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Power Series Rings Over Prüfer v-multiplication Domains. II

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Abstract. Let D be an integral domain, $X^1(D)$ be the set of height-one prime ideals of D, $\{X_\beta\}$ and $\{X_\alpha\}$ be two disjoint nonempty sets of indeterminates over D, $D[\{X_\beta\}]$ be the polynomial ring over D, and $D[\{X_\beta\}][[\{X_\alpha\}]]_1$ be the first type power series ring over $D[\{X_\beta\}]$. Assume that D is a Prüfer v-multiplication domain (PvMD) in which each proper integral t-ideal has only finitely many minimal prime ideals (*e.g.,* t-SFT PvMDs, valuation domains, rings of Krull type). Among other things, we show that if $X^1(D) = \varnothing$ or D_P is a DVR for all $P \in X^1(D)$, then $D[\{X_\beta\}][[\{X_\alpha\}]]_{1D-\{0\}}$ is a Krull domain. We also prove that if D is a t-SFT PvMD, then the complete integral closure of D is a Krull domain and $\operatorname{ht}(M[\{X_\beta\}][[\{X_\alpha\}]]_1) = 1$ for every height-one maximal t-ideal M of D.

1 Introduction

Let D be an integral domain with quotient field K. Let $\{X_\alpha\}$ be a nonempty set of indeterminates over D, $D[\{X_\alpha\}]$ be the polynomial ring over D, and $D[[\{X_\alpha\}]]_1$ be the first type power series ring over D; *i.e.,* $D[[\{X_\alpha\}]]_1 = \bigcup D[[X_1, \dots, X_n]]$, where $\{X_1, \dots, X_n\}$ runs over all finite subsets of $\{X_\alpha\}$, so $D[[\{X_\alpha\}]]_1 = D[[\{X_\alpha\}]]$ if and only if $|\{X_\alpha\}| < \infty$ (*cf.* [19, Section 1] for the power series ring). Let A be an ideal of D. Then $AD[[\{X_\alpha\}]]_1$ is the ideal of $D[[\{X_\alpha\}]]_1$ generated by A and $A[[\{X_\alpha\}]]_1 = \{f \in D[[\{X_\alpha\}]]_1 \mid c(f) \subseteq A\}$, where $c(f)$ is the ideal of D generated by the coefficients of f, so $A[[\{X_\alpha\}]]_1$ is an ideal of $D[[\{X_\alpha\}]]_1$ such that $AD[[\{X_\alpha\}]]_1 \subseteq A[[\{X_\alpha\}]]_1$. Also, $AD[[\{X_\alpha\}]]_1 = A[[\{X_\alpha\}]]_1$ if and only if A is finitely generated, and A is a prime ideal if and only if $A[[\{X_\alpha\}]]_1$ is a prime ideal.

Let $X^1(D)$ be the set of height-one prime ideals of D. A *Krull domain* D is an integral domain in which (i) $D = \bigcap_{P\in X^1(D)} D_P$, (ii) D_P is a rank-one discrete valuation ring (DVR) for all $P \in X^1(D)$, and (iii) the intersection $D = \bigcap_{P\in X^1(D)} D_P$ is locally finite; *i.e.,* each nonzero element of D lies in only a finite number of prime ideals in $X^1(D)$. It is clear that D is a Krull domain with $X^1(D) = \varnothing$ if and only if D is a field. Krull domains are very important because of, among other things, the following well-known results that D is a Dedekind domain if and only if D is a Krull domain of (Krull) dimension at most one; if D is a Krull domain, then $\operatorname{Div}(D)$, the monoid of v-ideals of D under $I * J = (IJ)_v$, is a free abelian group on $X^1(D)$ and $Cl(D) = \operatorname{Div}(D)/\operatorname{Prin}(D)$, where $\operatorname{Prin}(D)$ is the subgroup of nonzero principal fractional ideals of D, is the divisor class group of D; for every abelian group G, there is a Dedekind domain D with $Cl(D) = G$; D is a UFD if and only if D is a Krull

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domain with $Cl(D) = \{0\}$; the integral closure of a Noetherian domain is a Krull domain; and D is a Krull domain if and only if $D[\{X_\alpha\}]$ is a Krull domain, if and only if $D[[\{X_\alpha\}]]_1$ is a Krull domain (see, for example, [16]).

Clearly, $D[\{X_\alpha\}]_{D-\{0\}} = K[\{X_\alpha\}]$, and hence $D[\{X_\alpha\}]_{D-\{0\}}$ is a UFD (so a Krull domain), while the next example shows that $D[[\{X_\alpha\}]]_{1D-\{0\}}$ need not be a Krull domain.

Example 1.1 Let V be a rank-one nondiscrete valuation domain with maximal ideal M, and let $V[[\{X_\alpha\}]]_1$ be the first type power series ring over V. Note that if $X \in \{X_\alpha\}$, then $MV[[X]]$ is a prime ideal of $V[[X]]$ such that $V[[X]]_{MV[[X]]}$ is a rank-one valuation domain,

$$V[[X]]_{MV[[X]]} \cap V[[X]]_{V-\{0\}} = V[[X]],$$

and

$$V[[\{X_\alpha\}]]_{1V-\{0\}} \cap \mathrm{qf}(V[[X]]) = V[[X]]_{V-\{0\}},$$

where $\mathrm{qf}(V[[X]])$ is the quotient field of $V[[X]]$. Hence, if $V[[\{X_\alpha\}]]_{1V-\{0\}}$ is a Krull domain, then $V[[X]]_{V-\{0\}}$ is also a Krull domain, and thus $V[[X]]$ is a generalized Krull domain. (See Section 2 for the definition of a generalized Krull domain.) But, in this case, V must be a rank-one DVR [28, Theorem 2.5]. Thus, $V[[\{X_\alpha\}]]_{1V-\{0\}}$ is not a Krull domain.

However, in [3, Theorem 3.7], it was shown that if D is an SFT Prüfer domain, then $D[[\{X_\alpha\}]]_{1D-\{0\}}$ is a Krull domain. This was generalized in [8, Theorem 9(3)] to t-SFT Prüfer v-multiplication domains (PvMDs) as follows: If D is a t-SFT PvMD, then $D[[\{X_\alpha\}]]_{1D-\{0\}}$ is a Krull domain. Let $\{X_\beta\}$ and $\{X_\alpha\}$ be two disjoint nonempty sets of indeterminates over D and $D[\{X_\beta\}]$ be the polynomial ring over D. If D is a t-SFT PvMD, then $D_0 := D[\{X_\beta\}]$ is a t-SFT PvMD [8, Theorem 11]. Hence, $D_0[[\{X_\alpha\}]]_{1D_0-\{0\}}$ is a Krull domain for which it is natural to ask if $D_0[[\{X_\alpha\}]]_{1D-\{0\}}$ is a Krull domain.

Let D be a PvMD such that each proper integral t-ideal of D has a finite number of minimal prime ideals (*e.g.*, t-SFT PvMDs, valuation domains, rings of Krull type). In this paper, we modify the proof of [8, Lemma 8] (hence that of [5, Lemma 3.3]) to prove that if $X^1(D) = \varnothing$ or D_P is a DVR for all $P \in X^1(D)$, then both the complete integral closure of D and $D[[\{X_\alpha\}]]_{1D-\{0\}}$ are Krull domains. This also gives another proof of [3, Theorem 3.7] that if D is an SFT Prüfer domain, then $D[[\{X_\alpha\}]]_{1D-\{0\}}$ is a Krull domain. We then use this result to show that $D[\{X_\beta\}][[\{X_\alpha\}]]_{1D-\{0\}}$ is a Krull domain. Hence, if D is a t-SFT PvMD, then

$$D[[\{X_\alpha\}]]_{1D-\{0\}} \quad \text{and} \quad D[\{X_\beta\}][[\{X_\alpha\}]]_{1D-\{0\}}$$

are both Krull domains. As a corollary, we have that if D is a valuation domain such that either $X^1(D) = \varnothing$ or D has a height-one prime ideal P with $P^2 \neq P$, then $D[\{X_\beta\}][[\{X_\alpha\}]]_{1D-\{0\}}$ is a Krull domain. We finally prove that if M is a height-one maximal t-ideal of a t-SFT PvMD, then $\mathrm{ht}(M[\{X_\beta\}][[\{X_\alpha\}]]_1) = 1$. Although some of the proofs are similar to the proof of [8, Lemma 8], we include them for completeness.

We first review definitions related to the t-operation. A fractional ideal I of D is a D-submodule of K such that $dI \subseteq D$ for some $0 \neq d \in D$. Let $F(D)$ be the set of nonzero fractional ideals of D. For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$; then $I^{-1} \in F(D)$. The v-operation is defined by $I_v = (I^{-1})^{-1}$ and the t-operation by $I_t = \bigcup\{J_v \mid J \in F(D), J \text{ is finitely generated, and } J \subseteq I\}$. Clearly, if $I \in F(D)$, then $I \subseteq I_t \subseteq I_v$, and if I is finitely generated, then $I_t = I_v$. If $* = v$ or t, then I is called a *$*$-ideal* if $I = I_*$ and a *$*$-ideal of finite type* if $I = B_*$ for some finitely generated ideal $B \in F(D)$. A $*$-ideal of D is called a *maximal $*$-ideal* if it is maximal among proper integral $*$-ideals of D. Let $*$-Max(D) be the set of all maximal $*$-ideals of D. While v-Max(D) can be empty as in the case of a rank-one nondiscrete valuation domain D, it is well known that t-Max$(D) \neq \varnothing$ when D is not a field; a prime ideal minimal over a t-ideal is a t-ideal; each proper integral t-ideal is contained in a maximal t-ideal; each maximal t-ideal is a prime ideal; and $D = \bigcap_{P \in t\text{-Max}(D)} D_P$. An *overring* of D means a ring between D and K. We say that an overring R of D is *t-linked* over D if $I_v = D$ implies $(IR)_v = R$ for all finitely generated ideals $I \in F(D)$. It is easy to see that R is t-linked over D if and only if $(Q \cap D)_t \subsetneq D$ for each prime t-ideal Q of R [11, Proposition 2.1]. An $I \in F(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$, and we say that D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible. It is well known that D is a PvMD if and only if D_P is a valuation domain for each maximal t-ideal P of D [20, Theorem 5]. For more on basic properties of the v- and t-operations, see [19, Sections 32 and 34].

A nonzero ideal I of D is called an *SFT-ideal* (an ideal of strong finite type) (resp., a *t-SFT-ideal*) if there exist a finitely generated ideal $J \subseteq I$ and an integer $k \geq 1$ such that $a^k \in J$ for all $a \in I$ (resp., $a^k \in J_v$ for all $a \in I_t$). The ring D is called an *SFT-ring* (resp., a *t-SFT-ring*) if each nonzero ideal of D is an SFT-ideal (resp., a t-SFT-ideal). It is known that D is an SFT-ring (resp., a t-SFT-ring) if and only if each prime ideal (resp., prime t-ideal) of D is an SFT-ideal (resp., a t-SFT-ideal) [4, Proposition 2.2] (resp., [24, Proposition 2.1]). Note that D is a Prüfer domain if and only if D is a PvMD whose maximal ideals are t-ideals, and each nonzero ideal of a Prüfer domain is a t-ideal. Hence, SFT Prüfer domains $\Leftrightarrow$ t-SFT Prüfer domains $\Rightarrow$ t-SFT PvMDs. It is known that D is a Krull domain if and only if D is a t-SFT PvMD in which each prime t-ideal is a maximal t-ideal [8, Theorem 9(2)].

2 SFT Prüfer Domains, t-SFT PvMDs, and Rings of Krull Type

A valuation domain V is said to be *strongly discrete* if each nonzero prime ideal P of V is not idempotent, *i.e.,* $P^2 \neq P$. A *strongly discrete Prüfer domain* is an integral domain D in which D_M is a strongly discrete valuation domain for all maximal ideals M of D. We say that D is a *generalized Dedekind domain* if (i) D is a strongly discrete Prüfer domain and (ii) each prime ideal of D is the radical of a finitely generated ideal. The notion of generalized Dedekind domains was introduced by Popescu [29]. It is easy to see that D is a Dedekind domain if and only if D is a generalized Dedekind domain of dimension at most one. For more on generalized Dedekind domains, see [15, Chapter 5] or [17]. In [23, Theorem 2.4], Kang and Park showed the following lemma.

Lemma 2.1 *The concepts "SFT Prüfer domain" and "generalized Dedekind domain" are the same.*

Let F be a field with $K \subseteq F$, where K is the quotient field of D, and let X be an indeterminate. It is known that $R = D + XF[X]$ is an SFT Prüfer domain if and only if $F = K$ and D is an SFT Prüfer domain [17, Corollary 4.2]. More generally, we have the following proposition.

Proposition 2.2 *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a graded integral domain with $R_n \neq \{0\}$ for all $n \geq 0$. Then R is an SFT Prüfer domain if and only if $R \cong D + XK[X]$ for some SFT Prüfer domain D with quotient field K.*

Proof Recall from [10, Proposition 3.4] that $R = \bigoplus_{n=0}^{\infty} R_n$ is a Prüfer domain if and only $R \cong D + XK[X]$ for some Prüfer domain D with quotient field K. Thus, the result follows directly from [17, Corollary 4.2]. ∎

As the t-operation analog of generalized Dedekind domains, El Baghdadi [12] introduced the notion of generalized Krull domains as follows: D is a *generalized Krull domain* if D is a PvMD such that (i) D_P is strongly discrete for each maximal t-ideal P of D and (ii) each prime t-ideal of D is the radical of a t-ideal of finite type. We noted in the introduction that D is a Prüfer domain if and only if D is a PvMD whose maximal ideals are t-ideals, and each nonzero ideal of a Prüfer domain is a t-ideal. Thus, a generalized Dedekind domain is just a generalized Krull domain in which each maximal ideal is a t-ideal.

Recall from [19, Section 43] that D is a *generalized Krull domain* if (i) D_P is a valuation domain for each $P \in X^1(D)$, (ii) $D = \bigcap_{P \in X^1(D)} D_P$, and (iii) the intersection $D = \bigcap_{P \in X^1(D)} D_P$ is locally finite. A generalized Krull domain is a PvMD whose prime t-ideals are maximal t-ideals, and a Krull domain is a generalized Krull domain. Thus, a generalized Krull domain is a Krull domain if and only if it is a t-SFT-ring (*cf.* [8, Proposition 9(2)]). Clearly, this notion of generalized Krull domains is different from El Baghdadi's generalized Krull domains, so we denote by *GK-domains* El Baghdadi's generalized Krull domains. As in the case of SFT Prüfer domains, in [24, Theorem 2.5], Kang and Park proved the following lemma.

Lemma 2.3 *D is a GK-domain if and only if D is a t-SFT PvMD.*

An integral domain D is said to be of *finite character* (resp., *finite t-character*) if each nonzero element of D is contained in only finitely many maximal ideals (resp., maximal t-ideals) of D. Following [21], we say that D is a *ring of Krull type* if D is a locally finite intersection of essential valuation overrings of D; equivalently, D is a PvMD of finite t-character [20, Theorem 7]. Clearly, both Krull domains and Prüfer domains of finite character are rings of Krull type. For easy examples of t-SFT PvMDs and rings of Krull type, recall that a multiplicative subset S of D is *t-splitting* if for each $0 \neq d \in D$, we have $dD = (AB)_t$ for some integral ideals A, B of D such that $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. Let X be an indeterminate over D, S be a multiplicative subset of D, $D_S[X]$ be the polynomial ring over D_S, and

$$D + XD_S[X] = \{ f \in D_S[X] \mid f(0) \in D \},$$

so $D + XD_S[X]$ is a ring such that $D[X] \subseteq D + XD_S[X] \subseteq D_S[X]$.

Proposition 2.4 *Let S be a multiplicative subset of D and $R = D + XD_S[X]$.*

(i) *R is a t-SFT PvMD if and only if D is a t-SFT PvMD and S is t-splitting.*
(ii) *R is a ring of Krull type if and only if D is a ring of Krull type, S is t-splitting, and the set of maximal t-ideals of D that intersect S is finite.*

Proof

(i) See [13, Corollary 2.3].
(ii) See [2, Theorem 2.5]. ∎

Clearly, a Krull domain is both a t-SFT PvMD and a ring of Krull type. Also, it is easy to see that every multiplicative subset of a Krull domain is a t-splitting set [1, p. 8]. Thus, by Proposition 2.4, we have the following corollary.

Corollary 2.5 *Let D be a Krull domain, S be a multiplicative subset of D and $R = D + XD_S[X]$.*

(i) *R is a t-SFT PvMD.*
(ii) ([2, Corollary 2.6]) *If $|\{P \in t\text{-}Max(D) \mid P \cap S \neq \varnothing\}| < \infty$, then R is a ring of Krull type.*

We recall the following useful lemma by which it follows that each t-ideal of a t-SFT PvMD has only finitely many minimal prime ideals [12, Lemma 3.8].

Lemma 2.6 ([7, Lemma 2.1]) *Let I be a proper integral t-ideal of D. If every prime ideal of D minimal over I is the radical of a t-ideal of finite type, there are only finitely many prime ideals of D minimal over I.*

Let D be a ring of Krull type. If I is a proper integral t-ideal of D, then I is contained in only finitely many maximal t-ideals, and since each maximal t-ideal contains at most one prime ideal of D minimal over I, the number of minimal prime ideals of I is finite.

Proposition 2.7 *D is a PvMD in which each integral t-ideal has only finitely many minimal prime ideals if and only if $D[\{X_\alpha\}]$ is. In this case, D_P is a DVR for all $P \in X^1(D)$ if and only if $D[\{X_\alpha\}]_Q$ is a DVR for all $Q \in X^1(D[\{X_\alpha\}])$.*

Proof This result follows directly from the following observations: (i) D is a PvMD if and only if $D[\{X_\alpha\}]$ is; and (ii) if Q is a prime t-ideal of $D[\{X_\alpha\}]$, then either ht$Q = 1$ with $Q \cap D = (0)$ or $Q = (Q \cap D)[\{X_\alpha\}]$ and $Q \cap D$ is a prime t-ideal (*cf.* [22, Theorem 3.1] and [14, Lemma 2.3]).

The "in this case" part follows from the following two observations: (i) if P is a prime ideal of D, then ht$P = 1$ if and only if $P[\{X_\alpha\}] \in X^1(D[\{X_\alpha\}])$, and since

$D[\{X_\alpha\}]_{P[\{X_\alpha\}]} \cap K = D_P$, we have that $D[\{X_\alpha\}]_{P[\{X_\alpha\}]}$ is a DVR if and only if D_P is a DVR; and (ii) if $Q \in X^1(D[\{X_\alpha\}])$ with $Q \cap D = (0)$, then $D[\{X_\alpha\}]_Q$ is a DVR. ∎

We end this section with three examples that show that SFT Prüfer domains $\not\Rightarrow$ rings of Krull type; rings of Krull type $\not\Rightarrow$ t-SFT PvMDs; and integral domains in which each integral t-ideal has only finitely many minimal prime ideals $\not\Rightarrow$ t-SFT PvMDs or rings of Krull type.

Example 2.8 (i) The ring $R = \mathbb{Z} + X\mathbb{Q}[X]$ is an SFT Prüfer domain (hence a t-SFT PvMD), while R is not a ring of Krull type because $X \in R$ is contained in infinitely many maximal t-ideals $p\mathbb{Z} + X\mathbb{Q}[X]$ for all prime elements $p \in \mathbb{Z}$.

(ii) If V is a rank-one nondiscrete valuation domain, then V is a ring of Krull type but not a t-SFT PvMD.

(iii) Let D be a generalized Krull domain that is not a Krull domain and $R = D + XK[X]$. If $|X^1(D)| = \infty$, then each integral t-ideal of R has only finitely many minimal prime ideals but R is neither a t-SFT PvMD nor a ring of Krull type.

3 Power Series Rings Over PvMDs

In this section, we prove that if D is a PvMD such that each proper integral t-ideal has only finitely many minimal prime ideals and D_P is a DVR for all $P \in X^1(D)$, then $D[[\{X_\alpha\}]]_{1_{D-\{0\}}}$ is a Krull domain. Hence, we note that D is a PvMD in which each integral t-ideal has only finitely many minimal prime ideals if D is a t-SFT PvMD, D is a ring of Krull type, D is a Prüfer domain of finite character, or D is a valuation domain. Also, throughout this section, we use the following notation.

Notation 3.1
- D is a PvMD in which each integral t-ideal has only finitely many minimal prime ideals, and D is not a field.
- K is the quotient field of D.
- t-Spec(D) is the set of prime t-ideals of D.
- Λ is a nonempty set of prime t-ideals of D with the property that if $\{P_\delta\} \subseteq \Lambda$ is a chain under inclusion, then $\bigcup P_\delta \in \Lambda$.
- $\mathcal{F}(\Lambda)$ is the family of finite sets λ of prime t-ideals in Λ such that no two elements of λ are comparable under inclusion.
- $X^1(D)$ is the set of height-one prime ideals of D.
- $R = \bigcap_{P \in X^1(D)} D_P$ (where $R = K$ when $X^1(D) = \varnothing$).

If Θ is a set of prime t-ideals of an integral domain A, then $\bigcap_{P \in \Theta} A_P$ is called a *subintersection* of A. It is known that a subintersection of a PvMD is a PvMD [26, Proposition 5.1]. Thus, $R = \bigcap_{P \in X^1(D)} D_P$ is a PvMD.

Proposition 3.2 (i) *R is a generalized Krull domain.*
(ii) *R is a Krull domain if and only if D_P is a DVR for all $P \in X^1(D)$.*

Proof If $X^1(D) = \varnothing$, then $R = K$, so we can assume that $X^1(D) \neq \varnothing$.

(i) If $P \in X^1(D)$, then P is a t-ideal and $R_{PD_P \cap R} = D_P$, and since D is a PvMD, D_P is a rank-one valuation domain. Moreover, by assumption, each nonzero nonunit

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of D is contained in only finitely many height-one prime ideals of D, and hence $R = \bigcap_{P \in X^1(D)} D_P$ is locally finite. Thus, R is a generalized Krull domain.

(ii) This follows from (i) because a generalized Krull domain A is a Krull domain if and only if A_P is a DVR for each $P \in X^1(A)$. ∎

Corollary 3.3 (i) *If D is a t-SFT PvMD, then R is a Krull domain.*
(ii) *If D is an SFT Prüfer domain, then R is a Dedekind domain.*

Proof (i) Note that D_P is a DVR for all $P \in X^1(D)$ [8, Lemma 8(1)]. Thus, by Proposition 3.2(ii), R is a Krull domain.

(ii) By (i), R is a Krull domain. Also, since D is a Prüfer domain, R is a Prüfer domain [19, Theorem 26.1]. Thus, R is a Dedekind domain (note that Dedekind domain ⇔ Krull domain + Prüfer domain). ∎

A set $\mathfrak{S}$ of ideals of D is called a *multiplicatively closed set of ideals* if $AB \in \mathfrak{S}$ for all $A, B \in \mathfrak{S}$, and if $\mathfrak{S}$ is a multiplicatively closed set of ideals of D, then

$$D_{\mathfrak{S}} = \{x \in K \mid xA \subseteq D \text{ for some } A \in \mathfrak{S}\},$$

called a *generalized transform of D*, is a t-linked overring of D [22, Lemma 3.10]. For more on the ring $D_{\mathfrak{S}}$, see [6].

Proposition 3.4 *For $\lambda = \{P_1, \dots, P_r\} \in \mathcal{F}(\Lambda)$, let $\mathfrak{S}_\lambda$ be the set of all t-invertible ideals A of D such that $(\prod_{i=1}^r P_i)_t \subsetneq A_t \subseteq D$, but $A \nsubseteq P_i$ for $i = 1, \dots, r$.*

(i) *$\mathfrak{S}_\lambda$ is a multiplicatively closed set of ideals of D.*
(ii) *Let $D_\lambda = D_{\mathfrak{S}_\lambda}$. Then $(0) \neq \prod_{i=1}^r P_i \subseteq (D : D_\lambda)$.*
(iii) *Let $\mathfrak{S} = \bigcup_{\lambda \in \mathcal{F}(\Lambda)} \mathfrak{S}_\lambda$. Then $\mathfrak{S}$ is a multiplicatively closed set of ideals of D, $D_{\mathfrak{S}} = \bigcup_{\lambda \in \mathcal{F}(\Lambda)} D_\lambda$, and $D_{\mathfrak{S}}$ is a PvMD.*

Proof (i) If $A \in \mathfrak{S}_\lambda$, then

$$P_i \supseteq \Big(\prod_{i=1}^r P_i\Big)_t = \Big(\Big(\Big(\prod_{i=1}^r P_i\Big) A^{-1}\Big) A\Big)_t \quad \text{and} \quad \Big(\prod_{i=1}^r P_i\Big) A^{-1} \subseteq D.$$

But, since $A \nsubseteq P_i$ for $i = 1, \dots, r$, we have $(\prod_{i=1}^r P_i) A^{-1} \subseteq \bigcap_{i=1}^r P_i$. Note that $(P_i + P_j)_t = D$ for $i \neq j$, since D is a PvMD, so $\bigcap_{i=1}^r P_i = (\prod_{i=1}^r P_i)_t$, and therefore $(\prod_{i=1}^r P_i)_t = ((\prod_{i=1}^r P_i) A^{-1})_t$. Hence, if $A_1, A_2 \in \mathfrak{S}_\lambda$, then $A_1 A_2$ is t-invertible, $A_1 A_2 \nsubseteq P_i$ for $i = 1, \dots, r$, and

$$(A_1 A_2)_t \supsetneq \Big(\Big(\prod_{i=1}^r P_{\alpha_i}\Big) A_1 A_2\Big)_t = \Big(\Big(\Big(\prod_{i=1}^r P_{\alpha_i}\Big) A_2^{-1} A_1^{-1}\Big) A_1 A_2\Big)_t = \Big(\prod_{i=1}^r P_{\alpha_i}\Big)_t.$$

Thus, $A_1 A_2 \in \mathfrak{S}_\lambda$.

(ii) This follows because $\prod_{i=1}^r P_i \subseteq A_t$ for all $A \in \mathfrak{S}_\lambda$.

(iii) If $A_1, A_2 \in \mathfrak{S}$, then $A_i \in \mathfrak{S}_{\lambda_i}$ for some $\lambda_i \in \mathcal{F}(\Lambda)$ for $i = 1, 2$. Let λ be the set of minimal elements (under inclusion) of $\lambda_1 \cup \lambda_2$. Clearly, $\lambda \in \mathcal{F}(\Lambda)$. Also, $\prod_{P \in \lambda} P \subseteq \prod_{Q \in \lambda_i} Q$ for $i = 1, 2$, and hence $(\prod_{P \in \lambda} P)_t \subsetneq (A_i)_t$ and $A_i \nsubseteq P$ for all $P \in \lambda$. (For if $A_i \subseteq P$ for some $P \in \lambda$, then $P \notin \lambda_i$. Note that $\prod_{Q \in \lambda_i} Q \subsetneq (A_i)_t \subseteq P$; hence, $Q \subsetneq P$ for some $Q \in \lambda_i$, and in this case, $P \notin \lambda$, a contradiction.) Thus, $A_1, A_2 \in \mathfrak{S}_\lambda$, and therefore $A_1 A_2 \in \mathfrak{S}_\lambda \subseteq \mathfrak{S}$. Clearly, $D_{\mathfrak{S}} = \bigcup_{\lambda \in \mathcal{F}(\Lambda)} D_\lambda$, and since D is a PvMD, $D_{\mathfrak{S}}$ is a PvMD [22, Theorem 3.11]. ∎

Let Θ be a set of prime t-ideals of D. Clearly,

$$\bigcap_{P\in\Theta} D_P = \begin{cases} D & \text{if } \Theta = t\text{-Max}(D), \\ K & \text{if } \Theta = \varnothing. \end{cases}$$

Hence, if each prime t-ideal of D is a maximal t-ideal (*e.g.*, D is a Krull domain), then $t\text{-Max}(D) = X^1(D)$, and hence $R = D$.

Corollary 3.5 *Let the notation be as in Proposition 3.4, $\lambda = \{P_1, \dots, P_r\} \in \mathcal{F}(\Lambda)$, Ω be the set of nonzero prime ideals P of D such that P is a minimal element of Λ under inclusion or $P = \bigcap_\delta P_\delta$ for some chain $\{P_\delta\} \subseteq \Lambda$ with the property that $P' \in \Lambda$ with $P' \subseteq P_\delta$ for some P_δ implies $P' \in \{P_\delta\}$, and $\Delta = \{M \in t\text{-Max}(D) \mid P \nsubseteq M$ for all $P \in \Lambda\}$.*

(i) $D_\lambda = (\bigcap_{i=1}^r D_{P_i}) \cap (\bigcap\{D_M \mid M \in t\text{-}Max(D) \text{ and } \prod_{i=1}^r P_i \nsubseteq M\})$.
(ii) $D_{\mathfrak{S}} = (\bigcap_{P\in\Omega} D_P) \cap (\bigcap_{M\in\Delta} D_M)$.
(iii) *If* $\Lambda = t\text{-Spec}(D)$, *then* $R = \bigcup_{\lambda\in\mathcal{F}(\Lambda)} D_\lambda$.
(iv) *R is the complete integral closure of D.*

Proof (i) For convenience, let $\Delta_\lambda = \{M \in t\text{-Max}(D) \mid \prod_{i=1}^r P_i \nsubseteq M\}$ and $T_\lambda = (\bigcap_{i=1}^r D_{P_i}) \cap (\bigcap_{M\in\Delta_\lambda} D_M)$. ($\subseteq$): If $x \in D_\lambda$, then $xA \subseteq D$ for some $A \in \mathfrak{S}_\lambda$. Note that $\prod_{i=1}^r P_i \subseteq A_t$ and $A \nsubseteq P_i$ for $i = 1, \dots, r$, so $x \in (\bigcap_{i=1}^r xD_{P_i}) \cap (\bigcap_{M\in\Delta_\lambda} xD_M) = (\bigcap_{i=1}^r xAD_{P_i}) \cap (\bigcap_{M\in\Delta_\lambda} xAD_M) \subseteq T_\lambda$.

($\supseteq$): Let $0 \neq y \in T_\lambda$, and let $A_y = \{d \in D \mid dy \in D\}$. Clearly, $A_y \nsubseteq P_i$ for $i = 1, 2, \dots, r$. Note also that $A_y = (1, y)^{-1}$, so A_y is a t-invertible t-ideal of D. Let $I = \prod_{i=1}^r P_i$, and assume $M \in t\text{-Max}(D)$. If $A_y \nsubseteq M$, then $ID_M \subseteq D_M = A_yD_M$. Next, assume $A_y \subseteq M$. If $I \nsubseteq M$, *i.e.*, $P_i \nsubseteq M$ for $i = 1, \dots, r$, then, by assumption, $y \in D_M$, and so $A_y \nsubseteq M$, a contradiction. Hence, $P_j \subseteq M$ for some j, and since $A_y \nsubseteq P_j$ and D_M is a valuation domain, $ID_M = P_jD_M \subsetneq A_yD_M \subseteq D_M$. Thus, $I \subseteq \bigcap_{M\in t\text{-Max}(D)} ID_M \subseteq \bigcap_{M\in t\text{-Max}(D)} A_yD_M = (A_y)_t = A_y$ (*cf.* [22, Theorem 3.5] for the first equality). Clearly, $(\prod_{i=1}^r P_i)_t = I_t \neq A_y$, and hence $A_y \in \mathfrak{S}_\lambda$. Thus, $y \in D_\lambda$.

(ii) Let $T = (\bigcap_{P\in\Omega} D_P) \cap (\bigcap_{M\in\Delta} D_M)$. ($\subseteq$): If $x \in D_{\mathfrak{S}}$, then $x \in D_\lambda$ for some $\lambda = \{P_1, \dots, P_r\} \in \mathcal{F}(\Lambda)$. Hence, there exists an $A \in \mathfrak{S}_\lambda$ such that $xA \subseteq D$. Note that $\prod_{i=1}^r P_i \subseteq A_t$, so $A \nsubseteq P$ for all $P \in \Omega \cup \Delta$. Thus, $x \in (\bigcap_{P\in\Omega} xD_P) \cap (\bigcap_{M\in\Delta} xD_M) = (\bigcap_{P\in\Omega} xAD_P) \cap (\bigcap_{M\in\Delta} xAD_M) \subseteq T$.

($\supseteq$): For the reverse containment, let $0 \neq y \in T$ and $A_y = (1, y)^{-1}$. Then A_y is a t-invertible t-ideal of D. If $A_y = D$, then $y \in D \subseteq D_{\mathfrak{S}}$, so assume $A_y \subsetneq D$. Then there are only finitely many prime ideals of D minimal over A_y, say $Q_1, \dots, Q_n$. Let $\Theta = \{P \in \Lambda \mid P \subsetneq Q_i$ for some $i\}$, whence $A_y \nsubseteq P$ for all $P \in \Theta$. If M is a maximal t-ideal of D with $Q_i \subseteq M$ for some i, then $A_y \subseteq M$, and hence $M \notin \Delta$. Thus, M contains at least one prime ideal in Λ, and since D_M is a valuation domain, $P \subsetneq Q_i$ for some $P \in \Lambda$ by the choice of Ω and y. Hence, $\Theta \neq \varnothing$. Also, if $\{P_\delta\}$ is a chain of prime ideals in Θ, then $P := \bigcup P_\delta \in \Lambda$ by the property of Λ, and since $A_y \nsubseteq P_\delta$ for all δ and A_y is of finite type, $A_y \nsubseteq P$. Thus, each element of Θ is contained in at least one maximal element under inclusion, and Θ contains a finite number of maximal elements. Let μ be the set of maximal elements of Θ, and let $I = \prod_{P\in\mu} P$. Clearly,

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$\mu \in \mathcal{F}(\Lambda)$, and it is easy to see that $I_t \subsetneq A_y$ and $A_y \nsubseteq P$ for all $P \in \mu$ (*cf.* the proof of (i) above). Thus, $y \in D_\mu \subseteq D_{\mathfrak{S}}$.

(iii) It is obvious that t-Spec(D) satisfies the given property of Λ. Hence, if $\Lambda = t$-Spec(D), then $\Omega = X^1(D)$ and $\Delta = \varnothing$, and thus by (ii) and Proposition 3.4(iii), $R = \bigcup_{\lambda \in \mathcal{F}(\Lambda)} D_\lambda$.

(iv) Let D^* be the complete integral closure of D. Clearly, $D^* \subseteq R$, because $D \subseteq R$ and R is completely integrally closed. For the reverse containment, let $\alpha \in R$ and $\Lambda = t$-Spec(D). Then $\alpha \in D_\lambda$ for some $\lambda \in \mathcal{F}(\Lambda)$, and since D_λ is a ring, $\alpha^n \in D_\lambda$ for all integers $n \geq 1$. Note that $\prod_{P \in \lambda} P \subseteq (D : D_\lambda)$ by Proposition 3.4(ii), so if $0 \neq d \in \prod_{P \in \lambda} P$, then $d\alpha^n \in D$ for all $n \geq 1$. Thus, $\alpha \in D^*$. ∎

Remark 3.6 If D is a ring of Krull type, then each integral t-ideal of D has only a finite number of minimal prime ideals. Thus, by Corollary 3.5(iv), $R = \bigcap_{P \in X^1(D)} D_P$ is the complete integral closure of D. Also, if $X^1(D) \neq \varnothing$, then R is a generalized Krull domain by Proposition 3.2(i). This recovers Mott's results [25, Theorems 1 and 3].

It is known that the complete integral closure of an SFT Prüfer domain is a Dedekind domain [17, Corollary 3.2], and a completely integrally closed t-SFT PvMD is a Krull domain ([12, Theorem 3.11] or [24, Theorem 2.9]).

Corollary 3.7 *The complete integral closure of a t-SFT PvMD is a Krull domain.*

Proof By Corollary 3.5(iv), R is the complete integral closure of D. Thus, by Corollary 3.3, the complete integral closure of a t-SFT PvMD is a Krull domain. ∎

For brevity of notations, let $A[[X_1, \dots, X_n]] = A[[X_n]]$ for an integral domain A and an integer $n \geq 0$, $A[[X_0]] = A$, $\xi(X_1, \dots, X_n) = \xi(X_n)$ for any $\xi(X_1, \dots, X_n) \in A[[X_n]]$, and K_n be the quotient field of $D[[X_n]]$.

Lemma 3.8 *Let $\Lambda = t$-Spec(D). If $n \geq 0$ is an integer, $\{\xi_i(X_n)\}_{i=1}^\infty$ is a subset of $R[[X_n]]$, $\{m_i\}_{i=1}^\infty$ is a set of positive integers, and $0 \neq d(X_n) \in D[[X_n]]$ is such that $d(X_n)^{m_i}\xi_i(X_n) \in D[[X_n]]$ for all $i \geq 1$, then $\{\xi_i(X_n)\}_{i=1}^\infty \subseteq D_\lambda[[X_n]]$ for some $\lambda \in \mathcal{F}(\Lambda)$.*

Proof Let $\{\xi_i\}_{i=1}^\infty$ be a subset of R, and assume that there exist $0 \neq d \in D$ and positive integers $\{m_i\}_{i=1}^\infty$ such that $d^{m_i}\xi_i \in D$ for all $i \geq 1$. If $dD = D$, then $\xi_i \in D$, so we assume $dD \subsetneq D$. Hence, there are only finitely many minimal prime ideals of dD, say $Q_1, \dots, Q_m$. If ht$Q_j = 1$, let $P_j = Q_j$, and if ht$Q_j \geq 2$, then choose a prime ideal P_j such that $(0) \subsetneq P_j \subsetneq Q_j$. Let $\lambda = \{P_{\alpha_1}, \dots, P_{\alpha_r}\}$ be the set of distinct P_i's (it is possible that $P_i = P_j$ for $i \neq j$, so $r \leq m$), and let $A_{\xi_i} = \{a \in D \mid a\xi_i \in D\}$. Since $A_{\xi_i} = (1, \xi_i)^{-1}$, A_{ξ_i} is a t-invertible t-ideal of D. Since $\xi_i \in R$, we have $A_{\xi_i} \nsubseteq Q$ for all $Q \in X^1(D)$. Next, note that $d^{m_i} \in A_{\xi_i}$; so if ht$Q_j \geq 2$, then $P_j \subsetneq Q_j$, and hence $A_{\xi_i} \nsubseteq P_j$. Thus, $A_{\xi_i} \nsubseteq P_{\alpha_j}$ for $j = 1, \dots, r$. Let $p \in \prod_{j=1}^r P_{\alpha_j}$, and $M \in t$-Max(D). If $d \notin M$, then $p\xi_i \in D_M$. If $d \in M$, then $P_{\alpha_j} \subseteq M$ for some j, whence if ht$P_{\alpha_j} = 1$, then $p\xi_i \in pR \subseteq P_{\alpha_j}D_{P_{\alpha_j}} = P_{\alpha_j}D_M \subsetneq D_M$. If ht$P_{\alpha_j} \geq 2$, then $d \notin P_{\alpha_j}$, and so $p\xi_i \in$

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$p(d^{m_i}\xi_i)D_{P_{\alpha_j}} \subseteq pD_{P_{\alpha_j}} \subseteq P_{\alpha_j}D_{P_{\alpha_j}} \subseteq D_M$. Hence, $p\xi_i \in \bigcap_{M \in t\text{-Max}(D)} D_M = D$. Thus, $(\prod_{j=1}^r P_{\alpha_j})_t \subsetneq (A_{\xi_i})_t = A_{\xi_i}$, and so $A_{\xi_i} \in \mathfrak{S}_\lambda$. Therefore, $\xi_i \in D_\lambda$ for all $i \geq 0$.

Assume that if $k = n-1$ is a nonnegative integer, $\{\xi_i(X_k)\}_{i=1}^\infty$ is a subset of $R[[X_k]]$, $\{k_i\}_{i=1}^\infty$ is a set of positive integers, and $0 \neq d(X_k) \in D[[X_k]]$ is such that $d(X_k)^{k_i}\xi_i(X_k) \in D[[X_k]]$ for all $i \geq 1$, then $\{\xi_i(X_k)\}_{i=1}^\infty \subseteq D_\nu[[X_k]]$ for some $\nu \in \mathcal{F}(\Lambda)$. Let $\{\xi_i(X_n)\}_{i=1}^\infty$ be a subset of $R[[X_n]]$, $\{n_i\}_{i=1}^\infty$ be a set of positive integers, and $0 \neq d(X_n) \in D[[X_n]]$ be such that $d(X_n)^{n_i}\xi_i(X_n) \in D[[X_n]]$ for all $i \geq 1$. We can write

$$d(X_n) = \sum_{j=0}^\infty d_j(X_{n-1})X_n^j \quad \text{and} \quad \xi_i(X_n) = \sum_{j=0}^\infty \xi_{ij}(X_{n-1})X_n^j,$$

where $d_j(X_{n-1}) \in D[[X_{n-1}]]$ and $\xi_{ij}(X_{n-1}) \in R[[X_{n-1}]]$, and we can assume that $d_0(X_{n-1}) \neq 0$. Hence, $\{\xi_{ij}(X_{n-1})\}$ is a subset of $D[[X_{n-1}]]$ such that $d_0(X_{n-1})^{n_i(j+1)}\xi_{ij}(X_{n-1}) \in D[[X_{n-1}]]$ for all $j \geq 0$ (*cf.* the proof of [27, Proposition 2.5]), and thus $\{\xi_{ij}(X_{n-1})\}_{j=0}^\infty \subseteq D_\mu[[X_{n-1}]]$ for some $\mu \in \mathcal{F}(\Lambda)$ by assumption. Therefore, $\xi_i(X_n) \in D_\mu[[X_n]]$ for $i \geq 1$. ■

Lemma 3.9 *If $\Lambda = t\text{-Spec}(D)$, then $R[[X_n]] \cap K_n = \bigcup_{\lambda \in \mathcal{F}(\Lambda)} D_\lambda[[X_n]]$.*

Proof ($\supseteq$): Note that if $\lambda \in \mathcal{F}(\Lambda)$, then $(D : D_\lambda) \neq (0)$ by Proposition 3.4(ii), so $D_\lambda[[X_n]] \subseteq D[[X_n]]_{D-\{0\}} \subseteq K_n$. Hence, the result follows, because $R = \bigcup_{\lambda \in \mathcal{F}(\Lambda)} D_\lambda$ by Corollary 3.5(iii).

($\subseteq$): Let $\xi(X_n) = \frac{f(X_n)}{g(X_n)} \in R[[X_n]] \cap K_n$, where $0 \neq f(X_n), g(X_n) \in D[[X_n]]$, and write $\xi(X_n) = \sum_{i=0}^\infty \xi_i(X_{n-1})X_n^i$ and $g(X_n) = \sum_{i=0}^\infty d_i(X_{n-1})X_n^i$. We may assume that $d_0(X_{n-1}) \neq 0$; then

$$\xi(X_n)g(X_n) = \sum_{k=0}^\infty \Big(\sum_{i+j=k} \xi_i(X_{n-1})d_j(X_{n-1}) \Big) X_n^k \in D[[X_n]].$$

Hence, $d_0(X_{n-1})^{i+1}\xi_i(X_{n-1}) \in D[[X_{n-1}]]$ for all $i \geq 0$, and thus

$$\{\xi_i(X_{n-1})\} \subseteq D_\lambda[[X_{n-1}]]$$

for some $\lambda \in \mathcal{F}(\Lambda)$ by Lemma 3.8. Thus, $\xi(X_n) \in D_\lambda[[X_n]]$. ■

Theorem 3.10 *If $R = \bigcap_{P \in X^1(D)} D_P$ is a Krull domain, then $D[[\{X_\alpha\}]]_{1_{D-\{0\}}}$ is a Krull domain.*

Proof Since R is a Krull domain, $R[[\{X_\alpha\}]]_1$ is a Krull domain [18, Theorem 2.1], and hence $R[[\{X_\alpha\}]]_{1_{D-\{0\}}}$ is a Krull domain [19, Corollary 43.6]. Clearly, if we let $\text{qf}(D[[\{X_\alpha\}]]_1)$ be the quotient field of $D[[\{X_\alpha\}]]_1$, then $\text{qf}(D[[\{X_\alpha\}]]_1)$ is a Krull domain. Hence, by [19, Corollary 44.10], it suffices to show that

$$R[[\{X_\alpha\}]]_{1_{D-\{0\}}} \cap \text{qf}(D[[\{X_\alpha\}]]_1) = D[[\{X_\alpha\}]]_{1_{D-\{0\}}}.$$

The containment "$\supseteq$" is clear. For the reverse containment, note that if

$$u \in R[[\{X_\alpha\}]]_{1_{D-\{0\}}} \cap \text{qf}(D[[\{X_\alpha\}]]_1),$$

then $du \in R[[X_n]] \cap K_n$ for some $X_1, \dots, X_n \in \{X_\alpha\}$ and $0 \neq d \in D$. However, since $R[[X_n]] \cap K_n = D[[X_n]]$ by Lemma 3.9, we have $u \in D[[X_n]]_{D-\{0\}} \subseteq D[[\{X_\alpha\}]]_{1\,D-\{0\}}$. ∎

Corollary 3.11 *Let $\{X_\beta\}$ and $\{X_\alpha\}$ be two disjoint nonempty sets of indeterminates over D. If $R = \bigcap_{P \in X^1(D)} D_P$ is a Krull domain, then $D[\{X_\beta\}][[\{X_\alpha\}]]_{1\,D-\{0\}}$ is a Krull domain.*

Proof Let $D_0 = D[\{X_\beta\}]$. Note that D_P is a DVR for all $P \in X^1(D)$ by Proposition 3.2(ii), so D_0 is a PvMD in which each integral t-ideal has only finitely many minimal prime ideals and D_{0Q} is a rank-one DVR for all $Q \in X^1(D_0)$ by Proposition 2.7. Hence, again by Proposition 3.2, $\bigcap_{Q \in X^1(D_0)} D_{0Q}$ is a Krull domain, and thus $D_0[[\{X_\alpha\}]]_{1_{D_0-\{0\}}}$ is a Krull domain by Theorem 3.10.

Claim. $D_0[[\{X_\alpha\}]]_{1_{D_0-\{0\}}} \cap K[\{X_\beta\}][[\{X_\alpha\}]]_1 = D_0[[\{X_\alpha\}]]_{1\,D-\{0\}}$.

Proof of Claim Let $h = \frac{f}{g} \in D_0[[\{X_\alpha\}]]_{1_{D_0-\{0\}}} \cap K[\{X_\beta\}][[\{X_\alpha\}]]_1$, where $0 \neq f \in D_0[[\{X_\alpha\}]]_1$ and $0 \neq g \in D_0$. Let $h_i \in K[\{X_\beta\}]$ (resp., $f_i \in D_0$) be the coefficients of h (resp., f) such that $gh_i = f_i \in D_0 = D[\{X_\beta\}]$. Since D is a PvMD, $D \supseteq c(f_i)_v = c(gh_i)_v = (c(g)c(h_i))_v \supseteq a \cdot c(h_i) = c(ah_i)$ for all $0 \neq a \in c(g)$. Hence, $ah_i \in D[\{X_\beta\}]$ for all i, and thus $ah \in D_0[[\{X_\alpha\}]]_1$. Therefore, $h = \frac{ah}{a} \in D_0[[\{X_\alpha\}]]_{1\,D-\{0\}}$. The reverse containment is clear. ∎

Note that $K[\{X_\beta\}]$ is a Krull domain, so $K[\{X_\beta\}][[\{X_\alpha\}]]_1$ is a Krull domain [18, Theorem 2.1]. Thus, $D[\{X_\beta\}][[\{X_\alpha\}]]_{1_{D-\{0\}}}$ is a Krull domain by the claim and [19, Corollary 44.10]. ∎

If D is a t-SFT PvMD, then each proper integral t-ideal has only finitely many minimal prime ideals and $R = \bigcap_{P \in X^1(D)} D_P$ is a Krull domain. Thus, by Theorem 3.10 and Corollary 3.11, we have the following corollary.

Corollary 3.12 *Let $\{X_\beta\}$ and $\{X_\alpha\}$ be two disjoint nonempty sets of indeterminates over D. If D is a t-SFT PvMD, then $D[[\{X_\alpha\}]]_{1\,D-\{0\}}$ and $D[\{X_\beta\}][[\{X_\alpha\}]]_{1_{D-\{0\}}}$ are both Krull domains.*

Let D be a valuation domain, and assume that $|\{X_\beta\}| < \infty$. It is known that if $X^1(D) = \emptyset$, then $D[[\{X_\alpha\}]]_{1\,D-\{0\}}$ is a UFD [3, Proposition 2.1 and Corollary 3.4]. Also, if D has a height-one prime ideal P that is not idempotent, *i.e.*, $P \neq P^2$, then D_P is a rank-one DVR, and hence $D_P[\{X_\beta\}][[\{X_\alpha\}]]_1$ is a UFD (*cf.* [30, Theorem 2.1]). Note that

$$D[\{X_\beta\}][[\{X_\alpha\}]]_{1_{D-\{0\}}} = D_P[\{X_\beta\}][[\{X_\alpha\}]]_{1_{D_P-\{0\}}}.$$

Thus, $D[\{X_\beta\}][[\{X_\alpha\}]]_{1_{D-\{0\}}}$ is a UFD.

Corollary 3.13 *Let $\{X_\beta\}$ and $\{X_\alpha\}$ be two disjoint nonempty sets of indeterminates over a valuation domain V. If either $X^1(V) = \emptyset$ or V has a height-one prime ideal P with $P^2 \neq P$, then $V[\{X_\beta\}][[\{X_\alpha\}]]_{1_{V-\{0\}}}$ is a Krull domain.*

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Proof Let $R = \bigcap_{P \in X^1(V)} V_P$. Then either R is a field or $R = V_P$ is a rank-one DVR (so a Krull domain) by assumption. Thus, by Corollary 3.11, $V[\{X_\beta\}][[\{X_\alpha\}]]_{1_{V-\{0\}}}$ is a Krull domain. ■

We end this paper by a t-SFT PvMD analog of Arnold's result [5, Proposition 3.2] that if D is a finite dimensional Prüfer domain with the SFT-property and M is a height-one maximal ideal of D, then $\operatorname{ht}(M[[X_n]]) = 1$ for all integers $n \geq 1$. We first need two lemmas.

Lemma 3.14 (*cf.* [5, Lemma 3.1]) *Let D be a t-SFT PvMD and let A be a nonzero ideal of D with the property that each prime ideal of D minimal over A is a maximal t-ideal. Then A is t-invertible, and hence each maximal t-ideal is t-invertible.*

Proof Since D is a t-SFT-ring, A_t has only finitely many minimal prime ideals of D, say $M_1, \dots, M_k$, which are maximal t-ideals by assumption. Note that $M_i D_{M_i}$ is principal, so $AD_{M_i} = a_i D_{M_i}$ for some $a_i \in A$. Also, there exists a finitely generated ideal $J \subseteq A$ of D such that $\sqrt{A_t} = \sqrt{J_v}$. So if we let $B = J + (a_1, \dots, a_n)$, then $B \subseteq A$ is finitely generated, AD_M is principal, and $AD_M = BD_M$ for all maximal t-ideals M of D. Thus, $A_t = B_t$ and B_t is t-invertible [22, Theorem 3.5 and Corollary 2.7]. Thus, A is t-invertible. ■

Lemma 3.15 (*cf.* [5, Proposition 2.1(v)]) *Let D be a t-SFT PvMD and let Q be a prime t-ideal of $D[[\{X_\alpha\}]]_1$. If $Q \cap D = P$, then $P[[\{X_\alpha\}]]_1 \subseteq Q$.*

Proof If $P = (0)$, then $P[[\{X_\alpha\}]]_1 = (0) \subseteq Q$, and so assume $P \neq (0)$. Note that if $I \subseteq P$ is a nonzero finitely generated ideal of D, then $Q \supseteq (ID[[\{X_\alpha\}]]_1)_v = I_v[[\{X_\alpha\}]]_1$ [9, Lemma 3.1], and thus $P = Q \cap D \supseteq I_v[[\{X_\alpha\}]]_1 \cap D = I_v$. Thus, $P_t = P$, so there are a nonzero finitely generated ideal B and an integer $k \geq 1$ such that $a^k \in B_v$ for all $a \in P$. If $\overline{P} = P/B_v$, then each element of $\overline{P}[[\{X_\alpha\}]]_1$ is nilpotent (*cf.* the proof of [5, Proposition 2.1(v)]). Thus, $P[[\{X_\alpha\}]]_1 = \sqrt{B_v[[\{X_\alpha\}]]_1}$, and since $B_v[[\{X_\alpha\}]]_1 = (BD[[\{X_\alpha\}]]_1)_v \subseteq Q_t = Q$, we have $P[[\{X_\alpha\}]]_1 = \sqrt{B_v[[\{X_\alpha\}]]_1} \subseteq Q$. ■

Proposition 3.16 (*cf.* [5, Proposition 3.2]) *Let D be a t-SFT PvMD and let M be a maximal t-ideal of D. If $\operatorname{ht} M = 1$, then $\operatorname{ht}(M[[\{X_\alpha\}]]_1) = 1$.*

Proof By Lemma 3.14, $M = (a_1, \dots, a_k)_v$ and $MD_M = mD_M$ for some $a_1, \dots, a_k$, $m \in M$. Hence, there is an $s \in D - M$ such that $sM = s(a_1, \dots, a_k)_v = (sa_1, \dots, sa_k)_v \subseteq mD$, whence $s^r(M^r)_t \subseteq m^r D$ for all integers $r \geq 1$.

Assume that $\operatorname{ht}(M[[\{X_\alpha\}]]_1) > 1$, and let Q be a prime t-ideal of $D[[\{X_\alpha\}]]_1$ such that $(0) \subsetneq Q \subsetneq M[[\{X_\alpha\}]]_1$. Clearly, there are some $X_1, \dots, X_n \in \{X_\alpha\}$ so that $Q \cap D[[X_n]] \neq (0)$ and $Q \cap D[[X_n]]$ is a prime t-ideal. Replacing Q with $Q \cap D[[X_n]]$, we assume that $(0) \subsetneq Q \subsetneq M[[X_n]]$. If $Q \cap D \neq (0)$, then $Q \cap D \subseteq M$, and since $\operatorname{ht} M = 1$, $Q \cap D = M$. Thus, $M[[X_n]] = (MD[[X_n]])_v = (MD[[X_n]])_t \subseteq Q$, a contradiction. Hence, $Q \cap D = (0)$. Choose $0 \neq q \in Q$. Note that $(M^i)_t$ is M-primary for all integers $i \geq 1$; hence, $q \in \bigcap_{i=1}^{\infty} (M^i)_t[[X_n]]$ by an argument similar to the proof

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of [5, Proposition 3.2]. Thus, $\bigcap_{i=1}^{\infty} M^i D_M = \bigcap_{i=1}^{\infty}(M^i)_t D_M \supseteq \bigcap_{i=1}^{\infty}(M^i)_t \neq (0)$, and therefore $\text{ht}M = \text{ht}MD_M > 1$, a contradiction. ■

Corollary 3.17 *Let D be a t-SFT PvMD and $\{X_\beta\} \cup \{X_\alpha\}$ be the union of two disjoint nonempty sets of indeterminates over D. If M is a height-one maximal t-ideal of D, then $ht(M[\{X_\beta\}][[\{X_\alpha\}]]_1) = 1$.*

Proof This follows directly from Proposition 3.16, because $D[\{X_\beta\}]$ is a t-SFT PvMD by [8, Theorem 11], $M[\{X_\beta\}]$ is a maximal t-ideal by [14, Lemma 2.1], and $\text{ht}(M[\{X_\beta\}]) = 1$ (note that D_M is a one-dimensional valuation domain). ■

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