2: \mathscr{A}_0 \to \mathscr{A}_2,$$

there exist $\mathscr{A}_3 \in \mathscr{M}$ and monomorphisms

$$g_1: \mathscr{A}_1 \to \mathscr{A}_3 \quad \text{and} \quad g_2: \mathscr{A}_2 \to \mathscr{A}_3$$

for which the following diagram commutes.



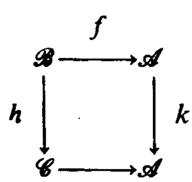
V. $\mathscr{M}$ is closed under unions of chains.

VI$_\lambda$. For some infinite cardinal λ, if $\mathscr{A} \in \mathscr{M}$, $S \subseteq |\mathscr{A}|$ and $c(S) < \lambda$, then there exists $\mathscr{B} \in \mathscr{M}$ with $\mathscr{B} \subseteq \mathscr{A}$, $S \subseteq |\mathscr{B}|$ and $c(\mathscr{B}) < \lambda$.

COMMENT. The status of this last condition is, perhaps, clarified by the observation that, if $\mathscr{M}$ satisfies conditions I to VI$_{\lambda_0}$ then one may show (by transfinite induction on cardinals) that $\mathscr{M}$ satisfies VI$_\lambda$ for every $\lambda \geqslant \lambda_0$.

The constructions which we shall use are obtained by forming transfinite chains from $\mathscr{M}$, repeatedly using the properties of $\mathscr{M}$, notably the amalgamation property, IV.

DEFINITION 3a. A structure $\mathscr{A} \in \mathscr{M}$ is said to be *$(\mathscr{M}, \kappa)$-homogeneous* if, whenever $\mathscr{B}, \mathscr{C} \in \mathscr{M}$, $f: \mathscr{B} \to \mathscr{A}$, $g: \mathscr{C} \to \mathscr{A}$ are monomorphisms, $h: \mathscr{B} \to \mathscr{C}$ is an isomorphism and $c(\mathscr{B})$, $c(\mathscr{C}) < \kappa$, then there exists an automorphism k of $\mathscr{A}$ for which the following diagram commutes.



PROPOSITION 3.1. If $\mathscr{M}$ is a class of structures satisfying Jonsson's conditions, then, for each cardinal κ, $\mathscr{M}$ contains an $(\mathscr{M}, \kappa)$-homogeneous structure.

DEFINITION 3b. $\mathscr{A} \in \mathscr{M}$ is *$(\mathscr{M}, \kappa)$-universal* if, whenever $\mathscr{A}_0 \in \mathscr{M}$ and $c(\mathscr{A}_0) < \kappa$, then there exists a monomorphism $f: \mathscr{A}_0 \to \mathscr{A}$.

$(\mathscr{M}, \kappa)$-homogeneous-universal means $(\mathscr{M}, \kappa)$-homogeneous and $(\mathscr{M}, \kappa)$-universal.

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PROPOSITION 3.2. *If $\mathcal{M}$ satisfies Jonsson's conditions then, for each cardinal κ, $\mathcal{M}$ contains an $(\mathcal{M}, \kappa)$-homogeneous-universal structure.*

The proof of Proposition 3.2 guarantees only a structure of power $\geqslant \kappa$. To minimize this power, some cardinal arithmetic is needed. Suppose that, for each cardinal μ, $\nu(\mu)$ denotes the number of non-isomorphic members of $\mathcal{M}$ of power μ.

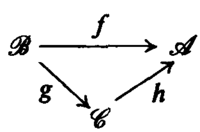
PROPOSITION 3.3. *Suppose that κ is a regular cardinal, $\kappa \geqslant \lambda$ and that, whenever $\mu < \kappa$, both $2^\mu \leqslant \kappa$ and $\nu(\mu) \leqslant \kappa$. Then, if $\mathcal{M}$ satisfies Jonsson's conditions* I *to* VI_λ, *$\mathcal{M}$ contains an $(\mathcal{M}, \kappa)$-homogeneous-universal structure of power κ.*

For the moment, we assume that such a κ exists. This assumption is not essential, as we comment later in Section 5, but avoids further complications. The significance of Proposition 3.3 is the following:

PROPOSITION 3.4. *If $\mathcal{M}$ satisfies Jonsson's conditions* I *to* VI_λ, *$\kappa \geqslant \lambda$, and $\mathcal{A}$, $\mathcal{B}$ are $(\mathcal{M}, \kappa)$-homogeneous-universal, both of power κ, then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.*

Finally, for such structures, we have the following simplification.

PROPOSITION 3.5. *If $\mathcal{M}$ satisfies Jonsson's conditions* I *to* VI_λ, *$\mathcal{A} \in \mathcal{M}$, $c(\mathcal{A}) \leqslant \kappa$ and $\kappa \geqslant \lambda$ then $\mathcal{A}$ is $(\mathcal{M}, \kappa)$-homogeneous-universal of power κ if and only if, whenever $\mathcal{B}, \mathcal{C} \in \mathcal{M}$, $c(\mathcal{B})$, $c(\mathcal{C}) < \kappa$ and $f: \mathcal{B} \to \mathcal{A}$ and $g: \mathcal{B} \to \mathcal{C}$ then there exists a monomorphism $h: \mathcal{C} \to \mathcal{A}$ for which the following diagram commutes.*



4. P has no least element: free P-semilattices

We now return to the problem of constructing full, uniform P-semilattices. We assume from now on that P is downward-directed and has *no* least element. Were it the case that the class of P-semilattices satisfied Jonsson's conditions, the problem would be immediately solved, using Proposition 3.1. This is not so, however, since the amalgamation property (IV) fails.† We therefore consider a different, but related, class of structures to which the results of the last section may be applied.

DEFINITION 4a. A *free P-semilattice* is a triple (L, X, l) for which (L, l) is a P-semilattice and X is a set of free generators for L.

† The author is indebted to T. E. Hall for demonstrating this fact to him.

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A *monomorphism* between free P-semilattices (L_1, X_1, l_1) and (L_2, X_2, l_2) is a semilattice monomorphism $f: L_1 \to L_2$ such that, for $x \in X_1$, $f(x) \in X_2$ and for $x \in L_1$, $l_1(x) = l_2(f(x))$.

Let $\mathscr{M}(P)$ denote the class of all free P-semilattices.

THEOREM 4.1. *The class* $\mathscr{M}(P)$ *satisfies Jonsson's conditions* I *to* $\mathrm{VI}_{\aleph_0}$.

PROOF. Conditions I, II, V, $\mathrm{VI}_{\aleph_0}$ are easily verified. Condition III may be shown in just the same way as IV, which we now establish.

Suppose that $\mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2 \in \mathscr{M}(P)$ and monomorphisms $f_1: \mathscr{A}_0 \to \mathscr{A}_1$ and $f_2: \mathscr{A}_0 \to \mathscr{A}_2$ are given. For simplicity of notation, we may suppose that the elements of $\mathscr{A}_1$ and $\mathscr{A}_2$ are renamed so that f_1 and f_2 are insertions. Thus, if $\mathscr{A}_i = (L_i, X_i, l_i)$, for $i = 0, 1, 2$, then

$$L_1 \cap L_2 = L_0, \quad X_1 \cap X_2 = X_0 \quad \text{and} \quad l_0 = l_1|_{L_0} = l_2|_{L_0}.$$

It is now sufficient to give $\mathscr{A}_3 = (L_3, X_3, l_3)$ for which $\mathscr{A}_3 \supseteq \mathscr{A}_1, \mathscr{A}_2$, since g_1 and g_2 may then be taken to be the corresponding insertions.

This is done as follows. From the properties of semilattices, L_3 clearly may be chosen to be free on $X_3 = X_1 \cup X_2$ and to include L_1 and L_2 as subsemilattices. It remains to define l_3 on L_3 extending l_1 and l_2. Each element a of L_3 may be expressed uniquely as the meet of a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X_3. We define $l_3(a)$ by induction on n as follows:

(i) If $a \in L_1$ then $l_3(a) = l_1(a)$.

(ii) If $a \in L_2$ then $l_3(a) = l_2(a)$.

(iii) Otherwise, $l_3(a)$ is arbitrarily chosen to be an element of P less than $l_3(b)$ for each $b > a$.

Clauses (i) and (ii) are compatible since l_1 and l_2 extend l_0. Clause (iii) is always applicable since the $b > a$ are exactly the meets of proper subsets of $\{x_1, \ldots, x_n\}$, so that each $l_3(b)$ is previously defined in the inductive procedure and also there are only finitely many such b so that the required choice may be made using the assumption that P is downward-directed with no least element.

It remains to observe that for any such l_3, (L_3, l_3) is a *P-semilattice*. This follows from the observation that if $a, b \in L_3$ and $a < b$, then either (iii) applies to the definition of $l_3(a)$, so that $l_3(a) < l_3(b)$, or $a \in L_i$ for $i = 1$ or 2, in which case also $b \in L_i$, so $l_3(a) = l_i(a) < l_i(b) = l_3(b)$.

From Proposition 3.1 we now have the following, for suitable κ,

COROLLARY 4.2. *There exists an* $(\mathscr{M}(P), \kappa)$*-homogeneous-universal structure* $\mathscr{A} = (L, X, l)$ *of power* κ.

We now claim that the corresponding P-semilattice (L, l) is full and uniform.

LEMMA 4.3. *If* $\mathscr{A} = (L, X, l)$ *is* $(\mathscr{M}(P), \kappa)$*-universal and* $\kappa \geqslant \aleph_0$*, then* (L, l) *is full.*

PROOF. Let $p, q \in P$ with $p \leqslant q$. If $p = q$, let $\mathscr{A}_0$ be the unique one-element free $\{p\}$-semilattice; if $p < q$ let $\mathscr{A}_0$ be $(\{x, y, x \wedge y\}, \{x, y\}, l_0)$, where $l_0(x) = l_0(y) = q$ and $l_0(x \wedge y) = p$. In either case, $\mathscr{A}_0 \in \mathscr{M}(P)$ and there exist $a, b \in \mathscr{A}_0$ with $a \leqslant b$, $l(a) = p$ and $l(b) = q$. By Definition 3b, there exists a monomorphism of $\mathscr{A}_0$ into $\mathscr{A}$, so the same statement applies to $\mathscr{A}$.

To show that, if $\mathscr{A} = (L, X, l)$ is $\mathscr{M}(P)$-universal-homogeneous, then (L, l) is uniform, we proceed as follows. Suppose that $a \in L$ and that $l(a) = p$. Then $(L^{<a}, l)$ is also a P-semilattice, in fact a $P^{<p}$-semilattice. Moreover, each element of $L^{<a}$ is uniquely expressible as the meet of a set $\{a, x_1, \ldots, x_n\}$ where $x_i \in X$ and $x_i \not> a$. It follows that $L^{<a}$ is freely generated by the set

$$X^a = \{a \wedge x \colon x \in X \quad \text{and} \quad x \not> a\}.$$

So the structure $\mathscr{A}^{<a} = (L^{<a}, X^a, l)$ is a member of the class $\mathscr{M}(P^{<p})$. We shall show that, for any such a, $\mathscr{A}^{<a}$ is $(\mathscr{M}(P^{<p}), \kappa)$-homogeneous-universal of power κ, using the criterion of Proposition 3.5. This requires the following lemma.

LEMMA 4.4. *If*

$$\mathscr{B} = (L_1, X_1, l_1) \in \mathscr{M}(P), \bar{\mathscr{C}} = (\bar{L}_2, \bar{X}_2, \bar{l}_2) \in \mathscr{M}(P^{<p}),\ b \in |\mathscr{B}|,\ l_1(b) = p$$

and $\bar{g} \colon \mathscr{B}^{<b} \to \bar{\mathscr{C}}$ *is a monomorphism, then there exist* $\mathscr{C} = (L_2, X_2, l_2) \in \mathscr{M}(P)$ *with* $c(\mathscr{C}) \leqslant \max(c(\mathscr{B}), c(\bar{\mathscr{C}}), \aleph_0)$, $c \in |\mathscr{C}|$ *with* $l_2(c) = p$, *an isomorphism* $\bar{\mathscr{C}} \cong \mathscr{C}^{<c}$ *and a monomorphism* $g \colon \mathscr{B} \to \mathscr{C}$ *with* $g(b) = c$ *for which the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{\quad g \quad} & \mathscr{C} \\ \cup\!| & & \cup\!| \\ \mathscr{B}^{<b} & \xrightarrow[\quad \bar{g} \quad]{} & \bar{\mathscr{C}} \cong \mathscr{C}^{<c} \end{array}$$

PROOF. For simplicity of notation, assume that $\mathscr{B}^{<b} \subseteq \bar{\mathscr{C}}$ and that $\bar{g}$ is the corresponding insertion. Let $\{y_\xi\}_{\xi < \gamma}$ be an arbitrary well-ordering of $\bar{X}_2 - X_1^b$. Let $\{z_\xi\}_{\xi < \gamma}$ be a sequence of new elements. We define $\mathscr{C}$ to be (L_2, X_2, l_2), where L_2 is free on $X_2 = X_1 \cup \{z_\xi\}_{\xi < \gamma}$ and l_2 remains to be defined. The isomorphism $\bar{\mathscr{C}} \cong \mathscr{C}^{<c}$ is that which identifies each y_ξ with the corresponding $b \wedge z_\xi$. We thus take $c = b$ and g to be the insertion of $\mathscr{B}$ into $\mathscr{C}$.

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It remains to define l_2 extending both l_1 and $\bar{l}_2$ in such a way that $\mathscr{C} \in \mathscr{M}(P)$.

Let $b = x_1 \wedge \ldots \wedge x_n$ where $x_1, \ldots, x_n$ are distinct members of X_1. Each $a \in L_2$ is uniquely expressible as $a = u \wedge z_{\xi_1} \wedge z_{\xi_2} \wedge \ldots \wedge z_{\xi_m}$, where $\xi_1 < \xi_2 < \ldots < \xi_m$ and u is the meet of some subset $\{u_1, \ldots, u_k\}$ of X_1. Let $\{x_{i_1}, x_{i_2}, \ldots, x_{i_l}\}$ be the members of $\{x_1, \ldots, x_n\}$ not occurring among $\{u_1, \ldots, u_k\}$, where $i_1 < i_2 < \ldots < i_l$. We now define l_2 as follows.

If $l \geqslant m$, let $\hat{a} = u \wedge x_{i_1} \wedge \ldots \wedge x_{i_m}$. Then $\hat{a} \in \mathscr{B}$ and we may define $l_2(a) = l_1(\hat{a})$.

If $l < m$, let $\hat{a} = u \wedge x_{i_1} \wedge \ldots \wedge x_{i_l} \wedge z_{\xi_{l+1}} \wedge \ldots \wedge z_{\xi_m}$. Then $\hat{a} \in \mathscr{C}^{<c}$ and we may define $l_2(a) = \bar{l}_2(\hat{a})$, according to the isomorphism $\bar{\mathscr{C}} \cong \mathscr{C}^{<c}$.

Thus, if $a \in \mathscr{B}$ or $a \in \mathscr{C}^{<c} = \mathscr{C}^{<b}$, then either $m = 0$ or $l = 0$, so that $\hat{a} = a$ and $l_2(a) = l_1(a)$ or $\bar{l}_2(a)$ accordingly. So l_2 extends l_1 to $\bar{l}_2$. Moreover, it is easy to check that, if $a_1 < a_2$, then $\hat{a}_1 < \hat{a}_2$. This may be seen by first considering the cases where $a_1 = a_2 \wedge x$ for some $x \in X_2$. From this, and the fact that, if $a_2 \in \mathscr{B}$ or $\mathscr{C}^{<c}$ and $a_1 < a_2$, then $a_1 \in \mathscr{B}$ or $\mathscr{C}^{<c}$, respectively, it follows that for all $a_1, a_2 \in \mathscr{C}$, if $a_1 < a_2$ then $l(a_1) < l(a_2)$, so that (L_2, l_2) is a P-semilattice and $\mathscr{C} \in \mathscr{M}(P)$, as desired.

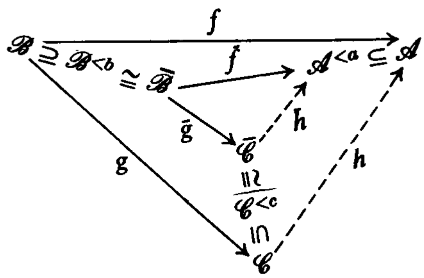
COROLLARY 4.5. *If $\mathscr{A} = (L, X, l)$ is $(\mathscr{M}(P), \kappa)$-homogeneous-universal of power κ, $a \in \mathscr{A}$ and $l(a) = p$, then $\mathscr{A}^{<a}$ is $(\mathscr{M}(P^{<p}), \kappa)$-homogeneous-universal of power κ.*

PROOF. We use Proposition 3.5. We have remarked that $\mathscr{A}^{<a} \in \mathscr{M}(P^{<p})$, and, since $|\mathscr{A}^{<a}| \subseteq |\mathscr{A}|$, $c(\mathscr{A}^{<a}) \leqslant \kappa$. $\mathscr{M}(P^{<p})$ satisfies Jonsson's conditions I–$\mathrm{VI}_{\aleph_0}$ by Theorem 4.1.

Now suppose that $\bar{\mathscr{B}}$, $\bar{\mathscr{C}} \in \mathscr{M}(P^{<p})$, $c(\bar{\mathscr{B}})$, $c(\bar{\mathscr{C}}) < \kappa$ and $\bar{f}: \bar{\mathscr{B}} \to \mathscr{A}^{<a}$ and $\bar{g}: \bar{\mathscr{B}} \to \bar{\mathscr{C}}$ are monomorphisms. By taking the substructure of $\mathscr{A}$ generated by the union of the range of $\bar{f}$ with the set of free generators of L whose meet is a, we obtain $\mathscr{B} \in \mathscr{M}(P)$, $b \in |\mathscr{B}|$, an isomorphism $\bar{\mathscr{B}} \cong \mathscr{B}^{<b}$ and a monomorphism $f: \mathscr{B} \to \mathscr{A}$ with $f(b) = a$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{\quad f \quad} & \mathscr{A} \\ \cup\!| & & \cup\!| \\ \mathscr{B}^{<b} \cong \bar{\mathscr{B}} & \xrightarrow{\quad \bar{f} \quad} & \mathscr{A}^{<a} \end{array}$$

By now applying Lemma 4.4, we obtain $\mathscr{C} \in \mathscr{M}(P)$, $c \in |\mathscr{C}|$, $\bar{\mathscr{C}} \cong \mathscr{C}^{<c}$ and a monomorphism $g: \mathscr{B} \to \mathscr{C}$ with $g(b) = c$ such that the solid part of the following diagram commutes.



A monomorphism h then exists, by Proposition 3.5 applied to $\mathscr{A}$, making the outer diagram commute. $f(b) = h(g(b))$, so $h(c) = a$. Thus, the restriction, $\bar{h}$, of h to $\bar{\mathscr{C}}$ is a monomorphism from $\bar{\mathscr{C}}$ to $\mathscr{A}^{<a}$ making the inner diagram commute, as required for the premises of Proposition 3.5 applied to $\mathscr{A}^{<a}$. Hence $\mathscr{A}^{<a}$ is $(\mathscr{M}(P^{<p}), \kappa)$-homogeneous-universal of power κ.

Since, by Proposition 3.4, all such structures are isomorphic, Corollary 4.2, Lemma 4.3 and Corollary 4.5 establish the following

THEOREM B. *If P is a downward-directed partially-ordered set with no least element, then there exists a full, uniform P-semilattice.*

Theorems A and B together clearly give our Main Theorem.

MAIN THEOREM. *If P is any downward-directed partially ordered set, then there exists a full, uniform P-semilattice.*

5. Further comments

Set theoretic aspects of the construction

In establishing Theorem B, we have assumed the existence of a cardinal κ satisfying the premises of Proposition 3.3. Since the number, $\nu(\mu)$, of free P-semilattices of power μ is at most $c(P)^{\mu} = 2^{\mu}$ for $\mu \geqslant c(P)$, this amounts to the existence of a regular cardinal κ for which $\kappa \geqslant c(P)$ and $\mu < \kappa$ implies $2^{\mu} \leqslant \kappa$. The existence of such a κ may *not* be proved in the usual Zermelo–Fraenkel set theory with the axiom of choice (ZFC), although the additional assumption of its existence would be acceptable. Two distinct conventional set theoretic assumptions which ensure the existence of such a cardinal κ for each P are: (a) the Generalized Continuum Hypothesis (GCH), that $\kappa^{+} = 2^{\kappa}$ for every infinite κ, and (b) that there are arbitrarily large strongly inaccessible cardinals, that is, regular cardinals, κ, for which $\mu < \kappa$ implies $2^{\mu} < \kappa$. Statement (a) is generally considered false, but is provably consistent, while (b) is considered true, but is not provably consistent.

https://doi.org/10.1017/S1446788700012520 Published online by Cambridge University Press

Avoiding set-theoretic assumptions

Our main theorem may, nevertheless, be proved in ZFC alone. Firstly, we observe that Theorem A is provable in ZF alone and therefore only the homogeneous-universal construction in the proof of Theorem B need be modified. Three different methods occur to the author and are now described.

(i) The use of *special* structures. These came into being precisely for this purpose, and are discussed in Jonsson (1960), Morley and Vaught (1962). For our purposes, it is sufficient to use Proposition 3.2 to construct a chain $\mathscr{A}_0 \subseteq \mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \dots$ of free P-semilattices for which $\mathscr{A}_n$ has power κ_n and each $\mathscr{A}_{n+1}$ is $(\mathscr{M}(P), \kappa_n)$-homogeneous-universal. The union of this chain is an example of an $\mathscr{M}(P)$-special structure and its associated P-semilattice is full and uniform, as may be shown by methods similar to, but longer than and involving heavier notation than, those already used.

(ii) The use of inner models of set theory. The class, $L[P']$, of all sets constructible (in the sense of Gödel) from an isomorphic copy, P', of P, all of whose elements are ordinals, is well known (see, for example, Cohen (1966)) to satisfy the axioms of ZFC plus the statement $\kappa^+ = 2^\kappa$ for all $\kappa \geqslant c(P)$. Thus, Theorem B holds when relativized to $L[P']$. Examination of the definitions reveals that a full, uniform P-semilattice in the sense of $L[P']$ is also one in the sense of the class of all sets.

(iii) A further possible variation is the use of the homogeneous-universal construction to prove, in ZFC alone, the existence of a *class-sized*, full, uniform P-semilattice. Only finitely many axioms of ZFC are used in this proof, and one may prove (in ZFC) the existence of a standard model, $M[P']$, of these axioms of ZFC, which contains an isomorphic, well-ordered copy, P', of P. The definition of the class-sized structure, when applied to the set $M[P']$, yields a *set-sized*, full, uniform P-semilattice.

Of these methods, (i) is clearly the most algebraic. In fact, in an earlier version of this paper, special structures were used throughout. We feel that the resulting simplification justifies the current presentation. Method (ii) is the standard technique for removing applications of the GCH and is probably the simplest and most generally acceptable. Method (iii), although more technical, seems preferable to the author from a metamathematical viewpoint, since it appears to be closer to the intuitive assumption of the existence of inaccessible cardinals. (The first ordinal not in $M[P']$ is 'as good as' an inaccessible cardinal, for this particular purpose.)

In contrast, the assumption of the axiom of choice or, at least, that P may be well-ordered, *does* appear to be necessary, although we have not proved this.

Reducing the size

Depending on the set-theoretic assumptions made, the power of the full, uniform P-semilattice constructed for Theorem B may be very large. However, one may prove the following, from first principles, in the case where P is infinite.

THEOREM 5.1. *Any full, uniform P-semilattice* $\mathscr{L} = (L, l)$ *has a full, uniform P-subsemilattice* $\mathscr{L}_0$ *of power* $c(P)$.

PROOF. Using the axiom of choice, let f and g be 3-ary operations on L such that, for $a, b \in L$ and $l(a) = l(b)$, the map $x \mapsto f(a, b, x)$ is an isomorphism from $\mathscr{L}^{<a}$ to $\mathscr{L}^{<b}$, and $x \mapsto g(a, b, x)$ is its inverse. Also, let S_0 be a subset of L of cardinality $c(P)$ containing, for all $p, q \in P$ with $p \leqslant q$, some a, b with $a \leqslant b$ and $l(a) < l(b)$. Let the chain $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ be defined inductively so that

$$S_{n+1} = S_n \cup \{f(a, b, c): a, b, c \in S_n\} \cup \{g(a, b, c): a, b, c \in S_n\} \cup \{a \wedge b: a, b \in S_n\}.$$

Let $L_0 = \bigcup S_n$. It is easily shown that $\mathscr{L}_0 = (L_0, l)$ is as stated.

If P is finite, then the semilattice obtained in the proof of Theorem A is finite, so our Main Theorem and Theorem 5.1 give the following

COROLLARY C. *If P is any downward-directed partially-ordered set, then there exists a full, uniform P-semilattice, finite if P is finite and of power* $c(P)$ *if P is infinite.*

It is interesting to note that the existence of a full, uniform P-semilattice of power $c(P)$, for P infinite, also follows more directly, using method (iii) of this section. If P has a least element then the proof of Theorem A gives the result immediately. If not, then, when using method (iii), the model $M[P']$ may be taken to have cardinality $c(P)$, with the desired result.

The method of proof

The reader may wonder, with the author, whether the use of homogeneous-universal structures is necessary for the proof of Theorem B. Similar constructions in model theory suggest that such methods are necessary for *some* theorems of this sort, but it is conceivable, and highly desirable, that a simpler proof may be found in this case.

Finally, it should be recorded that the author knows of no *a priori* reason why a full, uniform P-semilattice, if it exists, may be taken to be free. The class of free P-semilattices was conceived so that it, together with the universal-homogeneous construction, would realize a rather vague intuitive notion. If, however, such an *a priori* reason could be discovered, it might well clarify the status of our theorem.

Acknowledgements

The author wishes to thank J. N. Crossley for first introducing him to homogeneous-universal structures and T. E. Hall for describing to him a semigroup problem equivalent to the problem solved here and by which he has been intermittently preoccupied for longer than he cares to remember. The author also thanks Monash University and the University of Connecticut for employing him while thus preoccupied.

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https://doi.org/10.1017/S1446788700012520 Published online by Cambridge University Press