Bull. Aust. Math. Soc. **95** (2017), 412–423
doi:10.1017/S0004972716001374

EXTENSIONS OF THE HERMITE–HADAMARD INEQUALITY FOR *r*-PREINVEX FUNCTIONS ON AN INVEX SET

DAH-YAN HWANG✉ and SILVESTRU SEVER DRAGOMIR

(Received 2 September 2016; accepted 25 November 2016; first published online 6 March 2017)

Abstract

Necessary and sufficient conditions to characterise weakly r-preinvex functions on an invex set are obtained and used to establish generalisations of the Hermite–Hadamard inequality for such functions.

2010 *Mathematics subject classification*: primary 26D15; secondary 90C25.

Keywords and phrases: extended means, invex set, r-convex, r-preinvex, Hermite–Hadamard inequality.

1. Introduction

The classical Hermite–Hadamard inequality for convex functions states that if the function $f : [a, b] \to R$ is convex, then

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}.$$

In [5], Hanson introduced invex functions as a generalisation of convex functions. Hanson's result inspired subsequent work which established the role and applications of invexity in nonlinear optimisation and related fields. In [4], Ben-Israel and Mond introduced preinvex functions and showed that preinvexity implies invexity. The properties of preinvex functions in optimisation, equilibrium problems and variational inequalities were studied by Noor [8, 9] and Weir and Mond [12]. Antczak [1, 2] introduced r-invex and r-preinvex functions and gave a new method for solving nonlinear mathematical programming problems. Zhao *et al.* [14] characterised r-preinvex functions. In [10], Noor gave Hermite–Hadamard inequalities for preinvex and log-preinvex functions. Further, in [11], Ul-Haq and Iqbal established a Hermite–Hadamard inequality for r-preinvex functions.

The main purpose of this paper is to generalise the Hermite–Hadamard inequality to a relation between extended means of weakly r-preinvex functions on an invex set. The main tool is a characterisation of weakly r-preinvex functions on an invex set. We obtain new extended two-parameter mean inequalities for weakly r-preinvex functions on an invex set, which improve the results given in [10, 11].

2. Preliminary definitions and results for weakly r-preinvex functions

We begin with some definitions relating to invex sets and preinvex functions.

DEFINITION 2.1. Let $K \subset R^n$ be a nonempty set, let $\eta : K \times K \to R^n$ and let $u \in K$. Then the set K is said to be invex at u with respect to η if

$$u + \lambda\eta(v, u) \in K$$

for every $v \in K$ and $\lambda \in [0, 1]$. K is said to be an invex set with respect to η if K is invex at each $u \in K$ with respect to the same function η.

Definition 2.1 says that there is a path starting from u which is contained in K. It is not required that v should be an endpoint of the path. If we demand that v should be an endpoint of the path for every pair u, v, then $\eta(v, u) = v - u$ and invexity reduces to convexity. Thus every convex set is also an invex set with respect to $\eta(v, u) = v - u$, but the converse is not true (see [7, 8]).

In [3], Antczak introduced the following definition of an η-path on the basis of the consideration of invex sets.

DEFINITION 2.2. Let $K \subset R^n$ be a nonempty invex set with respect to η and let $u, v \in K$. For $x \in K$, the set $P_{ux} := \{u + \lambda\eta(v, u) : \lambda \in [0, 1]\}$ is the closed η-path joining the points u and $x = u + \eta(v, u)$ and $P^0_{ux} := \{u + \lambda\eta(v, u) : \lambda \in (0, 1)\}$ is the open η-path joining the points u and $x = u + \eta(v, u)$.

We note that if $\eta(v, u) = v - u$, then the set $P_{ux} = P_{uv} = \{\lambda v + (1 - \lambda)u : \lambda \in [0, 1]\}$ is the line segment with endpoints u and v.

In [4], Ben-Israel and Mond introduced the class of preinvex function with respect to η in optimisation theory.

DEFINITION 2.3. Let $K \subset R^n$ be a nonempty invex set with respect to η. A function $f : K \to R$ is said to be preinvex with respect to η if there is a vector-valued function $\eta : K \times K \to R^n$ such that

$$f(u + \lambda\eta(v, u)) \le \lambda f(v) + (1 - \lambda)f(u)$$

for every $u, v \in K$ and $\lambda \in [0, 1]$.

Every convex function is a preinvex function with respect to $\eta(v, u) = v - u$, but the converse may not always be true.

The detailed description of r-preinvex functions was given by Antczak in [1].

DEFINITION 2.4. Let $K \subset R^n$ be a nonempty invex set with respect to η. A function $f : K \to R^+$ is said to be r-preinvex with respect to η if there is a vector-valued function $\eta : K \times K \to R^n$ such that

$$f(u + \lambda\eta(v, u)) \le \begin{cases} (\lambda f(v)^r + (1 - \lambda)f(u)^r)^{1/r} & \text{if } r \ne 0, \\ f(v)^\lambda f(u)^{1-\lambda} & \text{if } r = 0, \end{cases}$$

for every $v, u \in K$ and $\lambda \in [0, 1]$.

https://doi.org/10.1017/S0004972716001374 Published online by Cambridge University Press

Note that 0-preinvex functions are logarithmic preinvex and that 1-preinvex functions are preinvex. It is obvious that if f is r-preinvex, then f^r is a preinvex function for positive r.

In [7], Mohan and Neogy showed that a differentiable invex function is also preinvex under the following Condition C.

CONDITION 2.5 (Condition C). *Let $K \subset R^n$ be a nonempty invex set with respect to $\eta : K \times K \to R^n$. We say that the function η satisfies Condition C if, for any $u, v \in K$ and $\lambda \in [0, 1]$, the following two identities hold.*

(i) $\eta(u, u + \lambda\eta(v, u)) = -\lambda\eta(v, u)$.

(i) $\eta(v, u + \lambda\eta(v, u)) = (1 - \lambda)\eta(v, u)$.

Applying Condition C, we have the following lemma.

LEMMA 2.6. *Let $K \subset R^n$ be a nonempty invex set with respect to $\eta : K \times K \to R^n$ and suppose that the function η satisfies Condition C. Then*

$$(\alpha - \beta)\eta(v, u) = \eta(u + \alpha\eta(v, u), u + \beta\eta(v, u))$$

for every $u, v \in K$ and $\alpha, \beta \in [0, 1]$.

PROOF. The identity obviously holds when $\alpha = \beta$. We will prove the case when $\alpha > \beta$. In this case, $0 < 1 - \beta \leq 1$ and $0 < (\alpha - \beta)/(1 - \beta) \leq 1$, so, by (i) and (ii) of Condition C,

$$\begin{aligned}(\alpha - \beta)\eta(v, u) &= \frac{\alpha - \beta}{1 - \beta}\eta(v, u + \beta\eta(v, u)) \\ &= \eta\Big(u + \beta\eta(v, u) + \frac{\alpha - \beta}{1 - \beta}\eta(v, u + \beta\eta(v, u)), u + \beta\eta(v, u)\Big).\end{aligned}$$

Using (i) of Condition C again,

$$\frac{1}{1 - \beta}\eta(v, u + \beta\eta(v, u)) = \eta(v, u).$$

These two results yield the desired identity immediately. The proof in the case when $\alpha < \beta$ is similar. This completes the proof of the lemma. □

In [13], Yang *et al.* gave the following Condition D to discuss the characterisation of prequasi-invex functions.

CONDITION 2.7 (Condition D). *Let $K \subset R^n$ be a nonempty invex set with respect to $\eta : K \times K \to R^n$ and let $f : K \to R$ be invex with respect to the same η. We say that the function f satisfies Condition D if the inequality*

$$f(u + \eta(v, u)) \leq f(v)$$

holds for any $u, v \in K$.

https://doi.org/10.1017/S0004972716001374 Published online by Cambridge University Press

The integral power mean, M_p, of a positive function on $[a, b]$ is given by

$$M_p(f; a, b) = \begin{cases} \left(\dfrac{1}{b-a}\displaystyle\int_a^b f^p(t)\,dt\right)^{1/p} & \text{if } p \neq 0, \\ \exp\left(\dfrac{1}{b-a}\displaystyle\int_a^b \ln f(t)\,dt\right) & \text{if } p = 0, \end{cases}$$

and the power mean, $M_r(x, y; \lambda)$, of order r of positive numbers x, y is defined by

$$M_r(x, y; \lambda) = \begin{cases} (\lambda x^r + (1-\lambda)y^r)^{1/r} & \text{if } r \neq 0, \\ x^\lambda y^{1-\lambda} & \text{if } r = 0 \end{cases}$$

(see [6]). In [6], Stolarsky introduced the mean values $E(r, s; x, y)$, to generalise the extended logarithmic mean $L_p(x, y)$, and the alternative extended logarithmic mean $F_r(x, y)$. The mean $E(r, s; x, y)$ is given by $E(r, s; x, x) = x$ if $x = y > 0$ and, for distinct numbers x, y,

$$\begin{aligned} E(r, s; x, y) &= \left(\frac{s}{r}\frac{y^r - x^r}{y^s - x^s}\right)^{1/(r-s)}, \quad rs(r-s) \neq 0, \\ E(r, 0; x, y) = E(0, r; x, y) &= \left(\frac{1}{r}\frac{y^r - x^r}{\ln y - \ln x}\right)^{1/r}, \quad r \neq 0, \\ E(r, r; x, y) &= e^{-1/r}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r - y^r)}, \quad r \neq 0, \\ E(0, 0; x, y) &= \sqrt{xy}. \end{aligned}$$

Clearly, for two positive real numbers x and y, $E(p+1, 1; x, y) = L_p(x, y)$ and $E(r+1, r; x, y) = F_r(x, y)$.

In order to obtain our results, we introduce the following new definitions related to power means.

Definition 2.8. Let $K \subset R^n$ be a nonempty invex set with respect to η. A function $f : K \to R$ is said to be weakly preinvex with respect to η if there is a vector-valued function $\eta : K \times K \to R^n$ such that

$$f(u + \lambda\eta(v, u)) \leq \lambda f(u + \eta(v, u)) + (1-\lambda)f(u)$$

for every $v, u \in K$ and $\lambda \in [0, 1]$.

Definition 2.9. Let $K \subset R^n$ be a nonempty invex set with respect to η. A function $f : K \to R^+$ is said to be weakly r-preinvex with respect to η if there is a vector-valued function $\eta : K \times K \to R^n$ such that

$$f(u + \lambda\eta(v, u)) \leq M_r(f(u + \eta(v, u)), f(u); \lambda)$$

for every $v, u \in K$ and $\lambda \in [0, 1]$.

https://doi.org/10.1017/S0004972716001374 Published online by Cambridge University Press

We note that if f is a weakly r-preinvex function, then f^r is weakly preinvex for positive r and, if f is a weakly 0-preinvex function, then $\log \circ f$ is weakly preinvex. We also note that, in Definitions 2.8 and 2.9, if f further satisfies Condition D, then f is a preinvex function and an r-preinvex function, respectively.

The extended two-parameter mean for a weakly r-preinvex function on an invex set is defined as follows.

Definition 2.10. Let $K \subset R^n$ be a nonempty invex set with respect to a vector-valued function $\eta : K \times K \to R^n$ and let $f : K \to R^+$ be integrable on the η-path P_{ux} for $x = u + \eta(v, u)$, where $v, u \in K$ and $\lambda \in [0, 1]$. Set $x(\lambda) = u + \lambda\eta(v, u)$. We define the two-parameter mean of the function $f(u + \lambda\eta(v, u))$ on $[0, 1]$ with respect to λ by

$$M_{p,q}(f; u, u + \eta(v, u)) = \begin{cases} \left(\int_0^1 f^p(x(\lambda))\, d\lambda \Big/ \int_0^1 f^q(x(\lambda))\, d\lambda \right)^{1/(p-q)} & \text{if } p \neq q, \\ \exp\left(\int_0^1 f^q(x(\lambda)) \ln f(x(\lambda))\, d\lambda \Big/ \int_0^1 f^q(x(\lambda))\, d\lambda \right) & \text{if } p = q. \end{cases}$$

In particular, when $q = 0$, $M_{p,0}(f; u, u + \eta(v, u)) = M_p(f; u, u + \eta(v, u))$ is the integral power mean.

We need the following properties of weakly r-preinvex functions.

Proposition 2.11. *Let $K \subset R^n$ be a nonempty invex set with respect to $\eta : K \times K \to R^n$ and suppose that η satisfies Condition C. Let $u \in K$ and $f : P_{ux} \to R$ for every $v \in K$, $\lambda \in [0, 1]$ and $x = u + \eta(v, u) \in K$. Suppose that $r \geq 0$. Then f is a weakly r-preinvex function with respect to η if and only if f^r is convex with respect to λ.*

Proof. Let $\phi(\lambda) = f^r(u + \lambda\eta(v, u))$ for $u, v \in K$, $\lambda \in [0, 1]$, $u + \lambda\eta(v, u) \in K$ and $r \geq 0$. First, assume that f is a weakly r-preinvex function with respect to η and that η satisfies Condition C. Obviously, f^r is a weakly preinvex function with respect to the same η. Now we will prove that $\phi(\lambda)$ is convex on $[0, 1]$. Since f^r is weakly preinvex, given $\alpha, \beta \in [0, 1]$ and for any $\lambda \in [0, 1]$,

$$\begin{aligned} \phi(\beta + \lambda(\alpha - \beta)) &= f^r(u + (\beta + \lambda(\alpha - \beta))\eta(v, u)) \\ &= f^r(u + \beta\eta(v, u) + \lambda(\alpha - \beta)\eta(v, u)) \\ &= f^r(u + \beta\eta(v, u) + \lambda(\eta(u + \alpha\eta(v, u), u + \beta\eta(v, u))) \quad \text{(by Lemma 2.6)} \\ &\leq \lambda f^r(u + \beta\eta(v, u) + \eta(u + \alpha\eta(v, u), u + \beta\eta(v, u))) \\ &\quad + (1 - \lambda) f^r(u + \beta\eta(v, u)) \\ &= \lambda f^r(u + \alpha\eta(v, u)) + (1 - \lambda) f^r(u + \beta\eta(v, u)) \quad \text{(by Lemma 2.6)} \end{aligned}$$

for $r > 0$, and, similarly,

$$\begin{aligned} \phi(\beta + \lambda(\alpha - \beta)) &\leq f^\lambda(u + \beta\eta(v, u) + \eta(u + \alpha\eta(v, u), u + \beta\eta(v, u))) f^{1-\lambda}(u + \beta\eta(v, u)) \\ &= f^\lambda(u + \alpha\eta(v, u)) f^{1-\lambda}(u + \beta\eta(v, u)) \end{aligned}$$

for $r = 0$. Therefore,

$$\phi(\beta + \lambda(\alpha - \beta)) \le \begin{cases} \lambda\phi(\alpha) + (1-\lambda)\phi(\beta) & \text{if } r > 0, \\ \phi^{\lambda}(\alpha)\phi^{1-\lambda}(\beta) & \text{if } r = 0. \end{cases}$$

Thus $f^r(u + \lambda\eta(v, u))$ is a convex function with respect to λ.

Second, assume that $f^r(u + \lambda\eta(v, u))$ is a convex function with respect to λ. We will prove that $f(u + \lambda\eta(v, u))$ is a weakly r-preinvex function with respect to η. Since $\phi(\lambda) = f^r(u + \lambda\eta(v, u))$ is convex with respect to λ,

$$\phi(\lambda \cdot 1 + (1-\lambda)\cdot 0) \le \begin{cases} \lambda\phi(1) + (1-\lambda)\phi(0) & \text{if } r > 0, \\ \phi^{\lambda}(1)\phi^{1-\lambda}(0) & \text{if } r = 0, \end{cases}$$

and then

$$f^r(u + \lambda\eta(v, u)) \le \begin{cases} \lambda f^r(u + \eta(v, u)) + (1-\lambda) f^r(u) & \text{if } r > 0, \\ f^{\lambda}(u + \eta(v, u)) f^{1-\lambda}(u) & \text{if } r = 0. \end{cases}$$

Thus f is weakly r-preinvex with respect to η. This completes the proof. □

Proposition 2.12. *In addition to the assumptions of Proposition 2.11, suppose that f is continuous on P_{ux} and is twice differentiable on P^0_{ux}. Then f is a weakly r-preinvex function with respect to η if and only if*

$$\begin{aligned} &r f^{r-2}(u)\{(r-1)[\eta(v,u)^T \nabla f(u)]^2 + f(u)\eta(v,u)^T \nabla^2 f(u)\eta(v,u)\} \ge 0 && \text{for } r > 0, \\ &\{\eta(v,u)^T \nabla^2 f(u)\eta(v,u) f(u) - [\eta(v,u)^T \nabla f(u)]^2\}/f^2(u) \ge 0 && \text{for } r = 0. \end{aligned}$$

Proof. Let $\phi(\lambda) = f^r(u + \lambda\eta(v, u))$ for $u, v \in K$, $\lambda \in [0, 1]$, $u + \lambda\eta(v, u) \in K$ and $r \ge 0$. Suppose that f is a weakly r-preinvex function with respect to η. Since f is continuous and twice differentiable,

$$\phi'(\lambda) = \begin{cases} r f^{r-1}(u + \lambda\eta(v,u))\eta(v,u)^T \nabla f(u + \lambda\eta(v,u)) & \text{if } r > 0, \\ \eta(v,u)^T \nabla f(u + \lambda\eta(v,u))/f(u + \lambda\eta(v,u)) & \text{if } r = 0, \end{cases}$$

and

$$\phi''(\lambda) = \begin{cases} r f^{r-2}(u + \lambda\eta(v,u))\{(r-1)[\eta(v,u)^T \nabla f(u + \lambda\eta(v,u))]^2 \\ \qquad + f(u + \lambda\eta(v,u))\eta(v,u)^T \nabla^2 f(u + \lambda\eta(v,u))\eta(v,u)\} & \text{if } r > 0, \\ \{\eta(v,u)^T \nabla^2 f(u + \lambda\eta(v,u))\eta(v,u) f(u + \lambda\eta(v,u)) \\ \qquad - [\eta(v,u)^T \nabla f(u + \lambda\eta(v,u))]^2\}/f^2(u + \lambda\eta(v,u)) & \text{if } r = 0. \end{cases}$$

Letting $\lambda \to 0^+$ gives

$$\phi''(0^+) = \begin{cases} r f^{r-2}(u)\{(r-1)[\eta(v,u)^T \nabla f(u)]^2 + f(u)\eta(v,u)^T \nabla^2 f(u)\eta(v,u)\} & \text{if } r > 0, \\ \{\eta(v,u)^T \nabla^2 f(u)\eta(v,u) f(u) - [\eta(v,u)^T \nabla f(u)]^2\}/f^2(u) & \text{if } r = 0. \end{cases}$$

By Proposition 2.11, for $r \ge 0$, $\phi(\lambda) = f^r(u + \lambda\eta(v, u))$ is a convex function with respect to λ and then $\phi''(\lambda) \ge 0$. This proves the necessity of the condition in the proposition.

https://doi.org/10.1017/S0004972716001374 Published online by Cambridge University Press

Conversely, assume that, for every $u, v \in K$,

$$rf^{r-2}(u)\{(r-1)[\eta(v,u)^T\nabla f(u)]^2 + f(u)\eta(v,u)^T\nabla^2 f(u)\eta(v,u)\} \geq 0 \quad \text{for } r > 0,$$
$$\{\eta(v,u)^T\nabla^2 f(u)\eta(v,u)f(u) - [\eta(v,u)^T\nabla f(u)]^2\}/f^2(u) \geq 0 \quad \text{for } r = 0.$$

For every $u, v \in K, \lambda$ in $[0, 1]$ and $u + \lambda\eta(v, u) \in K$,

$$rf^{r-2}(u+\lambda\eta(v,u))\{(r-1)[\eta(v,u+\lambda\eta(v,u))^T\nabla f(u+\lambda\eta(v,u))]^2 + f(u+\lambda\eta(v,u))\eta(v,u+\lambda\eta(v,u))^T\nabla^2 f(u+\lambda\eta(v,u))\eta(v,u+\lambda\eta(v,u))\} \geq 0$$

for $r > 0$, and

$$\{\eta(v,u+\lambda\eta(v,u))^T\nabla^2 f(u+\lambda\eta(v,u))\eta(v,u+\lambda\eta(v,u))f(u+\lambda\eta(v,u+\lambda\eta(v,u))) - [\eta(v,u+\lambda\eta(v,u))^T\nabla f(u+\lambda\eta(v,u))]^2\}/f^2(u+\lambda\eta(v,u)) \geq 0$$

for $r = 0$. By Condition C(ii),

$$rf^{r-2}(u+\lambda\eta(v,u))\{(r-1)[(1-\lambda)\eta(v,u)^T\nabla f(u+\lambda\eta(v,u))]^2 + f(u+\lambda\eta(v,u))(1-\lambda)\eta(v,u)^T\nabla^2 f(u+\lambda\eta(v,u))(1-\lambda)\eta(v,u)\} \geq 0$$

for $r > 0$, and

$$\{(1-\lambda)^2\eta(v,u)^T\nabla^2 f(u+\lambda\eta(v,u))\eta(v,u)f(u+\lambda\eta(v,u+\lambda\eta(v,u))) - [(1-\lambda)\eta(v,u)^T\nabla f(u+\lambda\eta(v,u))]^2\}/f^2(u+\lambda\eta(v,u)) \geq 0$$

for $r = 0$. Thus

$$\phi''(\lambda) = rf^{r-2}(u+\lambda\eta(v,u))\{(r-1)[\eta(v,u)^T\nabla f(u+\lambda\eta(v,u))]^2 + f(u+\lambda\eta(v,u))\eta(v,u)^T\nabla^2 f(u+\lambda\eta(v,u))\eta(v,u)\} \geq 0$$

for $r > 0$, and

$$\phi''(\lambda) = \{\eta(v,u)^T\nabla^2 f(u+\lambda\eta(v,u))\eta(v,u)f(u+\lambda\eta(v,u+\lambda\eta(v,u))) - [\eta(v,u)^T\nabla f(u+\lambda\eta(v,u))]^2\}/f^2(u+\lambda\eta(v,u)) \geq 0$$

for $r = 0$. Consequently, $\phi(\lambda) = f^r(u + \lambda\eta(v, u))$ is convex with respect to λ. By Proposition 2.11, f is weakly r-preinvex with respect to η. This completes the proof. □

3. Hermite–Hadamard inequality for weakly *r*-preinvex function

For simplicity, in this section, we assume that $K \subset R^n$ is a nonempty invex set with respect to a vector valued function $\eta : K \times K \to R^n$. Applying the definitions, conditions and results of Section 2, gives the following theorems.

THEOREM 3.1. *Let f be a weakly r-preinvex function on an invex set K with $r \geq 0$. Assume that f is positive and continuous on P_{ax} and is twice-differentiable on P^0_{ax} for every $a, b \in K$, $\lambda \in [0, 1]$ and $a < x = a + \eta(b, a)$, and let η satisfy Condition C. Let m and M be the minimum and maximum of f on P_{ax}, respectively. Further, let*

https://doi.org/10.1017/S0004972716001374 Published online by Cambridge University Press

$g_1, g_2 : (0, \infty) \to R$ and suppose that g_2 is positive and integrable on $[m, M]$ and that g_1/g_2 is integrable on $[m, M]$. If g_1/g_2 is increasing on $[m, M]$, then

$$\frac{\int_0^1 g_1(f(a+\lambda\eta(b,a)))\,d\lambda}{\int_0^1 g_2(f(a+\lambda\eta(b,a)))\,d\lambda} \le \frac{\int_{f(a)}^{f(a+\eta(b,a))} x^{r-1} g_1(x)\,dx}{\int_{f(a)}^{f(a+\eta(b,a))} x^{r-1} g_2(x)\,dx} \tag{3.1}$$

for $f(a) \neq f(a+\eta(b,a))$; the right-hand side of (3.1) is defined to be $g_1(f(a))/g_2(f(a))$ for $f(a) = f(a+\eta(b,a))$. If g_1/g_2 is decreasing, then the inequality (3.1) is reversed.

Proof. We will give the proof in the case when $r > 0$ and g_1/g_2 is increasing. The proof in the other cases is analogous. Let $\phi(\lambda) = f^r(a+\lambda\eta(b,a))$ for $r \neq 0$ and $\phi(\lambda) = \ln f(a+\lambda\eta(b,a))$ for $r = 0$. For convenience, let $\psi(\lambda) = f(a+\lambda\eta(b,a))$. Since f is weakly r-preinvex with respect to η, Proposition 2.12 gives

$$\phi''(\lambda) = rf^{(r-2)}(a)\{(r-1)[\eta(b,a)^T\nabla f(a)]^2 + f(a)\eta(b,a)^T\nabla^2 f(a)\eta(b,a)\} > 0.$$

When $f(a) \neq f(a+\eta(b,a))$, the inequality (3.1) is equivalent to

$$\frac{\int_0^1 g_1(\psi(\lambda))\,d\lambda}{\int_0^1 g_2(\psi(\lambda))\,d\lambda} \le \frac{\int_0^1 \psi^{r-1}(\lambda)g_1(\psi(\lambda))\psi'(\lambda)\,d\lambda}{\int_0^1 \psi^{r-1}(\lambda)g_2(\psi(\lambda))\psi'(\lambda)\,d\lambda}. \tag{3.2}$$

Consider

$$\begin{aligned} I &= \int_0^1 g_1(\psi(\lambda))\,d\lambda \int_0^1 \psi^{r-1}(\mu)g_2(\psi(\mu))\psi'(\mu)\,d\mu \\ &\quad - \int_0^1 g_2(\psi(\lambda))\,d\lambda \int_0^1 \psi^{r-1}(\mu)g_1(\psi(\mu))\psi'(\mu)\,d\mu \\ &= \int_0^1\int_0^1 g_2(\psi(\lambda))g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu)\Big(\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big)\,d\lambda\,d\mu. \end{aligned} \tag{3.3}$$

Interchanging λ and μ in (3.3) and adding the resulting equation to (3.3) gives

$$I = \frac{1}{2r}\int_0^1\int_0^1 g_2(\psi(\lambda))g_2(\psi(\mu))[(\psi^r(\mu))' - (\psi^r(\lambda))']\Big(\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big)\,d\lambda\,d\mu. \tag{3.4}$$

First, suppose that $\phi'(\lambda) = (\psi^r(\lambda))' \ge 0$ for all $\lambda \in (0,1)$. Since $\phi''(\lambda) = (\psi^r(\lambda))'' \ge 0$,

$$\frac{1}{r}[(\psi^r(\mu))' - (\psi^r(\lambda))']\Big(\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big) \le 0.$$

From (3.4), $I \le 0$. This implies that the inequality (3.2) holds and then (3.1) holds. If $\phi'(\lambda) = (\psi^r(\lambda))' \le 0$ for all $\lambda \in (0,1)$, a similar argument gives $I \ge 0$ and again the inequality (3.1) holds.

Now suppose that $\phi'(\lambda) = (\psi^r(\lambda))'$ changes sign and that $\phi(0) < \phi(1)$. Then $\psi^r(0) \le \psi^r(1)$ and there exists a point $\alpha \in (0,1)$ such that $\phi'(\alpha) = (\psi^r(\alpha))' = 0$ and $(\psi^r(\lambda))' \le 0$

https://doi.org/10.1017/S0004972716001374 Published online by Cambridge University Press

for all $\lambda \in [0, \alpha]$ and $(\psi^r(\lambda))' \geq 0$ for all $\lambda \in [\alpha, 1]$. Therefore, there exists $\beta \in (\alpha, 1)$ such that $\psi(0) = \psi(\beta)$. Thus

$$\int_0^\beta \psi^{r-1}(\lambda) g_1(\psi(\lambda))\psi'(\lambda)\, d\lambda = \int_{\psi(0)}^{\psi(\alpha)} x^{r-1} g_1(x)\, dx + \int_{\psi(\alpha)}^{\psi(\beta)} x^{r-1} g_1(x)\, dx = 0$$

and, similarly,

$$\int_0^\beta \psi^{r-1}(\lambda) g_2(\psi(\lambda))\psi'(\lambda)\, d\lambda = 0.$$

Consequently, the inequality (3.1) is equivalent to

$$\frac{\int_0^1 g_1(\psi(\lambda))\, d\lambda}{\int_0^1 g_1(\psi(\lambda))\, d\lambda} \leq \frac{\int_\beta^1 \psi^{r-1}(\lambda) g_1(\psi(\lambda))\psi'(\lambda)\, d\lambda}{\int_\beta^1 \psi^{r-1}(\lambda) g_2(\psi(\lambda))\psi'(\lambda)\, d\lambda}. \tag{3.5}$$

Consider

$$\begin{aligned} I_2 &= \int_0^1 g_1(\psi(\lambda))\, d\lambda \int_\beta^1 \psi^{r-1}(\mu) g_2(\psi(\mu))\psi'(\mu)\, d\mu \\ &\quad - \int_0^1 g_2(\psi(\lambda))\, d\lambda \int_\beta^1 \psi^{r-1}(\mu) g_1(\psi(\mu))\psi'(\mu)\, d\mu \\ &= \frac{1}{r}\int_0^1 \int_\beta^1 g_2(\psi(\lambda)) g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu)\Big(\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big)\, d\lambda\, d\mu. \end{aligned}$$

Split the double integral into two parts

$$I_2 = \frac{1}{r}\int_0^1 \int_\beta^1 \dots\, d\lambda\, d\mu = \frac{1}{r}\Big(\int_0^\beta \int_\beta^1 \dots\, d\lambda\, d\mu + \int_\beta^1 \int_\beta^1 \dots\, d\lambda\, d\mu\Big) = I_{21} + I_{22}.$$

When $(\lambda, \mu) \in [0, \beta] \times [\beta, 1]$, $\lambda \leq \mu$ and $(\psi^r(\mu))' = r\psi^{r-1}(\mu)\psi'(\mu) \geq 0$ for all $\mu \in (\beta, 1)$. Thus $\psi'(\mu) \geq 0$ for all $\mu \in (\beta, 1)$ and

$$\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} \leq \frac{g_1(\psi(\beta))}{g_2(\psi(\beta))} \leq \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}.$$

This gives $I_{21} \leq 0$. By the result proved in the case when $\phi'(\lambda) = (\psi^r(\lambda))' \geq 0$, we see that $I_{22} \leq 0$. Therefore, $I_2 = I_{21} + I_{22} \leq 0$. It follows that (3.5) and also (3.1) hold. Finally, if the sign of the derivative $\phi'(\lambda) = (\psi^r(\lambda))'$ changes and $\psi(0) \geq \psi(1)$, a similar proof again shows that (3.1) holds.

When $f(a) = f(a + \eta(b, a))$, $\psi(0) = \psi(1)$, so $\phi(0) = \phi(1)$. Since $\phi'' = (\psi^r(\lambda))'' \geq 0$, we see that $\phi' = (\psi^r(\lambda))'$ is continuous and increasing for $\lambda \in (0, 1)$. There exists a point $\alpha \in (0, 1)$ such that $(\psi^r(\alpha))' = 0$ and $(\psi^r(\lambda))' \leq 0$ for all $\lambda \in (0, \alpha)$ and $(\psi^r(\lambda))' \geq 0$ for all $\lambda \in (\alpha, 1)$. Hence

$$\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} \leq \frac{g_1(\psi(1))}{g_2(\psi(1))}$$

https://doi.org/10.1017/S0004972716001374 Published online by Cambridge University Press

for all $\lambda \in (0, 1)$. It follows that

$$\int_0^1 g_1(\psi(\lambda))\,d\lambda \leq \frac{g_1(\psi(1))}{g_2(\psi(1))}\int_0^1 g_2(\psi(\lambda))\,d\lambda.$$

Therefore, the inequality (3.1) is valid. This completes the proof of Theorem 3.1. □

Remark 3.2. If we take $g_1(x) = x^p$ and $g_2(x) = x^q$ for suitable real numbers p, q in (3.1), we get the extended mean inequality for the twice-differentiable and weakly r-preinvex function f on an invex set with respect to η satisfying Condition C given by

$$M_{p,q}(f; a, a + \eta(b, a)) \leq E(p + r, q + r; f(a), f(a + \eta(b, a))). \tag{3.6}$$

Moreover, if we take $q = 0$ in (3.6),

$$M_p(f; a, a + \eta(b, a)) \leq E(p + r, r; f(a), f(a + \eta(b, a))). \tag{3.7}$$

Taking $r = 1$ in (3.7) gives

$$M_p(f; a, a + \eta(b, a)) \leq L_p(f(a), f(a + \eta(b, a))),$$

and taking $p = 1$ in (3.7) gives

$$\frac{1}{\eta(b, a)}\int_a^{a+\eta(b,a)} f(x)\,dx \leq F_r(f(a), f(a + \eta(b, a))). \tag{3.8}$$

Further, if f satisfies the Condition D, (3.8) becomes

$$\frac{1}{\eta(b, a)}\int_a^{a+\eta(b,a)} f(x)\,dx \leq F_r(f(a), f(a + \eta(b, a))) \leq F_r(f(a), f(b)). \tag{3.9}$$

The inequality (3.9) is a refinement of the inequality given by Ul-Haq and Iqbal in [11]. For $r = 1$ or $r = 0$ in (3.9), the inequality (3.9) is a refinement of the inequality given by Noor in [10].

Theorem 3.3. *Let f be a weakly r-preinvex function on an invex set K with $r \geq 0$. Assume that f is positive and continuous on P_{ax} for given $a, b \in K$, $\lambda \in [0, 1]$ and $a < x = a + \eta(b, a)$. Further, let $g : (0, \infty) \to R$ be positive and integrable on $[m, M]$, where m, M are as in Theorem 3.1. If g is increasing on $[m, M]$, then*

$$\int_0^1 g(f(a + \lambda\eta(b, a)))\,d\lambda \leq \frac{r}{f^r(a + \eta(b, a)) - f^r(a)}\int_{f(a)}^{f(a+\eta(b,a))} x^{r-1}g(x)\,dx \tag{3.10}$$

for $f(a) \neq f(a + \eta(b, a))$; the right-hand side of (3.10) is defined to be $g(f(a))$ for $f(a) = f(a + \eta(b, a))$. If g is decreasing, the inequality (3.10) is reversed.

Proof. We consider only the case when $r > 0$ and g is increasing. The proof is analogous in the other cases. When $f(a) \neq f(a + \eta(b, a))$, the definition of a weakly r-preinvex function yields

$$\begin{aligned}\int_0^1 g(f(a + \lambda\eta(b, a)))\,d\lambda &\leq \int_0^1 g((\lambda f^r(a + \eta(b, a)) + (1 - \lambda)f^r(a))^{1/r})\,d\lambda \\ &= \frac{r}{f^r(a + \eta(b, a)) - f^r(a)}\int_{f(a)}^{f(a+\eta(b,a))} g(x)x^{r-1}\,dx.\end{aligned}$$

https://doi.org/10.1017/S0004972716001374 Published online by Cambridge University Press

Similarly, when $f(a) = f(a + \eta(b, a))$, it is immediate that

$$\int_0^1 g(f(a + \lambda\eta(b, a)))\, d\lambda \le \int_0^1 g((\lambda f^r(a + \eta(b, a)) + (1 - \lambda) f^r(a))^{1/r})\, d\lambda = g(f(a)).$$

The proof of Theorem 3.3 is complete. □

REMARK 3.4. Note that it is not necessary for the function f in Theorem 3.3 to be twice differentiable. Similarly to Remark 3.2, if we take $g(x) = x^p$ in (3.10), we obtain the extended mean inequality for the weakly r-preinvex function f on an invex set with respect to η given by

$$M_p(f; a, a + \eta(b, a)) \le E(p + r, r; f(a), f(a + \eta(b, a))). \tag{3.11}$$

Taking $r = 1$ in (3.11) gives

$$M_p(f; a, a + \eta(b, a)) \le L_p(f(a), f(a + \eta(b, a))),$$

and taking $p = 1$ in (3.11) gives

$$\frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, dx \le F_r(f(a), f(a + \eta(b, a))). \tag{3.12}$$

Further, if f satisfies Condition D, (3.12) yields

$$\frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, dx \le F_r(f(a), f(a + \eta(b, a))) \le F_r(f(a), f(b)). \tag{3.13}$$

The inequality (3.13) is a refinement of the inequality given by Ul-Haq and Iqbal in [11] and also a refinement of the inequality given by Noor in [10].

References

[1] T. Antczak, 'r-preinvexity and r-invexity in mathematical programming', *Comput. Math. Appl.* **50**(3–4) (2005), 551–566.

[2] T. Antczak, 'A new method of solving nonlinear mathematical programming problems involving r-invex functions', *J. Math. Anal. Appl.* **311**(1) (2005), 313–323.

[3] T. Antczak, 'Mean value in invexity analysis', *Nonlinear Anal.* **60** (2005), 1473–1484.

[4] A. Ben-Israel and B. Mond, 'What is invexity?', *J. Aust. Math. Soc. Ser. B* **28** (1986), 1–9.

[5] M. A. Hanson, 'On sufficiency of the Kuhn–Tucker conditions', *J. Math. Anal. Appl.* **80**(2) (1981), 545–550.

[6] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis* (Kluwer Academic, Dordrecht, 1993).

[7] S. R. Mohan and S. K. Neogy, 'On invex sets and preinvex functions', *J. Math. Anal. Appl.* **189** (1995), 901–908.

[8] M. A. Noor, 'Variational-like inequalities', *Optimization* **30**(4) (1994), 323–330.

[9] M. A. Noor, 'Invex equilibrium problems', *J. Math. Anal. Appl.* **302**(2) (2005), 463–475.

[10] M. A. Noor, 'Hermite–Hadamard integral inequalities for log-preinvex functions', *J. Math. Anal. Approx. Theory* **2**(2) (2007), 126–131.

[11] W. Ul-Haq and J. Iqbal, 'Hermite–Hadamard-type inequalities for r-preinvex functions', *J. Appl. Math.* **2013** (2013), Article ID 126457, 5 pages.

[12] T. Weir and B. Mond, 'Pre-invex functions in multiple objective optimization', *J. Math. Anal. Appl.* **136**(1) (1988), 29–38.
[13] X. M. Yang, X. Q. Yang and K. L. Teo, 'Characterizations and applications of prequasi-invex functions', *J. Optim. Theory Appl.* **110**(3) (2001), 645–668.
[14] K.-Q. Zhao, P.-J. Long and X. Wan, 'A characterization for *r*-preinvex function', *J. Chongqing Normal University (Natural Science)* **28**(2) (2011), 1–5.

DAH-YAN HWANG, Department of Information and Management,
Taipei City University of Science and Technology, No. 2, Xueyuan Road,
Beitou, 112, Taipei, Taiwan
e-mail: dyhuang@tpcu.edu.tw

SILVESTRU SEVER DRAGOMIR,
Mathematics, School of Engineering and Science, Victoria University,
PO Box 14428, Melbourne City, MC 8001, Australia
and
School of Computational and Applied Mathematics,
University of the Witwatersrand, Private Bag 3,
Johannesburg 2050, South Africa
e-mail: sever.dragomir@vu.edu.au