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A note on cyclic vectors in Dirichlet-type spaces in the unit ball of $\mathbb{C}^n$

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Abstract. We characterize model polynomials that are cyclic in Dirichlet-type spaces in the unit ball of $\mathbb{C}^n$, and we give a sufficient capacity condition in order to identify noncyclic vectors.

1 Introduction

Studying Dirichlet-type spaces in the unit ball of $\mathbb{C}^n$, we can draw conclusions for classical Hilbert spaces of holomorphic functions such as the Hardy, Bergman, and Dirichlet spaces. General introduction to this theory can be found in [18, 22].

The purpose of this note is to characterize model polynomials and to study special families of functions that are cyclic for the shift operators on these spaces. Moreover, we give a sufficient capacity condition in order to identify noncyclic functions. Norm comparisons, sharp decay of norms for special subspaces, capacity conditions studied in [3, 4, 6, 21] are the main motivation for this work. The cyclicity of a function f in a space of holomorphic functions is connected also with the problem of approximating $1/f$ (see [19, 20] for the study of this subject).

Full characterization of polynomials in more than two variables looks like a hard problem either in the unit ball or the polydisk. The cyclicity problem of polynomials for the bidisk was solved in [5] and shortly after extended in [13]. The corresponding problem in the setting of the unit ball of $\mathbb{C}^2$ was solved in [14].

1.1 Dirichlet-type spaces in the unit ball

Denote the unit ball by

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : ||z|| < 1\},$$

and its boundary, the unit sphere by

$$\mathbb{S}_n = \{z \in \mathbb{C}^n : ||z|| = 1\},$$

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where $||z|| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ is the associated norm of the usual *Euclidean inner product* $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$. Denote the *class of holomorphic functions* in $\mathbb{B}_n$ by $\mathrm{Hol}(\mathbb{B}_n)$. Any function $f \in \mathrm{Hol}(\mathbb{B}_n)$ has a power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1,\ldots,k_n} z_1^{k_1} \cdots z_n^{k_n}, \quad z \in \mathbb{B}_n, \tag{1}$$

where $k = (k_1, \ldots, k_n)$ is an n-tuple index of nonnegative integers, $k! = k_1! \cdots k_n!$ and $z^k = z_1^{k_1} \cdots z_n^{k_n}$. The power series in (1) exist, converges normal in $\mathbb{B}_n$ and it is unique since the unit ball is a connected Reinhardt domain containing the origin, i.e., $(z_1, \ldots, z_n) \in \mathbb{B}_n$ implies $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in \mathbb{B}_n$ for arbitrary real $\theta_1, \ldots, \theta_n$ (see [12]).

To simplify the notation, we may write (1) as follows:

$$f(z) = \sum_{m=0}^{\infty} \sum_{|k|=m}^{\infty} a_k z^k = \sum_{|k|=0}^{\infty} a_k z^k, \quad z \in \mathbb{B}_n, \tag{2}$$

where $|k| = k_1 + \cdots + k_n$.

Let $f \in \mathrm{Hol}(\mathbb{B}_n)$. We say that f belongs to the *Dirichlet-type space* $D_\alpha(\mathbb{B}_n)$, where $\alpha \in \mathbb{R}$ is a fixed parameter, if

$$||f||_\alpha^2 := \sum_{|k|=0}^{\infty} (n + |k|)^\alpha \frac{(n-1)!k!}{(n-1+|k|)!} |a_k|^2 < \infty. \tag{3}$$

General introduction to the theory of Dirichlet-type spaces in the unit ball of $\mathbb{C}^n$ can be found in [1, 2, 15, 16, 20–22]. One variable Dirichlet-type spaces are discussed in the textbook [11]. The weights in the norm in (3) are chosen in such a way that $D_0(\mathbb{B}_n)$ and $D_{-1}(\mathbb{B}_n)$ coincide with the *Hardy* and *Bergman spaces* of the ball, respectively. The *Dirichlet space* having Möbius invariant norm corresponds to the parameter choice $\alpha = n$.

By the definition, $D_\alpha(\mathbb{B}_n) \subset D_\beta(\mathbb{B}_n)$, when $\alpha \geq \beta$. Polynomials are dense in the spaces $D_\alpha(\mathbb{B}_n)$, $\alpha \in \mathbb{R}$, and $z_i \cdot f \in D_\alpha(\mathbb{B}_n)$, $i = 1, \ldots, n$ whenever $f \in D_\alpha(\mathbb{B}_n)$.

A *multiplier* in $D_\alpha(\mathbb{B}_n)$ is a holomorphic function $\phi : \mathbb{B}_n \to \mathbb{C}$ that satisfies $\phi \cdot f \in D_\alpha(\mathbb{B}_n)$ for all $f \in D_\alpha(\mathbb{B}_n)$. Polynomials, as well as holomorphic functions in a neighbourhood of the closed unit ball, are multipliers in every space $D_\alpha(\mathbb{B}_n)$.

1.2 Shift operators and cyclic vectors

Consider the bounded linear operators $S_1, \ldots, S_n : D_\alpha(\mathbb{B}_n) \to D_\alpha(\mathbb{B}_n)$ defined by $S_i : f \mapsto z_i \cdot f$. We say that $f \in D_\alpha(\mathbb{B}_n)$ is a *cyclic vector* if the closed invariant subspace, i.e.,

$$[f] := \mathrm{clos\,span}\{z_1^{k_1} \cdots z_n^{k_n} f : k_1, \ldots, k_n = 0, 1, \ldots\}$$

coincides with $D_\alpha(\mathbb{B}_n)$ (the closure is taken with respect to the $D_\alpha(\mathbb{B}_n)$ norm). An equivalent definition is that f is cyclic if and only if $1 \in [f]$.

Since $D_\alpha(\mathbb{B}_n)$ enjoys the *bounded point evaluation property* a function that is cyclic cannot vanish inside the unit ball. Thus, we focus on functions nonvanishing in the domain. Also, nonzero constant functions are cyclic in every space $D_\alpha(\mathbb{B}_n)$.

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More information regarding cyclic vectors in Dirichlet-type spaces over the disk, the polydisk and the unit ball can be found in [3–6, 8, 11, 13, 14, 19, 21].

Just as in the settings of the bidisk and the unit ball of two variables, the cyclicity of a function $f \in D_\alpha(\mathbb{B}_n)$ is inextricably linked with its *zero set*

$$\mathcal{Z}(f) = \{z \in \mathbb{C}^n : f(z) = 0\}.$$

The zeros of a function lying on the sphere are called the *boundary zeros*.

1.3 Plan of the paper

Section 3 studies Dirichlet-type spaces. In particular, we give a crucial relation among them. Using fractional radial derivatives and the Cauchy formula of functions lying in the *ball algebra* $A(\mathbb{B}_n)$ which contains functions that are continuous on the closed unit ball and holomorphic in its interior, we give an equivalent characterization of Dirichlet-type spaces for a wide range of parameters α.

Section 4 studies diagonal subspaces. In particular, we extend result from [21]. It makes sense to define functions $f \in \mathrm{Hol}(\mathbb{B}_n)$ using functions $\tilde{f} \in \mathrm{Hol}(\mathbb{D}(\mu))$ for a proper $\mu > 0$. Geometrically speaking, we are looking at a disk embedded in the ball but not in a coordinate plane. Thus, we may switch the problem of cyclicity from the ball to spaces of holomorphic functions of one variable that are well known. Then we use optimal approximants in order to identify cyclicity.

Moreover, we prove cyclicity for model polynomials for proper parameters. In the setting of the unit ball of two variables (see [21]), the model polynomials are the following: $1 - z_1$ which vanishes in the closed unit ball on a singleton, i.e., $\mathcal{Z}(1 - z_1) \cap \mathbb{S}_2 = \{(1, 0)\}$, and $1 - 2z_1z_2$ which vanishes along an analytic curve, i.e., $\mathcal{Z}(1 - 2z_1z_2) \cap \mathbb{S}_2 = \{(e^{i\theta}/\sqrt{2}, e^{-i\theta}/\sqrt{2}) : \theta \in \mathbb{R}\}$. In our case, the corresponding candidates are the following:

$$p(z) = 1 - m^{m/2} z_1 \cdots z_m, \quad 1 \le m \le n.$$

They vanish in the closed unit ball along the following analytic sets:

$$\mathcal{Z}(p) \cap \mathbb{S}_n = \{1/\sqrt{m}(e^{i\theta_1}, \ldots, e^{i\theta_{m-1}}, e^{-i(\theta_1 + \cdots + \theta_{m-1})}, 0, \ldots, 0) : \theta_i \in \mathbb{R}\}.$$

These polynomials are also studied with respect to the Drury–Arveson space in [20].

In two variables, $1 - z_1$ is cyclic in $D_\alpha(\mathbb{B}_2)$ precisely when $\alpha \le 2$, and $1 - 2z_1z_2$ is cyclic in $D_\alpha(\mathbb{B}_2)$ precisely when $\alpha \le 3/2$. Here, there are more than two fixed parameters. The characterization of cyclicity of these two polynomials was crucial in [14].

Section 5 studies the radial dilation of a polynomial. Using the equivalent characterization of Section 3, we identify cyclicity for the model polynomials via the powerful radial dilation method. In particular, we show that if $p/p_r \to 1$ weakly, where $p_r(z) = p(rz)$ is a radial dilation of p, then p is cyclic (see [13] for the bidisk settings and [14] for the unit ball in two variables). This method is quite interesting since it can be applied to an arbitrary polynomial. Note that in [13, 14], the radial dilation method is one of the main tools of solving cyclicity problem for polynomials. The main result of this section verifies the arguments made about polynomials in Section 4.

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Section 6 studies noncyclic vectors. We use the notion of Riesz α-capacity in order to identify noncyclic functions. Moreover, we study Cauchy transforms of Borel measures supported on zero sets of the radial limits of a given function $f \in D_\alpha(\mathbb{B}_n)$ and we give asymptotic expansions of their norms. Then employing a standard scheme due to Brown and Shields [8], we prove the main result. Note that this sufficient capacity condition for noncyclicity in Dirichlet-type spaces in the unit ball of two variables was proved by Sola in [21].

2 Standard tools

Let us give some standard tools which will be useful in the sequel.

The binomial series

$$\frac{1}{(1-x)^\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)k!} x^k,$$

where $|x| < 1$ is a complex number and α is a nonnegative real number. The asymptotic behavior of the Γ-function is the following: $\Gamma(k+\alpha) \asymp (k-1)!k^\alpha$, where the symbol $\asymp$ denotes that the ratio of the two quantities either tends to a constant as k tends to infinity or it is rather two sides bound by constants.

The multinomial formula

$$(x_1 + \cdots + x_n)^k = \sum_{|j|=k} \frac{k!}{j!} x_1^{j_1} \cdots x_n^{j_n},$$

where $j = (j_1, \ldots, j_n)$ is an n-tuple index of nonnegative integers and x_i are complex numbers.

The Stirling formula that describes the asymptotic behavior of the gamma function

$$k! \asymp k^{1/2} k^k / e^k.$$

Denote the normalized area measure on $\mathbb{C}^n = \mathbb{R}^{2n}$ by $du(z)$ and the normalized rotation-invariant positive Borel measure on $\mathbb{S}_n$ by $d\sigma(\zeta)$ (see [18, 22]). The measures $du(z)$ and $d\sigma(\zeta)$ are related by the formula

$$\int_{\mathbb{C}^n} f(z)du(z) = 2n \int_0^\infty \int_{\mathbb{S}_n} \varepsilon^{2n-1} f(\varepsilon\zeta) d\sigma(\zeta) d\varepsilon.$$

The holomorphic monomials are orthogonal to each other in $L^2(\sigma)$, that is, if k and l are multiindices such that $k \neq l$, then

$$\int_{\mathbb{S}_n} \zeta^k \bar{\zeta}^l d\sigma(\zeta) = 0.$$

Moreover,

$$\int_{\mathbb{S}_n} |\zeta^k|^2 d\sigma(\zeta) = \frac{(n-1)!k!}{(n-1+|k|)!} \quad \text{and} \quad \int_{\mathbb{B}_n} |z^k|^2 du(z) = \frac{n!k!}{(n+|k|)!}.$$

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3 Relation among Dirichlet-type spaces and an equivalent characterization

We study the structure of Dirichlet-type spaces. Note that

$$R(f)(z) = z_1\partial_{z_1}f(z) + \cdots + z_n\partial_{z_n}f(z)$$

is the *radial derivative* of a function f. The radial derivative plays a key role in the function theory of the unit ball. A crucial relation among these spaces is the following.

Proposition 1 *Let $f \in \mathrm{Hol}(\mathbb{B}_n)$ and $\alpha \in \mathbb{R}$ be fixed. Then*

$$f \in D_\alpha(\mathbb{B}_n) \quad \textit{if and only if} \quad n^q f + R^q(f) + q\sum_{i=1}^{q-1} n^i R^{q-i}(f) \in D_v(\mathbb{B}_n),$$

where $\alpha = 2q + v$, $q \in \mathbb{N}$, and R^q is the q-image of the operator R.

Proof Indeed, it is enough to check that

$$\|nf + R(f)\|^2_{\alpha-2} = \sum_{|k|=0}^{\infty} (n+|k|)^{\alpha-2}\frac{(n-1)!k!}{(n-1+|k|)!}(n+|k|)^2|a_k|^2 = \|f\|^2_\alpha. \qquad \blacksquare$$

We continue by giving an equivalent characterization of Dirichlet-type spaces. In Dirichlet-type spaces in the unit ball, one of the Dirichlet-type integrals is achieved in a limited range of parameters.

Lemma 2 (See [16]) *If $\alpha \in (-1,1)$, then $f \in D_\alpha(\mathbb{B}_n)$ if and only if*

$$|f|^2_\alpha := \int_{\mathbb{B}_n} \frac{\|\nabla(f)(z)\|^2 - |R(f)(z)|^2}{(1-\|z\|^2)^\alpha}du(z) < \infty.$$

Above, $\nabla(f)(z) = (\partial_{z_1}f(z), \ldots, \partial_{z_n}f(z))$ denotes the *holomorphic gradient* of a holomorphic function f. Note that Proposition 1 allows us to use Lemma 2 whenever $v \in (-1,1)$. Let $\gamma, t \in \mathbb{R}$ be such that neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer. If $f = \sum_{|k|=0}^{\infty} a_k z^k$ is the homogeneous expansion of a function $f \in \mathrm{Hol}(\mathbb{B}_n)$, then we may define an invertible continuous linear operator with respect to the topology of uniform convergence on compact subsets of $\mathbb{B}_n$, denoted by $R^{\gamma,t} : \mathrm{Hol}(\mathbb{B}_n) \to \mathrm{Hol}(\mathbb{B}_n)$ and having expression

$$R^{\gamma,t}f(z) = \sum_{|k|=0}^{\infty} C(\gamma,t,k)a_k z^k, \quad z \in \mathbb{B}_n,$$

where

$$C(\gamma,t,k) = \frac{\Gamma(n+1+\gamma)\Gamma(n+1+|k|+\gamma+t)}{\Gamma(n+1+\gamma+t)\Gamma(n+1+|k|+\gamma)} \asymp |k|^t. \tag{4}$$

See [22] for more information regarding these fractional radial derivatives.

Lemma 3 *Let $t \in \mathbb{R}$ be such that $n-1+t \geq 0$. If $f \in A(\mathbb{B}_n)$, then*

$$R^{-1,t}f(z) = \int_{\mathbb{S}_n} \frac{f(\zeta)}{(1-\langle z,\zeta\rangle)^{n+t}}d\sigma(\zeta), \quad z \in \mathbb{B}_n.$$

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Proof The continuous linear operator $R^{\gamma,t}$ (see [22]) satisfies

$$R^{\gamma,t}\Big(\frac{1}{(1-\langle z,w\rangle)^{n+1+\gamma}}\Big)=\frac{1}{(1-\langle z,w\rangle)^{n+1+\gamma+t}}$$

for all $w\in\mathbb{B}_n$. Next, define f_ε for $\varepsilon\in(0,1)$ by

$$f_\varepsilon(z)=\int_{\mathbb{S}_n}\frac{f(\zeta)}{(1-\langle z,\varepsilon\zeta\rangle)^n}d\sigma(\zeta),\quad z\in\mathbb{B}_n.$$

The Cauchy formula holds for $f\in A(\mathbb{B}_n)$ and hence $f=\lim_{\varepsilon\to1^-}f_\varepsilon$. It follows that

$$\begin{aligned}R^{-1,t}f(z)&=R^{-1,t}\Big(\lim_{\varepsilon\to1^-}\int_{\mathbb{S}_n}\frac{f(\zeta)}{(1-\langle z,\varepsilon\zeta\rangle)^n}d\sigma(\zeta)\Big)\\&=\lim_{\varepsilon\to1^-}R^{-1,t}\Big(\int_{\mathbb{S}_n}\frac{f(\zeta)}{(1-\langle z,\varepsilon\zeta\rangle)^n}d\sigma(\zeta)\Big)\\&=\lim_{\varepsilon\to1^-}\int_{\mathbb{S}_n}f(\zeta)R^{-1,t}\Big(\frac{1}{(1-\langle z,\varepsilon\zeta\rangle)^n}\Big)d\sigma(\zeta)\\&=\lim_{\varepsilon\to1^-}\int_{\mathbb{S}_n}\frac{f(\zeta)}{(1-\langle z,\varepsilon\zeta\rangle)^{n+t}}d\sigma(\zeta)\\&=\int_{\mathbb{S}_n}\frac{f(\zeta)}{(1-\langle z,\zeta\rangle)^{n+t}}d\sigma(\zeta),\end{aligned}$$

and the assertion follows. ∎

Theorem 4 *Let $\alpha\in\mathbb{R}$ be such that $n-1+\alpha/2\geq0$ and $f\in A(\mathbb{B}_n)$. Then $f\in D_\alpha(\mathbb{B}_n)$ if and only if*

$$\int_{\mathbb{B}_n}(1-\|z\|^2)\Big|\int_{\mathbb{S}_n}\frac{f(\zeta)\bar{\zeta}_p}{(1-\langle z,\zeta\rangle)^{n+\alpha/2+1}}d\sigma(\zeta)\Big|^2du(z)<\infty$$

and

$$\int_{\mathbb{B}_n}\Big|\int_{\mathbb{S}_n}\frac{\overline{(z_p\zeta_q-z_q\zeta_p)}f(\zeta)}{(1-\langle z,\zeta\rangle)^{n+\alpha/2+1}}d\sigma(\zeta)\Big|^2du(z)<\infty,$$

where $p,q=1,\dots,n$.

Proof Choose t so that $\alpha=2t$. Note that n,t are fixed and hence

$$\|f\|_\alpha^2\asymp\sum_{|k|=0}^{\infty}\frac{(n-1)!k!}{(n-1+|k|)!}||k|^ta_k|^2.$$

Thus, (4) implies that $\|R^{-1,t}f\|_0\asymp\|f\|_\alpha$. One can apply then the equivalent integral representation of Dirichlet-type norms to $R^{-1,t}f\in\mathrm{Hol}(\mathbb{B}_n)$, i.e., $R^{-1,t}f\in D_0(\mathbb{B}_n)$ if and only if $|R^{-1,t}f|_0<\infty$. According to Lemma 3, we get that

$$\partial_{z_p}(R^{-1,t}f)(z)=\int_{\mathbb{S}_n}\frac{f(\zeta)\bar{\zeta}_p}{(1-\langle z,\zeta\rangle)^{n+t+1}}d\sigma(\zeta),\quad z\in\mathbb{B}_n,$$

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where $p = 1, \dots, n$. Expand the term $\|\nabla(f)\|^2 - |R(f)|^2$ as follows:

$$\|\nabla(f)\|^2 - |R(f)|^2 = (1 - \|z\|^2)\|\nabla(f)\|^2 + \sum_{p,q} |\bar{z}_p \partial_{z_q} f - \bar{z}_q \partial_{z_p} f|^2.$$

The assertion follows by Lemma 2. ∎

4 Diagonal subspaces

In [3], a method of construction of optimal approximants via determinants in Dirichlet-type spaces in the unit disk is provided. Similarly, we may define optimal approximants in several variables (see [20]).

Fix $N \in \mathbb{N}$. We define the space of polynomials $p \in \mathbb{C}[z_1, \dots, z_n]$ with degree at most nN as follows:

$$P_N^n := \left\{ p(z) = \sum_{k_1=0}^{N} \cdots \sum_{k_n=0}^{N} a_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n} \right\}.$$

Remark 5 Let $(X, \|\cdot\|)$ be a normed space and fix $x \in X$, $C \subset X$. The distance between x and the set C is the following:

$$\operatorname{dist}_X(x, C) := \inf\{\|x - c\| : c \in C\}.$$

It is well known that if X is a Hilbert space and $C \subset X$ a convex closed subset, then for any $x \in X$, there exists a unique $y \in C$ such that $\|x - y\| = \operatorname{dist}_X(x, C)$. Let $f \in D_\alpha(\mathbb{B}_n)$ be nonzero constant. We deduce that for any $N \in \mathbb{N}$, there exists exactly one $p_N \in P_N^n$ satisfying

$$\|p_N f - 1\|_\alpha = \operatorname{dist}_{D_\alpha(\mathbb{B}_n)}(1, f \cdot P_N^n).$$

Let $f \in D_\alpha(\mathbb{B}_n)$. We say that a polynomial $p_N \in P_N^n$ is an *optimal approximant of order* N to $1/f$ if p_N minimizes $\|pf - 1\|_\alpha$ among all polynomials $p \in P_N^n$. We call $\|p_N f - 1\|_\alpha$ the *optimal norm of order* N associated with f.

Let $M = (M_1, \dots, M_n)$ be a multiindex, where M_i are nonnegative integers, and $m \in \{1, \dots, n\}$. Setting

$$\mu(m) := \frac{(M_1 + \cdots + M_m)^{M_1 + \cdots + M_m}}{M_1^{M_1} \cdots M_m^{M_m}},$$

we see that

$$\mu(m)^{1/2} |z_1|^{M_1} \cdots |z_m|^{M_m} \le 1, \quad z \in \mathbb{B}_n. \tag{5}$$

Using (5), we may construct polynomials that vanish in the closed unit ball along analytic subsets of the unit sphere.

Remark 6 Let $\tilde{f} \in \operatorname{Hol}(\mathbb{D}(\mu(m)^{-1/4}))$, where

$$\mathbb{D}(\mu) = \{z \in \mathbb{C} : |z| < \mu\}, \quad \mu > 0.$$

According to (5), we define the following function:

$$f(z) = f(z_1, \dots, z_n) = \tilde{f}(\mu(m)^{1/4} z_1^{M_1} \cdots z_m^{M_m}), \quad z \in \mathbb{B}_n.$$

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Then $f \in \mathrm{Hol}(\mathbb{B}_n)$ and it depends on m variables. Note that we may change the variables $z_1, \ldots, z_m$ by any other m variables. For convenience, we choose the m first variables. The power $1/4$ will be convenient in the sequel.

Thus, the question that arises out is if we may define closed subspaces of $D_\alpha(\mathbb{B}_n)$ passing through one variable functions. We shall see that these subspaces are called diagonal subspaces due to the nature of the power series expansion of their elements.

Instead of the classical one variable Dirichlet-type spaces of the unit disk, we may consider spaces d_β, $\beta \in \mathbb{R}$, consisting of holomorphic functions $\tilde{f} \in \mathrm{Hol}(\mathbb{D}(\mu^{-1/4}))$. Moreover, such functions with power series expansion $\tilde{f}(z) = \sum_{l=0}^{\infty} a_l z^l$ are said to belong to d_β if

$$\|\tilde{f}\|_{d_\beta}^2 := \sum_{l=0}^{\infty} \mu^{-l/2}(l+1)^\beta |a_l|^2 < \infty.$$

There is a natural identification between the function theories of $D_\beta(\mathbb{D})$: one variable Dirichlet-type spaces of the unit disk, and d_β, and one verifies that the results in [3] are valid for d_β.

We are ready to define diagonal closed subspaces. Set

$$\beta(\alpha) := \alpha - n + \frac{m+1}{2}.$$

Let α, M, m be as above. The diagonal closed subspace of $D_\alpha(\mathbb{B}_n)$ is the following:

$$J_{\alpha,M,m} := \{f \in D_\alpha(\mathbb{B}_n) : \exists \tilde{f} \in d_{\beta(\alpha)}, f(z) = \tilde{f}(\mu(m)^{1/4} z_1^{M_1} \cdots z_m^{M_m})\}.$$

The existence of a holomorphic function $\tilde{f}$ is unique by identity principle, and hence there is no any amiss in the definition. Any function $f \in J_{\alpha,M,m}$ has an expansion of the form

$$f(z) = \sum_{l=0}^{\infty} a_l (z_1^{M_1} \cdots z_m^{M_m})^l.$$

The relation of norms between one variable and diagonal subspaces follows.

Proposition 7 *If* $f \in J_{\alpha,M,m}$, *then* $\|f\|_\alpha \asymp \|\tilde{f}\|_{d_{\beta(\alpha)}}$.

Proof If $f \in J_{\alpha,M,m}$, then

$$\|f\|_\alpha^2 \asymp \sum_{l=0}^{\infty} (l+1)^\alpha \frac{(M_1 l)! \cdots (M_m l)!}{(n-1+(M_1+\cdots M_m)l)!} |a_l|^2.$$

By Stirling's formula, we obtain

$$\|f\|_\alpha^2 \asymp \sum_{l=0}^{\infty} (l+1)^{\alpha-n+m/2+1/2} \mu(m)^{-l} |a_l|^2.$$

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On the other hand, define the function $f'(z) = \sum_{l=0}^{\infty} \mu(m)^{-l/4} a_l z^l$. Then $f'(\mu(m)^{1/4} z_1^{M_1} \cdots z_m^{M_m}) = f(z_1, \ldots, z_n)$ and

$$\|f'\|^2_{d_{\beta(\alpha)}} \asymp \sum_{l=0}^{\infty} (l+1)^{\alpha - n + m/2 + 1/2} \mu(m)^{-l} |a_l|^2.$$

The assertion follows since f' coincides with $\tilde{f}$. ∎

The corresponding Lemma 3.4 of [4] in our case is the following.

Lemma 8 *Let $f \in J_{\alpha,M,m}$, where α, M, m be as above. Let $r_N \in P_N^n$ with expansion*

$$r_N(z) = \sum_{k_1=0}^{N} \cdots \sum_{k_n=0}^{N} a_{k_1,\ldots,k_n} z_1^{k_1} \ldots z_n^{k_n},$$

and consider its projection onto $J_{\alpha,M,m}$

$$\pi(r_N)(z) = \sum_{\{l : M_1 l, \ldots, M_m l \leq N\}} c_{M_1 l, \ldots, M_m l, 0, \ldots, 0} z_1^{M_1 l} \cdots z_m^{M_m l}.$$

Then

$$\|r_N f - 1\|_\alpha \geq \|\pi(r_N) f - 1\|_\alpha.$$

Moreover, just as in Proposition 7, there is a relation of optimal approximants between one variable and diagonal subspaces.

Proposition 9 *If $f \in J_{\alpha,M,m}$, then*

$$\operatorname{dist}_{D_\alpha(\mathbb{B}_n)}(1, f \cdot P_N^n) \asymp \operatorname{dist}_{d_{\beta(\alpha)}}(1, \tilde{f} \cdot P_N^1).$$

Proof Let $r_N, \pi(r_N)$ be as in Lemma 8. Then $\pi(r_N) f - 1 \in J_{\alpha,M,m}$. It follows that

$$\|r_N f - 1\|_\alpha \geq \|\pi(r_N) f - 1\|_\alpha \asymp \|\tilde{\pi}(r_N) \tilde{f} - 1\|_{d_{\beta(\alpha)}} \geq \operatorname{dist}_{d_{\beta(\alpha)}}(1, \tilde{f} \cdot P_N^1),$$

since $\tilde{\pi}(r_N) \in P_N^1$. On the other hand, let

$$\operatorname{dist}_{d_{\beta(\alpha)}}(1, \tilde{f} \cdot P_N^1) = \|q_N \tilde{f} - 1\|_{d_{\beta(\alpha)}}, \quad q_N(z) = \sum_{l=0}^{N} a_l z^l.$$

Then, the polynomial

$$q'_N(z_1, \ldots, z_n) = \sum_{l=0}^{N} \mu(m)^{-l/4} a_l z_1^{M_1 l} \cdots z_m^{M_m l}$$

satisfies $q'_N \in J_{\alpha,M,m} \cap P_N^n$ and $q'_N f - 1 \in J_{\alpha,M,m}$. Thus,

$$\|q_N \tilde{f} - 1\|_{d_{\beta(\alpha)}} = \|\tilde{q}'_N \tilde{f} - 1\|_{d_{\beta(\alpha)}} \asymp \|q'_N f - 1\|_\alpha \geq \operatorname{dist}_{D_\alpha(\mathbb{B}_n)}(1, f \cdot P_N^n)$$

and the assertion follows. ∎

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Define the function $\phi_\beta : [0,\infty) \to [0,\infty)$ by

$$\phi_\beta(t) = \begin{cases} t^{1-\beta}, & \beta < 1, \\ \log^+(t), & \beta = 1, \end{cases}$$

where $\log^+(t) := \max\{\log t, 0\}$. We have the following.

Theorem 10 *Let $\alpha \in \mathbb{R}$ be such that $\beta(\alpha) \le 1$. Let $f \in J_{\alpha,M,m}$ be as above, and suppose that the corresponding $\tilde{f}$ has no zeros inside its domain, has at least one zero on the boundary, and admits an analytic continuation to a strictly bigger domain. Then f is cyclic in $D_\alpha(\mathbb{B}_n)$ whenever $\alpha \le \frac{2n-m+1}{2}$ and*

$$\operatorname{dist}^2_{D_\alpha(\mathbb{B}_n)}(1, f \cdot P_N^n) \asymp \phi_{\beta(\alpha)}(N+1)^{-1}.$$

Proof It is an immediate consequence of the identification between $D_\beta(\mathbb{D})$ and d_β and previous lemmas and propositions. ∎

If we focus on polynomials, then the following is true.

Theorem 11 *Consider the polynomial $p(z) = 1 - m^{m/2} z_1 \cdots z_m$, where $1 \le m \le n$. Then p is cyclic in $D_\alpha(\mathbb{B}_n)$ whenever $\alpha \le \frac{2n+1-m}{2}$.*

Note that the Theorem 11 is not a characterization. We shall study the case $\alpha > \frac{2n+1-m}{2}$.

5 Cyclicity for model polynomials via radial dilation

The *radial dilation* of a function $f : \mathbb{B}_n \to \mathbb{C}$ is defined for $r \in (0,1)$ by $f_r(z) = f(rz)$. To prove Theorem 11, it is enough to prove the following lemma.

Lemma 12 *Consider the polynomial $p(z) = 1 - m^{m/2} z_1 \cdots z_m$, where $1 \le m \le n$. Then $\|p/p_r\|_\alpha < \infty$ as $r \to 1^-$ whenever $\alpha \le \frac{2n+1-m}{2}$.*

We follow the arguments of [13, 14]. Indeed, if Lemma 12 holds, then $\phi_r \cdot p \to 1$ weakly, where $\phi_r := 1/p_r$. This is a consequence of a crucial property of Dirichlet-type spaces: if $\{f_n\} \subset D_\alpha(\mathbb{B}_n)$, then $f_n \to 0$ weakly if and only if $f_n \to 0$ pointwise and $\sup_n\{\|f_n\|_\alpha\} < \infty$. Since ϕ_r extends holomorphically past the closed unit ball, ϕ_r are multipliers, and hence, $\phi_r \cdot p \in [p]$. Finally, 1 is weak limit of $\phi_r \cdot p$ and $[p]$ is closed and convex or, equivalently, weakly closed. It is clear that $1 \in [p]$, and hence p is cyclic.

Moreover, it is enough to prove that $\|p/p_r\|_\alpha < \infty$, as $r \to 1^-$, for $\alpha_0 = \frac{2n+1-m}{2}$. Then the case $\alpha < \alpha_0$ follows since the inclusion $D_{\alpha_0}(\mathbb{B}_n) \hookrightarrow D_\alpha(\mathbb{B}_n)$ is a compact linear map and weak convergence in $D_{\alpha_0}(\mathbb{B}_n)$ gives weak convergence in $D_\alpha(\mathbb{B}_n)$.

Proof of Lemma 12 By Theorem 4, it is enough to show the following:

$$I_p := \int_{\mathbb{B}_n} (1 - \|z\|^2) \left| \int_{\mathbb{S}_n} \frac{(1 - \lambda\zeta_1 \cdots \zeta_m)\bar{\zeta}_p}{(1 - r^m \lambda \zeta_1 \cdots \zeta_m)(1 - \langle z, \zeta\rangle)^\beta} d\sigma(\zeta) \right|^2 du(z)$$

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and

$$I_{p,q} := \int_{\mathbb{B}_n} \Big| \int_{\mathbb{S}_n} \frac{(\overline{z_p\zeta_q - z_q\zeta_p})(1-\lambda\zeta_1\cdots\zeta_m)}{(1-r^m\lambda\zeta_1\cdots\zeta_m)(1-\langle z,\zeta\rangle)^\beta} d\sigma(\zeta)\Big|^2 du(z)$$

are finite, as $r \to 1^-$, where $\beta = n+t+1$, $t = \frac{2n+1-m}{4}$, and $\lambda = m^{m/2}$.

Denote

$$S_p := \int_{\mathbb{S}_n} \frac{(1-\lambda\zeta_1\cdots\zeta_m)\bar{\zeta}_p}{(1-r^m\lambda\zeta_1\cdots\zeta_m)(1-\langle z,\zeta\rangle)^\beta} d\sigma(\zeta),$$

where the last integral is equal to

$$\frac{1}{2\pi}\int_{\mathbb{S}_n}\int_0^{2\pi} \frac{(1-\lambda e^{im\theta}\zeta_1\cdots\zeta_m)e^{-i\theta}\bar{\zeta}_p}{(1-r^m\lambda e^{im\theta}\zeta_1\cdots\zeta_m)(1-e^{-i\theta}\langle z,\zeta\rangle)^\beta} d\theta d\sigma(\zeta).$$

Let z, ζ be fixed. Then

$$\int_0^{2\pi} \frac{e^{-i\theta}}{(1-e^{-i\theta}\langle z,\zeta\rangle)^\beta} d\theta = 0.$$

Thus, replacing $p(e^{i\theta}\zeta)/p(re^{i\theta}\zeta)$ by $p(e^{i\theta}\zeta)/p(re^{i\theta}\zeta) - 1$, we obtain

$$S_p = \frac{\lambda(r^m-1)}{2\pi}\int_{\mathbb{S}_n}\int_0^{2\pi} \frac{\bar{\zeta}_p\zeta_1\cdots\zeta_m e^{i(m-1)\theta}}{(1-r^m\lambda e^{im\theta}\zeta_1\cdots\zeta_m)(1-e^{-i\theta}\langle z,\zeta\rangle)^\beta} d\theta d\sigma(\zeta).$$

Next, expand the binomials

$$\begin{aligned}
&\int_0^{2\pi} \frac{e^{i(m-1)\theta}}{(1-r^m\lambda e^{im\theta}\zeta_1\cdots\zeta_m)(1-e^{-i\theta}\langle z,\zeta\rangle)^\beta} d\theta \\
&\quad = \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{\Gamma(k+\beta)}{\Gamma(\beta)k!}(r^m\lambda\zeta_1\cdots\zeta_m)^l\langle z,\zeta\rangle^k \int_0^{2\pi} e^{i(m(l+1)-k-1)\theta} d\theta \\
&\quad = 2\pi\sum_{k=0}^\infty \frac{\Gamma(m(k+1)-1+\beta)}{\Gamma(\beta)(m(k+1)-1)!}(r^m\lambda\zeta_1\cdots\zeta_m)^k\langle z,\zeta\rangle^{m(k+1)-1} \\
&\quad = 2\pi\sum_{k=0}^\infty \sum_{|j|=m(k+1)-1} \frac{\Gamma(m(k+1)-1+\beta)}{\Gamma(\beta)j!}(r^m\lambda\zeta_1\cdots\zeta_m)^k z^j\bar{\zeta}^j.
\end{aligned}$$

Therefore,

$$S_p = \lambda(r^m-1)\sum_{k=0}^\infty \sum_{|j|=m(k+1)-1} \frac{\Gamma(m(k+1)-1+\beta)}{\Gamma(\beta)j!}(r^m\lambda)^k z^j c(k),$$

where $c(k) = \int_{\mathbb{S}_n} \zeta^{\alpha(k)}\bar{\zeta}^{b(k)} d\sigma(\zeta)$, $\alpha(k) = (k+1,\dots,k+1(\text{m-comp.}),0,\dots,0)$ and $b(k) = (j_1,\dots,j_{p-1},j_p+1,j_{p+1},\dots,j_n)$. Whence, $1 \le p \le m$. Since the holomorphic monomials are orthogonal to each other in $L^2(\sigma)$, we get that

$$|S_p| \asymp (1-r^m)\Big|z_p'\sum_{k=0}^\infty (k+1)^{\beta-n}(r^m\lambda z_1\cdots z_m)^k\Big|,$$

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where $z_p' = z_1 \cdots z_{p-1} z_{p+1} \cdots z_m$. Hence, we obtain

$$I_p \asymp (1-r^m)^2 \sum_{k=0}^{\infty} (k+1)^{2(\beta-n)} (r^m \lambda)^{2k} \int_{\mathbb{B}_n} (1-\|z\|^2) |z_p'|^2 |z_1 \cdots z_m|^{2k} du(z),$$

where has been used again the orthogonality of the holomorphic monomials in $L^2(\sigma)$. To handle the integral above, we use polar coordinates

$$\begin{aligned} &\int_{\mathbb{B}_n} (1-\|z\|^2) |z_p'|^2 |z_1 \cdots z_m|^{2k} du(z) \\ &\quad \asymp \int_0^1 \int_{\mathbb{S}_n} \varepsilon^{2n-1} (1-\varepsilon^2) \varepsilon^{2km+2m-2} |\zeta_p'|^2 |\zeta_1 \cdots \zeta_m|^{2k} d\sigma(\zeta) d\varepsilon \\ &\quad \asymp \frac{[(k+1)!]^{m-1} k!}{(n+m(k+1)-2)!} \cdot \frac{1}{(k+1)^2}. \end{aligned}$$

If we recall that $\beta = n + t + 1$, $t = \frac{2n+1-m}{4}$ and $\lambda^{2k} = m^{mk}$, then applying the Stirling formula more than one time, we see that

$$I_p \asymp (1-r^m)^2 \sum_{k=0}^{\infty} (k+1) r^{2mk}.$$

This proves the assertion made about I_p.

It remains to estimate the following term:

$$I_{p,q} = \int_{\mathbb{B}_n} \Big| \int_{\mathbb{S}_n} \frac{(\overline{z_p \zeta_q - z_q \zeta_p})(1-\lambda \zeta_1 \cdots \zeta_m)}{(1-r^m \lambda \zeta_1 \cdots \zeta_m)(1-\langle z, \zeta \rangle)^{\beta}} d\sigma(\zeta) \Big|^2 du(z).$$

We shall show that $I_{p,q} \asymp I_p$. Denote again the inner integral by $S_{p,q}$ which is convenient to expand it as $S_{p,q} = \bar{z}_p S_q - \bar{z}_q S_p$. Recall that $z_p' = z_1 \cdots z_{p-1} z_{p+1} \cdots z_m$. Similar calculations to the one above lead to

$$|S_{p,q}| \asymp (1-r^m) |\bar{z}_p z_q' - \bar{z}_q z_p'| \Big| \sum_{k=0}^{\infty} (k+1)^{\beta-n} (r^m \lambda z_1 \cdots z_m)^k \Big|.$$

Moreover, the orthogonality of the holomorphic monomials in $L^2(\sigma)$ gives the following estimation:

$$I_{p,q} \asymp (1-r^m)^2 \sum_{k=0}^{\infty} (k+1)^{2\beta-2n} (r^m \lambda)^{2k} \int_{\mathbb{B}_n} |\bar{z}_p z_q' - \bar{z}_q z_p'|^2 |z_1 \cdots z_m|^{2k} du(z).$$

It is easy to see that $|\bar{z}_p z_q' - \bar{z}_q z_p'|^2 = |z_p|^2 |z_q'|^2 + |z_q|^2 |z_p'|^2 - 2|z_1 \cdots z_m|^2$. Let us estimate the integral

$$\int_{\mathbb{B}_n} (|z_p|^2 |z_q'|^2 - |z_1 \cdots z_m|^2) |z_1 \cdots z_m|^{2k} du(z).$$

Passing through polar coordinates, we get, for $p \neq q$, that

$$\begin{aligned} &\int_{\mathbb{B}_n} |z_p|^2 |z_q'|^2 |z_1 \cdots z_m|^{2k} du(z) \\ &\qquad \asymp 2n(n-1)! \frac{[(k+1)!]^{m-1} k!}{(mk+n+m-1)!} \frac{k+2}{2km+2n+2m} \end{aligned}$$

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and

$$\int_{\mathbb{B}_n} |z_1 \cdots z_m|^{2(k+1)} du(z) = 2n(n-1)! \frac{[(k+1)!]^{m-1} k!}{(mk+n+m-1)!} \frac{k+1}{2km+2n+2m}.$$

Hence, we obtain

$$\int_{\mathbb{B}_n} (|z_p|^2 |z_q'|^2 - |z_1 \cdots z_m|^2) |z_1 \cdots z_m|^{2k} du(z) \asymp \frac{[(k+1)!]^{m-1} k!}{(mk+n+m-2)!(k+1)^2}.$$

Again, applying the Stirling formula to the one above estimates, we obtain

$$I_{p,q} \asymp (1-r^m)^2 \sum_{k=0}^{\infty} (k+1) r^{2mk}.$$

This proves the assertion made about $I_{p,q}$. ∎

6 Sufficient conditions for noncyclicity via Cauchy transforms and α-capacities

We consider the *Cauchy transform* of a complex Borel measure μ on the unit sphere by

$$C_{[\mu]}(z) = \int_{\mathbb{S}_n} \frac{1}{(1-\langle z, \bar{\zeta} \rangle)^n} d\mu(\zeta), \quad z \in \mathbb{B}_n.$$

Note that this definition differs from the classical one.

Let $f \in D_\alpha(\mathbb{B}_n)$ and put a measure μ on $\mathcal{Z}(f^*)$: the zero set in the sphere of the radial limits of f. The results in [21] about Cauchy transforms and noncyclicity are valid in our settings. We deduce that $[f] \neq D_\alpha(\mathbb{B}_n)$, and hence noncyclicity, whenever $C_{[\mu]} \in D_{-\alpha}(\mathbb{B}_n)$. Thus, it is important to compute the Dirichlet-type norm of the Cauchy transform.

Let μ be a Borel measure on $\mathbb{S}_n$ and set $\mu^*(j) = \int_{\mathbb{S}_n} \zeta^j d\mu(\zeta)$, $\bar{\mu}^*(j) = \int_{\mathbb{S}_n} \bar{\zeta}^j d\mu(\zeta)$. We have the following.

Lemma 13 *Let μ be a Borel measure on $\mathbb{S}_n$. Then*

$$\|C_{[\mu]}\|_{-\alpha}^2 \asymp \sum_{k=0}^{\infty} \sum_{|j|=k} \frac{(k+1)^{n-1-\alpha} k!}{j!} |\bar{\mu}^*(j)|^2.$$

Proof Our Cauchy integral of μ on $\mathbb{B}_n$ has the following expansion:

$$C_{[\mu]}(z) = \sum_{k=0}^{\infty} \sum_{|j|=k} \frac{\Gamma(k+n)}{\Gamma(n) j!} \bar{\mu}^*(j) z^j.$$

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Therefore, one can compute the norm of $C_{[\mu]}$ in the space $D_{-\alpha}(\mathbb{B}_n)$. The assertion follows. ∎

The following lemma is crucial in the sequel. It is probably known, but we were not able to locate it in the literature, and hence we include its proof.

Lemma 14 *Let $j_1, \dots, j_n, k$ be nonnegative integers satisfying $j_1 + \cdots + j_n = nk$. Then*

$$j_1! \cdots j_n! \geq (k!)^n.$$

Proof The Γ-function is logarithmically convex, and hence we may apply the Jensen inequality to it:

$$\log \Gamma\Big(\frac{x_1}{n} + \cdots + \frac{x_n}{n}\Big) \leq \frac{\log \Gamma(x_1)}{n} + \cdots + \frac{\log \Gamma(x_n)}{n}.$$

Set $x_i := j_i + 1$, $i = 1, \dots, n$. Since $j_1 + \cdots + j_n = nk$, the assertion follows. ∎

We may identify noncyclicity for model polynomials via Cauchy transforms.

Lemma 15 *Consider the polynomial $p(z) = 1 - m^{m/2} z_1 \cdots z_m$, where $1 \leq m \leq n$. Then p is not cyclic in $D_\alpha(\mathbb{B}_n)$ whenever $\alpha > \frac{2n+1-m}{2}$.*

Proof Recall that the model polynomials vanish in the closed unit ball along analytic sets of the form:

$$\mathcal{Z}(p) \cap \mathbb{S}_n = \{1/\sqrt{m}(e^{i\theta_1}, \dots, e^{i\theta_{m-1}}, e^{-i(\theta_1 + \cdots + \theta_{m-1})}, 0, \dots, 0) : \theta_i \in \mathbb{R}\}.$$

It is easy to see that for a proper measure μ, $\mu^*(j)$ is nonzero, when $mj_m = k$ and $\mu^*(j) \asymp m^{-k/2}$. By Stirling's formula and Lemma 14, we get that

$$\|C_{[\mu]}\|^2_{-\alpha} \leq C \sum_{k=0}^{\infty} \frac{(mk+1)^{n-1-\alpha}(mk)!}{(k!)^m m^{mk}} \asymp \sum_{k=0}^{\infty} (k+1)^{1/2(2n-m-1)-\alpha}.$$

Thus, p is not cyclic in $D_\alpha(\mathbb{B}_n)$ for $\alpha > \frac{2n+1-m}{2}$. ∎

We consider Riesz α-capacity for a fixed parameter $\alpha \in (0, n)$ with respect to the *anisotropic distance* in $\mathbb{S}_n$ given by

$$d(\zeta, \eta) = |1 - \langle \zeta, \eta \rangle|^{1/2}$$

and the nonnegative *kernel* $K_\alpha : (0, \infty) \to [0, \infty)$ given by

$$K_\alpha(t) = \begin{cases} t^{\alpha - n}, & \alpha \in (0, n), \\ \log(e/t), & \alpha = n. \end{cases}$$

Note that we may extend the definition of K to 0 by defining $K(0) := \lim_{t \to 0^+} K(t)$.

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Let μ be any Borel probability measure supported on some Borel set $E \subset \mathbb{S}_n$. Then the *Riesz α-energy* of μ is given by

$$I_\alpha[\mu] = \iint_{\mathbb{S}_n} K_\alpha(|1 - \langle \zeta, \eta \rangle|) d\mu(\zeta) d\mu(\eta)$$

and the *Riesz α-capacity* of E by

$$\mathrm{cap}_\alpha(E) = \inf\{I_\alpha[\mu] : \mu \in \mathcal{P}(E)\}^{-1},$$

where $\mathcal{P}(E)$ is the set of all Borel probability measures supported on E. Note that if $\mathrm{cap}_\alpha(E) > 0$, then there exist at least one probability measure supported on E having finite Riesz α-energy. Moreover, any $f \in D_\alpha(\mathbb{B}_n)$ has finite radial limits f^* on $\mathbb{S}_n$, except possibly, on a set E having $\mathrm{cap}_\alpha(E) = 0$. Theory regarding to the above standard construction in potential theory can be found in [1, 9, 11, 17].

The relation between noncyclicity of a function and the Riesz α-capacity of the zeros of its radial limits follows.

Theorem 16 *Fix $\alpha \in (0, n]$ and let $f \in D_\alpha(\mathbb{B}_n)$. If $\mathrm{cap}_\alpha(\mathcal{Z}(f^*)) > 0$, then f is not cyclic in $D_\alpha(\mathbb{B}_n)$.*

Proof Let μ be a probability measure supported in $\mathcal{Z}(f^*)$, with finite Riesz n-energy. If $r \in (0,1)$, then

$$\begin{aligned} \log \frac{e}{|1 - r\langle \zeta, \eta \rangle|} &= 1 + \mathrm{Re}\Big(\log \frac{1}{1 - r\langle \zeta, \eta \rangle} \Big) \\ &= 1 + \mathrm{Re} \sum_{k=1}^{\infty} \sum_{|j|=k} \frac{r^k k!}{kj!} \zeta^j \overline{\eta}^j. \end{aligned}$$

Note that μ is a probability measure, and hence

$$\iint_{\mathbb{S}_n} \log \frac{e}{|1 - r\langle \zeta, \eta \rangle|} d\mu(\zeta) d\mu(\eta) = 1 + \sum_{k=1}^{\infty} \sum_{|j|=k} \frac{r^k k!}{kj!} |\mu^*(j)|^2.$$

Since $|1 - w|/|1 - rw| \leq 2$ for $r \in (0,1)$ and $w \in \overline{\mathbb{D}}$, the dominated convergence theorem and Lemma 13 give

$$\|C_{[\mu]}\|^2_{-n} \asymp \sum_{k=1}^{\infty} \sum_{|j|=k} \frac{k!}{kj!} |\mu^*(j)|^2 < \infty.$$

The assertion follows.

We continue setting a probability measure μ, supported in $\mathcal{Z}(f^*)$, with finite Riesz α-energy, where $\alpha \in (0, n)$. If $r \in (0,1)$, then

$$\frac{1}{(1 - r\langle \zeta, \eta \rangle)^{n-\alpha}} = \sum_{k=0}^{\infty} \sum_{|j|=k} \frac{\Gamma(k + n - \alpha) k! r^k}{k! \Gamma(n - \alpha) j!} \zeta^j \overline{\eta}^j.$$

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Similar arguments to the one above show that

$$\begin{aligned} I_\alpha[\mu] &\geq \Big| \iint_{\mathbb{S}_n} \mathrm{Re}\Big(\frac{1}{(1-r\langle\zeta,\eta\rangle)^{n-\alpha}}\Big) d\mu(\zeta) d\mu(\eta)\Big| \\ &= \Big| \sum_{k=0}^{\infty} \sum_{|j|=k} \frac{\Gamma(k+n-\alpha)k!r^k}{k!\Gamma(n-\alpha)j!} \iint_{\mathbb{S}_n} \zeta^j \overline{\eta}^j d\mu(\zeta) d\mu(\eta)\Big| \\ &\asymp \sum_{k=0}^{\infty} \sum_{|j|=k} \frac{(k+1)^{n-1-\alpha}k!}{j!} r^k |\mu^*(j)|^2. \end{aligned}$$

Again, letting $r \to 1^-$ by Lemma 13, we obtain that $C_{[\mu]} \in D_{-\alpha}(\mathbb{B}_n)$. The assertion follows. ∎

Remark 17 According to [14], one can expect that the cyclicity problem of polynomials in the unit ball of $\mathbb{C}^n$ depends on the real dimension of their zero set restricted on the unit sphere: $\dim_{\mathbb{R}}(\mathcal{Z}(p) \cap \mathbb{S}_n)$.

Let us point out the nature of the boundary zeros of a polynomial nonvanishing in the ball. See [14] for the two-dimensional case where had been used the Curve Selection Lemma of [10].

Let $p \in \mathbb{C}[z_1, \ldots, z_n]$ be a polynomial nonvanishing in the ball. Looking at $\mathcal{Z}(p) \cap \mathbb{S}_n$ as at a semi-algebraic set, we conclude that it is the disjoint union of a finite number of Nash manifolds M_i, each Nash diffeomorphic to an open hypercube $(0,1)^{\dim(M_i)}$. Note that the Nash diffeomorphisms over the closed field of the real numbers satisfy some additional properties (see [7, Proposition 2.9.10]).

One can expect then that the characterization of cyclicity and the nature of the boundary zeros of the model polynomials, as well as, the unitary invariance of the Dirichlet norm and the sufficient capacity condition, will be crucial in the characterization of cyclic polynomials in arbitrary dimension.

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