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STRATEGIC CUSTOMERS IN MARKOVIAN QUEUES WITH VACATIONS AND SYNCHRONIZED ABANDONMENT

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Abstract

We study impatient customers' joining strategies in a single-server Markovian queue with synchronized abandonment and multiple vacations. Customers receive the system information upon arrival, and decide whether to join or balk, based on a linear reward-cost structure under the acquired information. Waiting customers are served in a first-come-first-serve discipline, and no service is rendered during vacation. Server's vacation becomes the cause of impatience for the waiting customers, which leads to synchronous abandonment at the end of vacation. That is, customers consider simultaneously but independent of others, whether to renege the system or to remain. We are interested to study the effect of both information and reneging choice on the balking strategies of impatient customers. We examine the customers' equilibrium and socially optimal balking strategies under four cases of information: fully/almost observable and fully/almost unobservable cases, assuming the linear reward-cost structure. We compare the social benefits under all the information policies.

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1. Introduction

Waiting is an unavoidable annoyance for most people in this digital age, but they become more sensitive towards the level of information and the quality of service they receive. They might spare a little more time in a queue, if they receive proper care from the management. When their waiting times get extremely large, the impatience rises and they may decide to leave the queue before receiving service. This phenomenon in queueing literature is referred to as reneging or abandonment, which takes place in many real-life situations. Customers' impatience might be due to their long wait

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in queue, poor quality of service they receive or the absence of server (vacations). Queueing models with customer abandonment and various vacation policies were investigated thoroughly because of their diverse applications in productions, computer communications and other fields.

Markovian queueing systems with server vacations where the source of customers' impatience is the unavailability of the server, have been studied by Altman and Yechiali [2] and Yechiali [20]. The main presumption in their work is that customers carry out independent abandonments, that is, each customer upon encountering the vacation state, initiates his own patience timer which is independent of the patience times of other customers, and abandons the system once the timer expires. Similar to these works, Adan et al. [1], Economou and Kapodistria [7] and Kapodistria [13] considered vacation queueing models with impatient customers, but instead of independent abandonments, they assumed that impatient customers wait for a certain transport facility to abandon the system. They called this type of abandonment as synchronized or binomial abandonment. A third type of reneging, known as sequential or geometric abandonment, was presented by Dimou et al. [6], Dimou and Economou [5]. Both synchronized and sequential abandonments are incited by remote systems, where users have to look for the arrival of the secondary transport facility (with different capacity depending on the type of abandonment) to abandon the system.

The analysis of queueing systems with strategic customer behaviour has attracted interest in recent years, as it provides invaluable insight into the economic aspects of real-world systems. Extensive work on strategic queues were reported by Hassin and Haviv [11] and Hassin [9]. The introduction of server vacations into the strategic queueing games was initiated by Burnetas and Economou [4]. They studied equilibrium balking strategies of rational customers in an M/M/1 queue with setup times under four information cases. Variants of Markovian vacation queues with optimal strategic behaviour have been discussed in the literature (see [8, 14, 15, 19, 21] and the references therein).

The strategic behaviour of impatient customers in a Markovian setup was first studied by Assaf and Haviv [3]. They considered a processor sharing single-server Markovian queue, where customers know the number of customers waiting in front of them before making their abandonment decisions. They may choose to abandon the queue when it grows beyond a specific threshold value. Assaf and Haviv derived the symmetric ϵ-Nash equilibrium reneging strategies, and showed the stationarity with respect to the queue length. Several researchers complemented their work by studying the abandonment behaviour in unobservable queues. Hassin and Haviv [10] studied customers' abandonment in a single serve Markovian queue, where customers are rewarded if served within a fixed time and there is no reward if served after that. They assumed that a customer either completes service or reneges while in service, but he never reneges while waiting. Customers select a deadline and renege if their services do not end within that deadline. Haviv and Ritov [12] studied the symmetric Nash equilibrium reneging strategies under an increasing and convex waiting cost structure. Motivated by the remote service systems such as telephone call centres, several

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researchers [16, 18, 22] studied the equilibrium abandonment behaviour of impatient customers in a multi-server Markovian setup. All the above models presented the equilibrium abandonment behaviour of impatient customers in Markovian queues under the unobservable and observable cases separately. Recently, Panda et al. [17] presented a thorough analysis of customers' abandonment behaviour in both the cases. They analysed the impatient customers' strategic behaviour in an M/M/1 queue with server vacations and sequential abandonment where the source of customer impatience is the absence of server (vacation). During the server's vacation, waiting customers become impatient and decide sequentially whether they will abandon the system or not, depending on the availability of a secondary transport facility. This type of abandonment is motivated by the operation of a perishable inventory system.

In this paper, we are complementing the initial work of Panda et al. [17] to include synchronized abandonments. In the case of synchronized abandonments, every arriving customer at an abandonment epoch decides, independent of others, either to abandon or stay in the system with a certain probability. Thus, the probabilities associated with customers are simultaneously determined, and the number of customers in the system reduces in accordance with a binomial distribution. From an economic point of view, this model has applications in the areas of wireless sensor networks and cloud computing. This prompts us to extend previous research [17] featuring the equilibrium and optimal social behaviour under different scenarios of information. We study a Markovian queue with multiple vacations and synchronized abandonments under these scenarios. We assume four cases of information: the fully observable, almost observable, fully unobservable and almost unobservable cases. Whenever the system becomes empty, the server takes a vacation. At the vacation completion epochs, if the system is empty it takes another vacation. Otherwise, each present customer decides independently of the others, whether to abandon the system with probability p or remain in the system with probability $q = 1 - p$. We obtain customers' equilibrium balking and optimal social behaviour under each information level. Then we compare their strategies and compute the optimal social benefit. We find the stationary analysis of the system-length distribution and the mean sojourn times in all of the cases. The impact of various parameters on the equilibrium and optimal social thresholds have been represented through numerical experiments.

The main contributions of this work may be outlined as follows.

- We derive closed-form expressions for the computation of the customer equilibrium and social optimal strategies under the fully/almost observable and almost/fully unobservable cases. Equilibrium strategies are nondecreasing in the observable cases, and corresponding strategies are concave in the unobservable cases.
- A mixed dominance between the equilibrium threshold strategies ($n_e(0)$ in vacation and $n_e(1)$ in regular service) is observed. The condition of dominance

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is established. This behaviour is due to the synchronized abandonment of customers in the vacation state.

- Social benefit of the almost unobservable case dominates that of the fully observable case. Thus, revealing more information to the impatient customers is not beneficial for the social benefit of the system. Social benefit function is always concave in the unobservable case, whereas in the observable case, it is concave for specific model parameters.

The paper is organized as follows. We outline the dynamics of the model in Section 2. In Section 3, we study the model for observable cases and separately obtain the stationary analysis of the two information (fully and almost observable) cases. In Section 4, we present the strategic behaviour of customers in the almost and fully unobservable cases. We also find the equilibrium threshold and socially optimal behaviour of customers in all the information cases. Section 5 depicts some numerical results, and discusses the corresponding findings under each information policy. Section 6 concludes the paper.

2. Description of the model

We assume a single-server queueing system with multiple vacations, wherein customers enter the system according to a Poisson process with rate λ. Service is rendered to customers individually by a single server, and service times are independent and exponentially distributed with rate μ. The server can be in two modes: regular service (active) mode or vacation (inactive) mode. Customers are served with rate μ in the active mode, and no service is provided in the vacation mode. The server takes a vacation, if the system becomes empty at the end of a service completion epoch. At the vacation completion epoch, the server goes on another vacation if the server gets the system empty; otherwise, it switches to active mode and starts serving the present customers exhaustively. We assume multiple server vacations of exponentially distributed vacation times with rate ϕ. Customers become impatient during the vacation mode and carry out synchronized abandonments on the availability of a secondary transport facility. The transport facility arrives at the system at the end of each vacation period. When a vacation period ends, the customers take a decision whether to abandon or remain in the system. Each present customer at the abandonment/vacation completion epoch stays in the system with probability q or leaves the system with probability $p = 1 - q$, independently of the others. At every abandonment epoch, the number of customers is reduced according to a binomial distribution as a result of the synchronized departure of some of the present customers. We assume that the arrival process, the service and vacation times, and the abandonment process are mutually independent.

Let $N(t)$ be the number of customers in the system at time t, and

$$\zeta(t) = \begin{cases} 0 & \text{if the server is on inactive mode (vacation),} \\ 1 & \text{if the server is in active mode (busy).} \end{cases}$$

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The process $\{(N(t), \zeta(t)) \mid t \geq 0\}$ is a continuous time Markov chain (CTMC) with state space $\Omega = \{(n, i) \mid i = 0, 1, \ n \geq i\}$. The nonzero transition rates are

$$\hat{q}_{(n,i),(n+1,i)} = \lambda \quad n \geq i, \ i = 0, 1, \quad \hat{q}_{(n+1,1),(n,1)} = \mu \quad n \geq 1,$$
$$\hat{q}_{(1,1),(0,0)} = \mu, \quad \hat{q}_{(n,0),(n,1)} = \phi q^n \quad n \geq 1,$$
$$\hat{q}_{(n,0),(n-k,1)} = \phi \binom{n}{k} p^k q^{n-k} \quad 1 \leq k \leq n-1, \ n \geq 1, \quad \hat{q}_{(n,0),(0,0)} = \phi p^n \quad n \geq 1.$$

Using these transition rates, the steady-state balance equations for the model are

$$\lambda \pi_{0,0} = \mu \pi_{1,1} + \phi \sum_{j=1}^{\infty} p^j \pi_{j,0},$$
$$(\lambda + \phi)\pi_{n,0} = \lambda \pi_{n-1,0} \quad n \geq 1,$$
$$(\lambda + \mu)\pi_{1,1} = \mu \pi_{2,1} + \phi \sum_{j=1}^{\infty} jqp^{j-1} \pi_{j,0},$$
$$(\lambda + \mu)\pi_{n,1} = \mu \pi_{n+1,1} + \lambda \pi_{n-1,1} + \phi \sum_{j=n}^{\infty} \binom{j}{j-n} q^n p^{j-n} \pi_{j,0} \quad n \geq 2,$$

where $\pi_{n,i} = \lim_{t\to\infty} P\{N(t) = n, \zeta(t) = i \mid i = 0, 1 \text{ and } n \geq i\}$ is the stationary probability distribution of the CTMC $\{(N(t), \zeta(t))\}$. The normalization condition for solving the balance equations is

$$\sum_{n=0}^{\infty} \pi_{n,0} + \sum_{n=1}^{\infty} \pi_{n,1} = 1. \tag{2.1}$$

Figure 1 presents the transition rate diagram of the CTMC. We are concerned in the behaviour of customers, when upon their arrival they decide whether to enter or balk the system. We presume that each customer gets a reward of R units for finishing service. Further, there is a waiting cost of C units per time unit that gather. during which they remain in the system. We define a linear cost-reward function to analyse a customer's expected net benefit subsequent to completion of service as $\Delta = R - CT(n, i)$ (or $\Delta = R - CE(W)$), where $T(n, i)(E(W))$ corresponds to the mean sojourn time of an arriving customer in observable (unobservable) queue. The customers take decisions only at their arrival instants to maximize their expected net benefit. Because customers are permitted to take their decisions, and there is a homogeneous reward-cost structure for all customers, the system can model as a noncooperative and symmetric game among the customers. We are interested only in the balking/joining strategies of impatient customers, not in the reneging strategies of impatient customers. We assume that their decision is irrevocable, that is, retrials of customers (both balking and reneging customers) are not allowed. Further, the net benefit of balking customers and reneging customers is zero.

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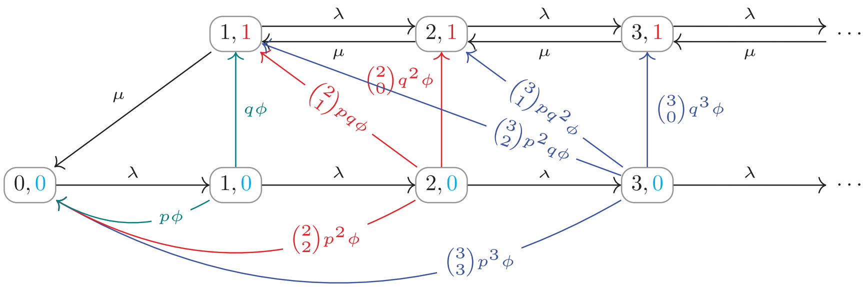

FIGURE 1. Transition rate diagram of the original model.

3. Analysis of the observable queues

In this section, we assume that upon arrival, customers are informed of the system length (observable queues). Further, it can be studied under two levels: (1) fully observable case, that is, arriving customers observe the state of the server and the system length; (2) almost observable case, that is, arriving customers observe only the system length. Based on the information available, customers take a decision at their arrival instant whether to join or to balk the system. We will study the fully observable case followed by the almost observable case in the next section.

3.1. Fully observable queue In this case, arriving customers have exact system state information before making decision, whether to join or to balk. The equilibrium balking strategy in the fully observable case is a pure strategy of threshold type. We are interested to find the balking threshold $n_e(i)$, such that an arriving customer enters the system at state i $(i = 0, 1)$, if the number of the customers present upon arrival does not exceed the specified threshold. Thus, a pure threshold strategy defined by the pair $(n_e(0), n_e(1))$ and the form of balking strategy is "observe $(N(t), \zeta(t))$ when arriving at time t; join if $N(t) \le n_e(\zeta(t))$ and balk otherwise".

THEOREM 3.1. *In a fully observable M/M/1 queue with synchronized abandonments and multiple vacations, there exists a unique equilibrium threshold strategy*

$$(n_e(0), n_e(1)) = \left(\left\lfloor \frac{1}{q}\left(\frac{\mu R}{Cq} - \frac{\mu}{\phi q} - 1\right)\right\rfloor, \left\lfloor \frac{\mu R}{C} \right\rfloor - 1\right),$$

such that a customer who observes the system at state $(N(t), \zeta(t))$ upon arrival, enters if $N(t) \le n_e(\zeta(t))$ and balks otherwise.

PROOF. An arriving customer who observes the system in state (n, i) will join the system, if the service completion reward is higher than his total waiting cost. Let $\Delta_{fo}(n, i)$ represent the expected net benefit of a tagged customer who observes the system state (n, i) on arrival and decides to join, which is given by

$$\Delta_{fo}(n, i) = R - CT(n, i),$$

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where $T(n, i) = E(S \mid N = n, \zeta = i)$ is the expected mean sojourn time of the tagged customer who finds the system in state (n, i) upon arrival. Conditioning on system state, we have the following system that represents the mean sojourn time of the tagged customer:

$$T(0,0) = \frac{1}{\phi} + \frac{q}{\mu}, \tag{3.1a}$$

$$T(n,0) = \frac{1}{\phi} + \frac{nq^2}{\mu} + \frac{q}{\mu}, \quad n \geq 1, \tag{3.1b}$$

$$T(n,1) = \frac{n+1}{\mu}, \quad n \geq 1, \tag{3.1c}$$

where $(n+1)/\mu$ is the mean service time of the tagged customer who finds n customers in the system on arrival, and $1/\phi$ is the mean residual vacation time. The primary presumption regarding the reward cost structure is that a customer is willing to join the queue, upon arrival to an empty system. Specifically, the reward for service must be higher than the total waiting cost of an arriving customer who finds the system empty (state $(0, 0)$). When an arriving customer encounters the system on state $(0, 0)$, his sojourn time will be p/ϕ or $q(1/\phi + 1/\mu)$ if he abandons or remains in the system, respectively. Thus, his mean sojourn time is $(1/\phi + q/\mu)$, which is given in (3.1a). We assume

$$R > C\left(\frac{1}{\phi} + \frac{q}{\mu}\right)$$

for the equilibrium analysis of the balking strategies in all cases. If the tagged customer encounters system state $(n, 0)$ and decides to abandon the system with probability p, his mean sojourn time will be his waiting time till vacation completion. If he decides to remain in the system with probability q, then his mean sojourn time will be the sum of his waiting time till service completions of n customers and his own service time. His waiting time will depend on the number of customers that decide to remain in the system at the vacation completion or abandonment epoch. If m customers remain in the system with probability q, and the rest $n - m$ customers abandon the system with probability p, then his mean sojourn time is

$$\frac{1}{\phi} + \sum_{m=0}^{n} \binom{n}{m} p^{n-m} q^m \frac{m+1}{\mu}$$

with probability q. Thus, we have

$$T(n,0) = \frac{p}{\phi} + q\left(\frac{1}{\phi} + \sum_{m=0}^{n} \binom{n}{m} p^{n-m} q^m \frac{m+1}{\mu}\right),$$

which on simplification results in (3.1b). Similarly, if the tagged customer finds the system state $(n, 1)$ upon arrival, then his mean sojourn time will be the sum of $(n + 1)$ service times with mean μ^{-1}. As the customers are indistinguishable and only interested in maximizing their own benefit, there is a symmetric game among the

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customers. We are interested to compute the Nash equilibrium solution of this game. That is, under this equilibrium (threshold strategy) all joining customers are benefited, and a customer deviating from this will not be benefited.

Thus, an arbitrary customer who confronts n customers in the system on arrival, strictly prefers to enter the system if $\Delta_{fo}(n, i) > 0$ and is indifferent between entering and balking if $\Delta_{fo}(n, i) = 0$, and does not join the system if $\Delta_{fo}(n, i) < 0$. Hence, the customer arriving at time t will join the system if and only if $N(t) \le n_e(j)$, where $(n_e(0), n_e(1))$ are found by solving $\Delta_{fo}(n, 0) = 0$ and $\Delta_{fo}(n, 1) = 0$. □

REMARK 3.2. The literature on queueing games with vacation reveals that in the fully observable case, the equilibrium balking threshold in the busy period always dominates the equilibrium balking threshold in the vacation period. Because the queue forms quicker in case of a vacation period as opposed to regular service period, the relation among the thresholds $n_e(0) < n_e(1)$ is prevalent in literature. Here, this dominance relation holds under certain restrictions on the behaviour of the impatient customers. We derive the condition of dominance $(n_e(1) > n_e(0))$ to be $q > q^*$, where

$$q^* = \frac{\mu}{\phi p} \frac{C - R\phi p}{\mu R - C}.$$

Whenever the dominance condition fails, that is, $q \le q^*$, the threshold of an inactive server is higher than that of an active server. This peculiar behaviour of the thresholds is the effect of synchronized abandonment of impatient customers in the inactive mode of the server that brings down even higher thresholds in vacation to very small values. Some numerical experiments also agree with it under certain variation in the model parameters.

3.1.1 *Socially optimal balking strategy.* Next, we are interested in the social benefit of all joining customers under the equilibrium threshold $(n_e(0), n_e(1))$ and the socially optimal joining thresholds $(n^*(0), n^*(1))$, where $n^*(i)$ is the threshold that optimizes the social benefit of all customers when they join the system in state (n, i). To compute this, we need the stationary analysis of the fully observable system under the equilibrium threshold strategy $(n_e(0), n_e(1))$. In our Markov chain analysis, we have assumed the relation $n_e(1) > n_e(0)$.

LEMMA 3.3. *Consider a fully observable M/M/1 queue with synchronized abandonments and multiple vacations, in which customers follow the threshold policy* $(n_e(0), n_e(1))$. *Then the stationary probabilities* $\{\pi_{n,i} \mid (n, i) \in \Omega_{fo}\}$ *are given by*

$$\pi_{n,0} = \left(\frac{\lambda}{\phi}\right)^{\delta_{n,n_e(0)+1}} \left(\frac{\lambda}{\lambda + \phi}\right)^{n_e(0)} \pi_{0,0}, \quad n = 0, 1, \dots, n_e(0) + 1,$$
$$\pi_{n,1} = \psi_n \pi_{0,0}, \quad 1 \le n \le n_e(1) + 1,$$

where ψ_n *and* $\pi_{0,0}$ *are given in* (3.3), *and* $\delta_{i,j} = 1$ *for* $i = j$ *and 0 otherwise, is the Kronecker delta function.*

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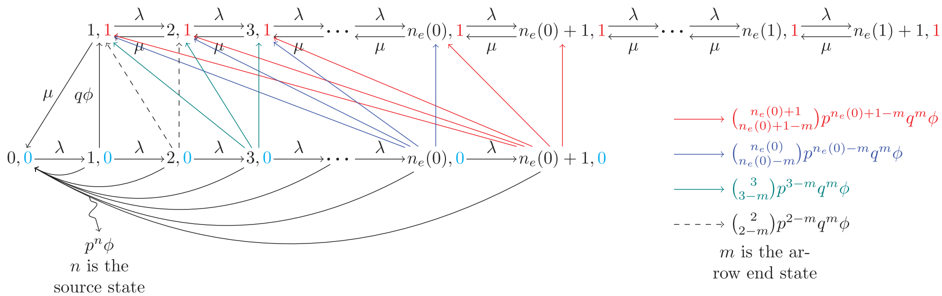

FIGURE 2. Transition rate diagram for the threshold strategy $(n_e(0), n_e(1))$.

PROOF. Assuming that all customers follow the same threshold strategy $(n_e(0), n_e(1))$, the fully observable model works as a finite buffer M/M/1 queue, with buffer capacities $n_e(0)$ during vacation and $n_e(1)$ during normal service. The transition rate diagram of the underlying CTMC is depicted in Figure 2.

In the limiting case, define the stationary probability distributions as

$$\pi_{k,0} = \lim_{t\to\infty} P\{N(t) = k, \zeta(t = 0)\}, \quad 0 \le k \le n_e(0) + 1,$$
$$\pi_{k,1} = \lim_{t\to\infty} P\{N(t) = k, \zeta(t = 1)\}, \quad 1 \le k \le n_e(1) + 1,$$

where $\pi_{k,0}$ ($\pi_{k,1}$) is the probability that there are k customers in the system in steady-state when the server is on vacation (busy). We obtain the stationary probabilities from the system of balance equations given below

$$\lambda\pi_{0,0} = \mu\pi_{1,1} + \phi \sum_{j=1}^{n_e(0)+1} p^j \pi_{j,0}, \tag{3.2a}$$

$$(\lambda + \phi)\pi_{n,0} = \lambda\pi_{n-1,0}, \quad 1 \le i \le n_e(0), \tag{3.2b}$$

$$\phi\pi_{n_e(0)+1,0} = \lambda\pi_{n_e(0),0}, \tag{3.2c}$$

$$(\lambda + \mu)\pi_{1,1} = \mu\pi_{2,1} + \phi q \sum_{j=1}^{n_e(0)+1} j p^{j-1} \pi_{j,0}, \tag{3.2d}$$

$$(\lambda + \mu)\pi_{n,1} = \mu\pi_{n+1,1} + \lambda\pi_{n-1,1} + \phi \sum_{j=n}^{n_e(0)+1} \binom{j}{j-n} q^n p^{j-n} \pi_{j,0}, \quad 2 \le n \le n_e(0) + 1, \tag{3.2e}$$

$$(\lambda + \mu)\pi_{n,1} = \lambda\pi_{n-1,1} + \mu\pi_{n+1,1}, \quad n_e(0) + 2 \le n \le n_e(1), \tag{3.2f}$$

$$\mu\pi_{n_e(1)+1,1} = \lambda\pi_{n_e(1),1}. \tag{3.2g}$$

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From (3.2b) and (3.2c),

$$\pi_{n,0} = \left(\frac{\lambda}{\lambda+\phi}\right)^n \pi_{0,0}, \quad 1 \le n \le n_e(0),$$
$$\pi_{n_e(0)+1,0} = \frac{\lambda}{\phi}\left(\frac{\lambda}{\lambda+\phi}\right)^{n_e(0)} \pi_{0,0}.$$

Using the equations (3.2a) and (3.2d)–(3.2f), we have

$$\pi_{n,1} = \psi_n \pi_{0,0}, \quad 1 \le n \le n_e(1)+1,$$

where

$$\psi_1 = \frac{\rho q\,(\lambda+\phi)}{\phi+\lambda q}\left[1-\left(\frac{\lambda p}{\lambda+\phi}\right)^{n_e(0)+1}\right],$$
$$\psi_2 = (1+\rho)\psi_1 - \frac{\phi q}{p\mu}\sum_{j=1}^{n_e(0)} j\left(\frac{\lambda p}{\lambda+\phi}\right)^j - \rho q\,(n_e(0)+1)\left(\frac{\lambda p}{\lambda+\phi}\right)^{n_e(0)},$$
$$\psi_{n+1} = (1+\rho)\psi_n - \rho\psi_{n-1} - \left(\frac{q}{p}\right)^n \frac{\phi}{\mu}\sum_{j=n}^{n_e(0)} \binom{j}{j-n}\left(\frac{\lambda p}{\lambda+\phi}\right)^j$$
$$- \rho p\left(\frac{q}{p}\right)^n \binom{n_e(0)+1}{n_e(0)+1-n}\left(\frac{\lambda p}{\lambda+\phi}\right)^{n_e(0)}, \quad n = 2,3,\ldots,n_e(0)+1,$$
$$\psi_{n+1} = (1+\rho)\psi_n - \rho\psi_{n-1}, \quad n = n_e(0)+2,\ldots,n_e(1).$$

We have thus expressed all stationary probabilities in terms of the only unknown $\pi(0,0)$, which can be obtained using the normalization condition

$$\pi_{0,0} = \left[\frac{\lambda+\phi}{\phi} + \sum_{j=1}^{n_e(1)+1} \psi_j\right]^{-1}. \tag{3.3}$$

□

The probability of balking is equal to $P_{\text{balk}} = \pi_{n_e(0)+1,0} + \pi_{n_e(1)+1,1}$ due to the PASTA property (Poisson Arrivals See Time Averages). Thus the effective arrival rate is $\lambda(1 - P_{\text{balk}})$. If all joining customers follow the equilibrium threshold strategy $(n_e(0), n_e(1))$, then the social benefit per time unit in equilibrium, denoted by $\Delta_s(n_e(0), n_e(1))$, can be expressed as

$$\Delta_s(n_e(0),n_e(1)) = R\lambda(1-\psi_{n_e(0)+1}\pi_{0,0}) - C\left(\frac{\lambda(\lambda+\phi)}{\phi^2} + \sum_{n=1}^{n_e(1)+1} n\psi_n\right)\pi_{0,0}$$
$$- \left(R-\frac{C}{\phi}\right)\frac{\lambda^2}{\phi}\left(\frac{\lambda}{\lambda+\phi}\right)^{n_e(0)} \pi_{0,0}.$$

A social planner is interested to find the joining thresholds for which the social benefit of the whole system is optimized. Denoting the socially optimal threshold strategy

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by $(n^*(0), n^*(1))$, the social planner wants to optimize $\Delta_s(n(0), n(1))$. A closed form expression for the socially optimal strategy is intractable. However, it is possible to evaluate these thresholds numerically using softwares that helps to solve unconstrained integer programming problems.

Remark 3.4. When the dominance condition fails, that is, $q \leq \mu(C - R\phi p)/\phi p(\mu R - C)$, the equilibrium thresholds satisfy $n_e(1) \leq n_e(0)$. Under this situation, the steady-state distribution in this case can be computed by solving the following system of balance equations:

$$\lambda\pi_{0,0} = \mu\pi_{1,1} + \phi \sum_{j=1}^{n_e(0)+1} p^j \pi_{j,0} \tag{3.4a}$$

$$(\lambda + \phi)\pi_{n,0} = \lambda\pi_{n-1,0}, \quad 1 \leq i \leq n_e(0), \tag{3.4b}$$

$$\phi\pi_{n_e(0)+1,0} = \lambda\pi_{n_e(0),0}, \tag{3.4c}$$

$$(\lambda + \mu)\pi_{1,1} = \mu\pi_{2,1} + \phi q \sum_{j=1}^{n_e(0)+1} j p^{j-1} \pi_{j,0}, \tag{3.4d}$$

$$(\lambda + \mu)\pi_{n,1} = \mu\pi_{n+1,1} + \lambda\pi_{n-1,1} + \phi \sum_{j=n}^{n_e(0)+1} \binom{j}{j-n} q^n p^{j-n} \pi_{j,0}, \quad 2 \leq n \leq n_e(1), \tag{3.4e}$$

$$\mu\pi_{n_e(1)+1,1} = \lambda\pi_{n_e(1),1} + \phi \sum_{k=n_e(1)+1}^{n_e(0)+1} \sum_{j=k}^{n_e(0)+1} \binom{j}{j-k} q^k p^{j-k} \pi_{j,0}. \tag{3.4f}$$

From (3.4b) and (3.4c)

$$\pi_{n,0} = \left(\frac{\lambda}{\lambda + \phi}\right)^n \pi_{0,0}, \quad 1 \leq n \leq n_e(0),$$

$$\pi_{n_e(0)+1,0} = \frac{\lambda}{\phi}\left(\frac{\lambda}{\lambda + \phi}\right)^{n_e(0)} \pi_{0,0}.$$

Using the equations (3.4a) and (3.4d)–(3.4f), we have

$$\pi_{n,1} = \psi_n \pi_{0,0}, \quad 1 \leq n \leq n_e(1) + 1,$$

where

$$\psi_1 = \frac{\rho q(\lambda + \phi)}{\phi + \lambda q}\left[1 - \left(\frac{\lambda p}{\lambda + \phi}\right)^{n_e(0)+1}\right],$$

$$\psi_2 = (1 + \rho)\psi_1 - \frac{\phi q}{p\mu} \sum_{j=1}^{n_e(0)} j\left(\frac{\lambda p}{\lambda + \phi}\right)^j - \rho q\,(n_e(0) + 1)\left(\frac{\lambda p}{\lambda + \phi}\right)^{n_e(0)},$$

$$\psi_{n+1} = (1 + \rho)\psi_n - \rho\psi_{n-1} - \left(\frac{q}{p}\right)^n \frac{\phi}{\mu} \sum_{j=n}^{n_e(0)} \binom{j}{j-n} \left(\frac{\lambda p}{\lambda + \phi}\right)^j$$
$$- \rho p\left(\frac{q}{p}\right)^n \binom{n_e(0) + 1}{n_e(0) + 1 - n} \left(\frac{\lambda p}{\lambda + \phi}\right)^{n_e(0)}, \quad n = 2, 3, \ldots, n_e(1).$$

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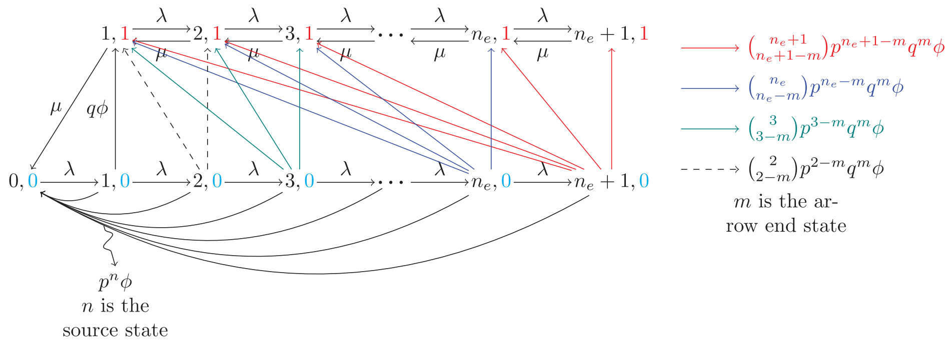

FIGURE 3. Transition rate diagram for the threshold strategy n_e.

3.2. Almost observable queue In this section, we examine the almost observable queues. The customers upon arrival before making decisions, observe the system length, but they are not informed about the server state, that is, they cannot differentiate whether the server is on vacation or busy period. A pure threshold strategy is defined by (all arriving customers apply the indistinguishable threshold for joining) considering $n_e(0) = n_e(1) = n_e$ in the fully observable case. The balking strategy has the form "observe $N(t)$ at arrival instant t; join if $N(t) \le n_e$ and balk otherwise". The state space of the representing Markov chain is

$$\Omega_{ao} = \{(k,0) \mid 0 \le k \le n_e + 1\} \cup \{(k,1) \mid 1 \le k \le n_e + 1\},$$

and the transition rate diagram is specified in Figure 3.

The system-length distribution can be obtained by solving the balance equations (3.5), which are deduced from the system of balance equations (3.2) by substituting $n_e(0) = n_e(1) = n_e$:

$$\lambda\pi_{0,0} = \mu\pi_{1,1} + \phi \sum_{j=1}^{n_e+1} p^j \pi_{j,0}, \tag{3.5a}$$

$$(\lambda+\phi)\pi_{n,0} = \lambda\pi_{n-1,0}, \quad 1 \le i \le n_e, \tag{3.5b}$$

$$\phi\pi_{n_e+1,0} = \lambda\pi_{n_e,0}, \tag{3.5c}$$

$$(\lambda+\mu)\pi_{1,1} = \mu\pi_{2,1} + \phi q \sum_{j=1}^{n_e+1} j p^{j-1}\pi_{j,0}, \tag{3.5d}$$

$$(\lambda+\mu)\pi_{n,1} = \mu\pi_{n+1,1} + \lambda\pi_{n-1,1} + \phi q^n \sum_{j=n}^{n_e+1} \binom{j}{j-n} p^{j-n}\pi_{j,0}, \quad 2 \le n \le n_e, \tag{3.5e}$$

$$\mu\pi_{n_e+1,1} = \lambda\pi_{n_e,1} + \phi q^{n_e+1}\pi_{n_e+1,0}. \tag{3.5f}$$

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Following the analysis similar to the fully observable case, we get the stationary distributions of the above system of balance equations

$$\begin{aligned}\pi_{n,0} &= \left(\frac{\lambda}{\lambda+\phi}\right)^{n}\pi_{0,0}, \quad n = 0, 1, \ldots, n_e,\\ \pi_{n_e+1,0} &= \frac{\lambda}{\phi}\left(\frac{\lambda}{\lambda+\phi}\right)^{n_e}\pi_{0,0},\\ \pi_{n,1} &= \psi_n\pi_{0,0}, \quad 1 \le n \le n_e+1,\end{aligned}$$

where

$$\begin{aligned}\psi_1 &= \frac{\rho q\,(\lambda+\phi)}{\phi+\lambda q}\left[1-\left(\frac{\lambda p}{\lambda+\phi}\right)^{n_e+1}\right],\\ \psi_2 &= (1+\rho)\psi_1 - \frac{\phi q}{\mu p}\sum_{j=1}^{n_e} j\left(\frac{\lambda p}{\lambda+\phi}\right)^{j} - \rho q(n_e+1)\left(\frac{\lambda p}{\lambda+\phi}\right)^{n_e},\\ \psi_{n+1} &= (1+\rho)\psi_n - \rho\psi_{n-1} - \frac{\phi}{\mu}\left(\frac{q}{p}\right)^{n}\sum_{j=n}^{n_e}\binom{j}{j-n}\left(\frac{\lambda p}{\lambda+\phi}\right)^{j}\\ &\quad - \rho p\left(\frac{q}{p}\right)^{n}\binom{n_e+1}{n_e+1-n}\left(\frac{\lambda p}{\lambda+\phi}\right)^{n_e}, \quad n = 2, 3, \ldots, n_e.\end{aligned}$$

Now, using the normalization condition, we compute

$$\pi_{0,0} = \left(\frac{\lambda+\phi}{\phi} + \sum_{j=1}^{n_e+1}\psi_j\right)^{-1}.$$

3.2.1 *Equilibrium and socially optimal balking strategy.* Let $T(n)$ be the mean sojourn time of a tagged customer who notices n customers in the system upon arrival. The expected net benefit of an arriving customer who decides to join when observes n customers ahead, is $\Delta_{ao}(n) = R - CT(n)$. Conditioning on the state of the server which has been withstood by the tagged customer, we get

$$T(n) = T(n,0)P(\zeta = 0 \mid N = n) + T(n,1)P(\zeta = 1 \mid N = n),$$

where $P(\zeta = i \mid N = n)$ is the probability that the tagged customer gets the server at state i, given that there are n customers in the system. Applying the PASTA property, we have

$$P(\zeta = 0 \mid N = n) = \frac{\pi_{n,0}}{\pi_{n,0}+\pi_{n,1}} = \begin{cases}\dfrac{\lambda^{n}}{\lambda^{n}+\psi_n(\lambda+\phi)^{n}} & n = 0, 1, \ldots, n_e,\\[2ex] \dfrac{\lambda^{n_e+1}}{\lambda^{n_e+1}+\psi_{n_e+1}\phi(\lambda+\phi)^{n_e}} & n = n_e+1,\end{cases}$$

$$P(\zeta = 1 \mid N = n) = \frac{\pi_{n,1}}{\pi_{n,0}+\pi_{n,1}} = 1 - P(\zeta = 0 \mid N = n), \quad n = 1, 2, \ldots, n_e+1.$$

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Thus, the mean sojourn time of the tagged customer who observes n customers in the system, becomes

$$T(n) = \begin{cases} \left(\dfrac{\mu+\phi q}{\phi\mu}+\dfrac{q^2 n}{\mu}\right)\dfrac{\lambda^n}{\lambda^n+\psi_n(\lambda+\phi)^n}+\dfrac{n+1}{\mu}\dfrac{\psi_n(\lambda+\phi)^n}{\lambda^n+\psi_n(\lambda+\phi)^n}, & 0\le n\le n_e, \\ \left(\dfrac{\mu+\phi q}{\phi\mu}+\dfrac{q^2 n}{\mu}\right)\dfrac{\lambda^n}{\lambda^n+\psi_n\phi(\lambda+\phi)^{n-1}}+\dfrac{n+1}{\mu}\dfrac{\psi_n\phi(\lambda+\phi)^{n-1}}{\lambda^n+\psi_n\phi(\lambda+\phi)^{n-1}}, & n=n_e+1. \end{cases}$$

Clearly, the number of customers in the system, $T(n)$, is an increasing function of n. This is intuitive, as a customer who notices more numbers in the system after joining will have to wait longer than the customer who encounters fewer numbers in the system for a fixed abandonment rate. Alternatively, one can find the derivative

$$\frac{d}{dn}T(n)=\frac{1}{\mu}\frac{\lambda^n q^2+\psi_n(\lambda+\phi)^n}{\lambda^n+\psi_n(\lambda+\phi)^n}+\left[\frac{(1-q)(qn+n+1)}{\mu}-\frac{1}{\phi}\right]\frac{\psi_n\lambda^n(\lambda+\phi)^n}{\{\lambda^n+\psi_n(\lambda+\phi)^n\}^2},$$

which is always positive for $n\ge 1$. Now for the tagged customer who finds n customers in the system and decides to join, given that all other customers follow a threshold strategy n_e, his expected net benefit will be

$$\Delta_{n_e,ao}(n)=R-C\,\frac{n+1}{\mu}\,\frac{\psi_n(\lambda+\phi)^n}{\lambda^n+\psi_n(\lambda+\phi)^n}-C\left(\frac{\mu+\phi q}{\phi\mu}+\frac{q^2 n}{\mu}\right)\frac{\lambda^n}{\lambda^n+\psi_n(\lambda+\phi)^n}.$$

Since $T(n)$ is an increasing function of n, we have $\Delta_{n_e,ao}(n)$ a decreasing function of n for a fixed threshold strategy n_e. To show that such a threshold exists, we define the functions

$$f(n)=\left(\frac{\mu+\phi q}{\phi\mu}+\frac{q^2 n}{\mu}\right)\frac{\lambda^n}{\lambda^n+\psi_n(\lambda+\phi)^n}+\frac{n+1}{\mu}\frac{\psi_n(\lambda+\phi)^n}{\lambda^n+\psi_n(\lambda+\phi)^n},\quad n\ge 0,$$
$$g(n)=\left(\frac{\mu+\phi q}{\phi\mu}+\frac{q^2 n}{\mu}\right)\frac{\lambda^n}{\lambda^n+\psi_n\phi(\lambda+\phi)^{n-1}}+\frac{n+1}{\mu}\frac{\psi_n\phi(\lambda+\phi)^{n-1}}{\lambda^n+\psi_n\phi(\lambda+\phi)^{n-1}},\quad n\ge 1,$$

One can check that $f(n)\le g(n)$ for $n\ge 1$. Now, define the following decreasing sequences $S_1(n)=R-Cf(n), n=0,1,2,\dots$ and $S_2(n)=R-Cg(n), n=1,2,\dots$, such that both satisfy $S_2(n)\le S_1(n)$ for $n\ge 1$. We have $S_1(0)=R-CT(0)>0$; otherwise, no customer will join when the system is empty. Note that

$$\lim_{n\to\infty}S_1(n)=-\infty=\lim_{n\to\infty}S_2(n).$$

Hence, there exists a finite nonnegative integer n_U, such that $S_1(i)>0$ for $i=0,1,2,\dots,n_U$ and $S_1(n_U+1)\le 0$. Since, $S_2(n)\le S_1(n)$ for $n\ge 1$, we have, $S_2(n_U+1)<S_1(n_U+1)\le 0$. Using the similar arguments for the sequence $S_2(n)$, that is, $S_2(0)>0$ and $S_2(n_U+1)\le 0$, we get a finite nonnegative integer $n_L\le n_U$ such that $S_2(n_U+1), S_2(n_U), S_2(n_U-1),\dots,S_2(n_L+1)\le 0$ and $S_2(n_L)>0$. Hence, the threshold strategy n_e satisfies $n_e\in\{n_L,\dots,n_U\}$.

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If the tagged customer finds $n(\leq n_e)$ customers ahead of him and decides to enter when all other customers follow the same threshold strategy n_e, his expected net benefit is equal to $\Delta_{n_e,ao}(n) \geq \Delta_{n_e,ao}(n_e) = S_1(n_e) > 0$. So, in this case, the customer's decision to join brings a positive net benefit for him. Hence, the tagged customer prefers to join the system. On the other hand, if the tagged customer finds $n_e + 1$ (higher than the threshold n_e) customers ahead of him, and decides to enter when all other customers follow the same threshold strategy n_e, his expected net benefit is equal to $\Delta_{n_e,ao}(n_e + 1) = S_2(n_e + 1) \leq 0$. So, in this case, the tagged customer's decision to enter is not beneficial for him. Hence, he prefers to balk. Therefore, every threshold value n_e in between n_L and n_U is an equilibrium threshold in the almost observable case.

To calculate the benefit of all the present customers in the system, we need to find the effective arrival rate $\lambda_{\text{eff}} = \lambda(1 - P_{\text{balk}})$, where P_{balk} is the probability of balking and is given by

$$P_{\text{balk}} = \left(\frac{\lambda}{\phi}\left(\frac{\lambda}{\lambda + \phi}\right)^{n_e} + \psi_{n_e+1}\right)\pi_{0,0}.$$

Now, the social benefit per time unit, when all the customers follow the same equilibrium threshold strategy n_e is

$$\Delta_s(n_e) = R\lambda(1 - \psi_{n_e+1}\pi_{0,0}) - C\left(\frac{\lambda(\lambda + \phi)}{\phi^2} + \sum_{n=1}^{n_e+1} n\psi_n\right)\pi_{0,0} - \left(R - \frac{C}{\phi}\right)\frac{\lambda^2}{\phi}\left(\frac{\lambda}{\lambda + \phi}\right)^{n_e}\pi_{0,0}.$$

Similar to the fully observable case, a socially optimal threshold strategy n^* under which the social benefit of the whole system is optimized, can be computed numerically by solving $\Delta_s(n) = 0$. Although this computation is less complex than that of the fully observable case, a closed form expression is still intractable.

4. Analysis of unobservable queues

In the unobservable queues, customers are not aware of the information on the system length at their decision making stages (whether to balk or join the system), but they have the information about the state of the server. Based on the server state information, unobservable queues can be categorized into the following: (i) almost unobservable, that is, arriving customers know the server state $\zeta(t)$ before making join or balk decision; (ii) fully unobservable, that is, arriving customers do not have any information about the server state before making decisions. In the almost unobservable case, a mixed equilibrium strategy is specified by a pair (d_0, d_1), where d_0 $(0 \leq d_0 \leq 1)$ is the joining probability, when the customer wishes to join the system during vacation mode and $d_1(0 \leq d_1 \leq 1)$ is the joining probability, when the customer wishes to join the system during normal service mode. Similarly, a mixed strategy is assigned by the joining probability d, in the fully unobservable case. In both the cases, we are interested to find the Nash equilibrium mixed joining strategies.

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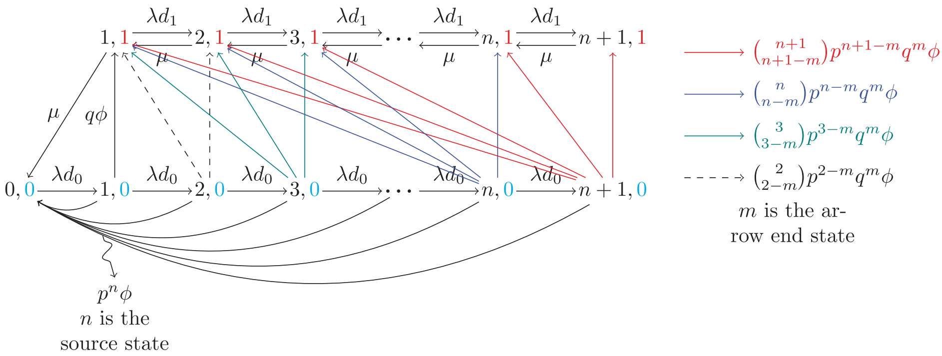

FIGURE 4. Transition rate diagram for the mixed strategy (d_1, d_0).

4.1. Almost unobservable queue We first consider the almost unobservable case, where, upon arrival, customers are aware of the state of the server. Suppose that all customers in the system follow a mixed joining strategy (d_0, d_1). The associated queueing model is the same as the original one with potential arrival rates λd_i to the system when server state is i. The state space of the Markov chain $\{(N(t), \zeta(t)), t \geq 0\}$ is $\Omega_{au} = \{(n, 0), (n + 1, 1) \mid n \geq 0\}$, and the state transitions are demonstrated in Figure 4. The existence of the stationary probabilities $\pi_{k,i}$ for $(k, i) \in \Omega_{au}$ is guaranteed by the stability criteria $\lambda < \mu$, and the probabilities can be computed by solving the following balance equations:

$$\lambda d_0\pi_{0,0} = \mu\pi_{1,1} + \phi\sum_{j=1}^{\infty} p^j\pi_{j,0},$$

$$(\lambda d_0 + \phi)\pi_{n,0} = \lambda d_0\pi_{n-1,0}, \quad n \geq 1, \tag{4.1a}$$

$$(\lambda d_1 + \mu)\pi_{1,1} = \mu\pi_{2,1} + \phi q\sum_{j=1}^{\infty} jp^{j-1}\pi_{j,0}, \tag{4.1b}$$

$$(\lambda d_1 + \mu)\pi_{n,1} = \mu\pi_{n+1,1} + \lambda d_1\pi_{n-1,1} + \phi\sum_{j=n}^{\infty}\binom{j}{j-n}q^n p^{j-n}\pi_{j,0}, \quad n \geq 2. \tag{4.1c}$$

We define the probability generating functions (PGFs) as

$$\Pi_0(z) = \sum_{n=0}^{\infty}\pi_{n,0}z^n \quad \text{and} \quad \Pi_1(z) = \sum_{n=1}^{\infty}\pi_{n,1}z^n, \quad |z| \leq 1.$$

Multiplying (4.1a) by z^n, $n \geq 1$ and then simplifying, we obtain

$$\Pi_0(z) = \frac{\phi + \lambda d_0}{\phi + \lambda d_0 - \lambda d_0 z}\pi_{0,0} = \sum_{n=0}^{\infty}\left(\frac{\lambda d_0 z}{\phi + \lambda d_0}\right)^n\pi_{0,0}. \tag{4.2}$$

Multiplying (4.1b) and (4.1c) by an appropriate power of z^n and summing over n,

$$(\lambda d_1 + \mu)\Pi_1(z) = \lambda d_1 z\Pi_1(z) + \frac{\mu\Pi_1(z)}{z} - \mu\pi_{1,1} + \phi\Pi_0(p + qz) - \phi\Pi_0(p). \tag{4.3}$$

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Using (4.2) and simplifying (4.3) for $\Pi_1(z)$, we get

$$\Pi_1(z) = \frac{\lambda d_0 q z(\phi + \lambda d_0)\pi_{0,0}}{(\phi + \lambda d_0 q - \lambda d_0 q z)(\mu - \lambda d_1 z)}. \tag{4.4}$$

Extending equation (4.4) in partial fractions,

$$\Pi_1(z) = \frac{\lambda d_0 q(\phi + \lambda d_0)\pi_{0,0}}{\lambda d_1(\phi + \lambda d_0 q) - \lambda d_0 q \mu}\Big[\sum_{n=0}^{\infty}\Big(\frac{\lambda d_1 z}{\mu}\Big)^n - \sum_{n=0}^{\infty}\Big(\frac{\lambda d_0 z}{\phi + \lambda d_0 q}\Big)^n\Big]. \tag{4.5}$$

Further simplification of (4.3) and (4.5) yields

$$\pi_{n,0} = \Big(\frac{\lambda d_0}{\phi + \lambda d_0}\Big)^n \pi_{0,0}, \quad n \geq 0, \tag{4.6}$$

$$\pi_{n,1} = \frac{d_0 q(\phi + \lambda d_0)\pi_{0,0}}{\phi d_1 + d_0 q(\lambda d_1 - \mu)}\Big[\Big(\frac{\lambda d_1}{\mu}\Big)^n - \Big(\frac{\lambda d_0 q}{\phi + \lambda d_0 q}\Big)^n\Big], \quad n \geq 1, \tag{4.7}$$

Using the normalization equation (2.1), we get

$$\pi_{0,0} = \frac{\phi(\mu - \lambda d_1)}{(\phi + \lambda d_0)(\mu - \lambda d_1 + \lambda d_0 q)}.$$

These results can be summarized in the following lemma.

Lemma 4.1. *In the almost unobservable queue with synchronized abandonment wherein all customers follow a mixed balking strategy* (d_0, d_1)*, the stationary distribution is*

$$\pi_{n,0} = \frac{\phi(\mu - \lambda d_1)}{\lambda d_0(\mu - \lambda d_1 + \lambda d_0 q)}\Big(\frac{\lambda d_0}{\phi + \lambda d_0}\Big)^{n+1}, \quad n \geq 0,$$

$$\pi_{n,1} = \frac{d_0 q \phi(\mu - \lambda d_1)}{(\phi d_1 + d_0 q(\lambda d_1 - \mu))(\mu - \lambda d_1 + \lambda d_0 q)}\Big[\Big(\frac{\lambda d_1}{\mu}\Big)^n - \Big(\frac{\lambda d_0 q}{\phi + \lambda d_0 q}\Big)^n\Big], \quad n \geq 1.$$

The probability that the system is on vacation (or busy) state, denoted by π_0 (π_1), is computed as

$$\pi_0 = \sum_{n=0}^{\infty}\pi_{n,0} = \frac{\mu - \lambda d_1}{\mu - \lambda d_1 + \lambda d_0 q}, \tag{4.8}$$

$$\pi_1 = \sum_{n=1}^{\infty}\pi_{n,1} = \frac{\lambda d_0 q}{\mu - \lambda d_1 + \lambda d_0 q}. \tag{4.9}$$

The conditional mean system length when the server is on vacation state and on busy state, are denoted by $E(N \mid \zeta = 0)$ and $E(N \mid \zeta = 1)$, respectively, and are derived as

$$E(N \mid \zeta = 0) = \frac{\sum_{n=1}^{\infty} n\pi_{n,0}}{\pi_0} = \frac{\Pi_0'(1)}{\pi_0} = \frac{\lambda d_0}{\phi},$$

$$E(N \mid \zeta = 1) = \frac{\sum_{n=1}^{\infty} n\pi_{n,1}}{\pi_1} = \frac{\Pi_1'(1)}{\pi_1} = \frac{\phi\mu + \lambda d_0 q(\mu - \lambda d_1)}{\phi(\mu - \lambda d_1)},$$

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where $\Pi_0'(z)$ and $\Pi_1'(z)$ represent the first derivatives of $\Pi_0(z)$ and $\Pi_1(z)$, respectively, with respect to z. The average number of the customers in the system is

$$E(N) = \sum_{n=1}^{\infty} n\pi_{n,0} + \sum_{n=1}^{\infty} n\pi_{n,1} = \frac{\lambda d_0(\mu - \lambda d_1)}{\phi(\mu - \lambda d_1 + \lambda d_0 q)}\left[1 + \frac{q(\phi\mu + \lambda d_0 q(\mu - \lambda d_1))}{(\mu - \lambda d_1)^2}\right].$$

4.1.1 *Equilibrium and socially optimal balking strategy.* Now consider a tagged customer who finds the server on state i ($i = 0, 1$) upon arrival. The conditional mean sojourn time of the tagged customer who decides to join the system with server state i, given that others follow the same mixed strategy (d_0, d_1) is

$$W(0, d_0, d_1) = \frac{\sum_{n=0}^{\infty} T(n, 0)\, \pi_{n,0}}{\sum_{n=0}^{\infty} \pi_{n,0}}, \quad W(1, d_0, d_1) = \frac{\sum_{n=1}^{\infty} T(n, 1)\, \pi_{n,1}}{\sum_{n=1}^{\infty} \pi_{n,1}}.$$

Using (3.1b) and (4.6), we obtain

$$W(0, d_0, d_1) = \frac{1}{\phi} + \frac{q}{\mu} + \frac{\lambda d_0 q^2}{\mu\phi}.$$

Similarly, using (3.1c) and (4.7) yields

$$W(1, d_0, d_1) = \frac{1}{\mu}\left(1 + \frac{\phi\mu + \lambda d_0 q(\mu - \lambda d_1)}{\phi(\mu - \lambda d_1)}\right).$$

The expected net benefits of a customer that finds the server on state i upon arrival and decides to join the system when all customers follow the same joining strategy (d_0, d_1) are

$$\Delta_{au}(0, d_0, d_1) = R - C\left(\frac{1}{\phi} + \frac{q}{\mu} + \frac{\lambda d_0 q^2}{\mu\phi}\right),$$
$$\Delta_{au}(1, d_0, d_1) = R - \frac{C}{\mu} - C\left(\frac{\phi\mu + \lambda d_0 q(\mu - \lambda d_1)}{\mu\phi(\mu - \lambda d_1)}\right). \tag{4.10}$$

Consider the different cases of equilibrium analysis for the joining probabilities d_0 and d_1 with the corresponding Nash equilibrium solutions denoted by $d_e(0)$ and $d_e(1)$, respectively. First, we consider a tagged customer who discovers the server on vacation mode upon arrival and joins the system with probability d_0 if he earns a positive net benefit. The equilibrium joining probability $d_e(0)$ is analysed under the following two cases.

CASE 1 ($C((\mu + q\phi)/\phi\mu) < R \le C((\mu + q\phi + q^2\lambda)/\phi\mu)$). In this case, if all customers who discover the server at state 0 enter with probability $d_0 = 1$, then the tagged customer who decides to enter has $\Delta_{au}(0, d_0, d_1) \le 0$. Hence, $d_0 = 1$ can not be an equilibrium strategy. On the other hand, if all other customers join with probability $d_0 = 0$, then the tagged customer entering the system has $\Delta_{au}(0, d_0, d_1) > 0$. The tagged customer is benefited more by joining than balking. Hence, $d_0 = 0$ can not be an equilibrium strategy. Therefore, a unique mixed Nash equilibrium strategy $d_0 = d_e(0)$ exists for which customers are indifferent between entering and balking the queue.

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This unique strategy $d_e(0)$ is obtained by solving $\Delta_{au}(0, d_0, d_1) = 0$ for d_0, and is given by

$$d_e(0) = \frac{1}{\lambda q^2}\Big(\frac{R\mu\phi}{C} - \mu - \phi q\Big).$$

CASE 2 ($C((\mu + q\phi + q^2\lambda)/\phi\mu) < R$). In this case, the best response is 1, that is, the tagged customer is always benefited by joining the system irrespective of the decisions taken by all other customers. Hence, $d_e(0) = 1$ is the only equilibrium strategy.

Next, we consider the equilibrium mixed strategies for a tagged customer who discovers the server on normal service mode upon arrival. From (4.10), the expected net benefit of the tagged customer is given by

$$R - \frac{C}{\mu} - C\Big(\frac{\phi\mu + \lambda d_0 q(\mu - \lambda d_1)}{\mu\phi(\mu - \lambda d_1)}\Big) = \begin{cases} \dfrac{C}{q}\Big(\dfrac{1}{\phi} - \dfrac{q}{\mu - \lambda d_1}\Big) - R\dfrac{p}{q} & \text{in Case 1,} \\ R - \dfrac{C}{\mu} - \dfrac{C}{\mu - \lambda d_1} - \dfrac{C\lambda q}{\mu\phi} & \text{in Case 2.} \end{cases}$$

To find $d_e(1)$, we study some subcases of both Cases 1 and 2:

Case 1a:

$$C\Big(\frac{\mu + q\phi}{\phi\mu}\Big) < R \le C\Big(\frac{\mu + q\phi + q^2\lambda}{\phi\mu}\Big) \quad \text{and} \quad R < C\Big(\frac{\mu - \lambda - \phi q}{p\phi(\mu - \lambda)}\Big),$$
$$(d_e(0), d_e(1)) = \Big(\frac{1}{\lambda q^2}\Big(\frac{R\mu\phi}{C} - \mu - \phi q\Big), 0\Big);$$

Case 1b:

$$C\Big(\frac{\mu + q\phi}{\phi\mu}\Big) < R \le C\Big(\frac{\mu + q\phi + q^2\lambda}{\phi\mu}\Big) \quad \text{and} \quad C\Big(\frac{\mu - \lambda - \phi q}{p\phi(\mu - \lambda)}\Big) \le R \le C\Big(\frac{\mu - \phi q}{p\mu\phi}\Big),$$
$$(d_e(0), d_e(1)) = \Big(\frac{1}{\lambda q^2}\Big(\frac{R\mu\phi}{C} - \mu - \phi q\Big), \frac{\mu}{\lambda} - \frac{C\phi q}{\lambda(C - R\phi p)}\Big);$$

Case 1c:

$$C\Big(\frac{\mu + q\phi}{\phi\mu}\Big) < R \le C\Big(\frac{\mu + q\phi + q^2\lambda}{\phi\mu}\Big) \quad \text{and} \quad C\Big(\frac{\mu - \phi q}{p\mu\phi}\Big) < R,$$
$$(d_e(0), d_e(1)) = \Big(\frac{1}{\lambda q^2}\Big(\frac{R\mu\phi}{C} - \mu - \phi q\Big), 1\Big);$$

Case 2a:

$$C\Big(\frac{\mu + q\phi + q^2\lambda}{\phi\mu}\Big) < R \quad \text{and} \quad R < C\Big(\frac{2}{\mu} + \frac{\lambda q}{\mu\phi}\Big),$$
$$(d_e(0), d_e(1)) = (1, 0);$$

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Case 2b:

$$C\Big(\frac{\mu+q\phi+q^2\lambda}{\phi\mu}\Big)<R \quad\text{and}\quad C\Big(\frac{2}{\mu}+\frac{\lambda q}{\mu\phi}\Big)\le R\le C\Big(\frac{1}{\mu}+\frac{1}{\mu-\lambda}+\frac{\lambda q}{\mu\phi}\Big),$$
$$(d_e(0),d_e(1))=\Big(1,\frac{\mu}{\lambda}-\frac{C\mu\phi}{\lambda(R\mu\phi-C(\phi+\lambda q))}\Big);$$

Case 2c:

$$C\Big(\frac{\mu+q\phi+q^2\lambda}{\phi\mu}\Big)<R \quad\text{and}\quad C\Big(\frac{1}{\mu}+\frac{1}{\mu-\lambda}+\frac{\lambda q}{\mu\phi}\Big)<R,$$
$$(d_e(0),d_e(1))=(1,1).$$

REMARK 4.2. If we assume $\lambda\ge\mu$, a unique Nash equilibrium mixed strategy $(d_e(0),d_e(1))$ "observe $\zeta(t)$ and join with probability $d_e(\zeta(t))$" exists, where the mixed equilibrium strategy is $(d_e(0),d_e(1))$. Here Case 1a and Case 2a are the same as the previous case when $\lambda\le\mu$. Other cases are:

Case 1b:

$$C\Big(\frac{\mu+q\phi}{\phi\mu}\Big)<R\le C\Big(\frac{\mu+q\phi+q^2\lambda}{\phi\mu}\Big) \quad\text{and}\quad C\Big(\frac{\lambda-\mu+\phi q}{p\phi(\lambda-\mu)}\Big)\le R,$$
$$(d_e(0),d_e(1))=\Big(\frac{1}{\lambda q^2}\Big(\frac{R\mu\phi}{C}-\mu-\phi q\Big),\frac{\mu}{\lambda}-\frac{C\phi\, q}{\lambda(C-R\phi p)}\Big);$$

Case 2b:

$$R>C\Big(\frac{\mu+q\phi+q^2\lambda}{\phi\mu}\Big) \quad\text{and}\quad C\Big(\frac{2}{\mu}+\frac{\lambda q}{\mu\phi}\Big)\le R,$$
$$(d_e(0),d_e(1))=\Big(1,\frac{\mu}{\lambda}-\frac{C\mu\phi}{\lambda(R\mu\phi-C(\phi+\lambda q))}\Big).$$

REMARK 4.3. Substituting $p=0$ and $q=1$, the model reduces to M/M/1 queue with multiple vacations. Taking $\lambda<\mu$, a unique Nash equilibrium mixed strategy $(d_e(0),d_e(1))$ "observe $\zeta(t)$ and join with probability $d_e(\zeta(t))$" exists, where $(d_e(0),d_e(1))$ is given as:

CASE 1 ($C((\mu+\phi)/\phi\mu)<R\le C((\mu+\phi+\lambda)/\phi\mu)$).

$$d_e(0)=\frac{1}{\lambda}\Big(\frac{R\mu\phi}{C}-\mu-\phi\Big);$$

CASE 2 ($C((\mu+\phi+\lambda)/\phi\mu)<R$).

$$R-\frac{C}{\mu-\lambda d_1}-\frac{C(\phi+\lambda d_0)}{\mu\phi}=\begin{cases}\dfrac{C}{\phi}-\dfrac{C}{\mu-\lambda d_1} & \text{in Case 1},\\[2ex] R-\dfrac{C}{\mu-\lambda d_1}-\dfrac{C(\phi+\lambda)}{\mu\phi} & \text{in Case 2}.\end{cases}$$

The above expressions match exactly with the expressions of Burnetas and Economou [4, p. 222] by replacing the vacation parameter ϕ with θ.

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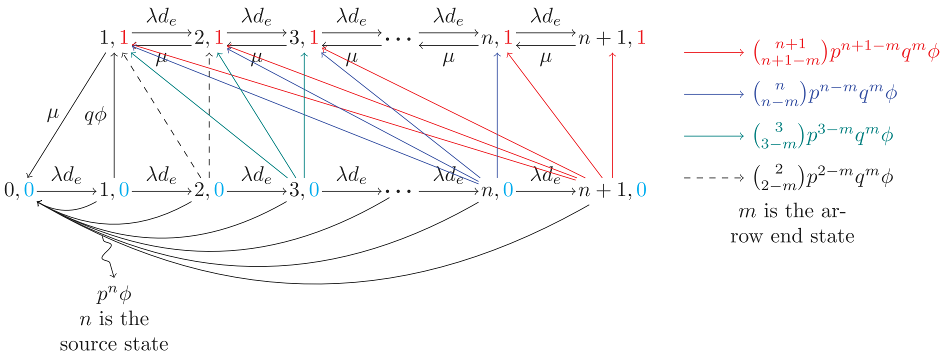

FIGURE 5. Transition rate diagram for the mixed strategy d_e.

The social benefit of the system when all customers follow the same joining strategy (d_0, d_1), is

$$\begin{aligned}\Delta_s(d_0, d_1) &= \lambda(\pi_0 d_0 + \pi_1 d_1)R - C\,E(N) \\ &= R\,\frac{\lambda d_0(\mu - \lambda d_1 p)}{\mu - \lambda d_1 + \lambda d_0 q} - C\,\frac{\lambda d_0(\mu - \lambda d_1)}{\phi(\mu - \lambda d_1 + \lambda d_0 q)}\left[1 + \frac{q(\phi\mu + \lambda d_0 q(\mu - \lambda d_1))}{(\mu - \lambda d_1)^2}\right] \\ &= \frac{\lambda d_0}{\mu - \lambda d_1 + \lambda d_0 q}\left[R(\mu - \lambda d_1 p) - \frac{C(\mu - \lambda d_1)}{\phi}\left\{1 + \frac{q(\phi\mu + \lambda d_0 q(\mu - \lambda d_1))}{(\mu - \lambda d_1)^2}\right\}\right],\end{aligned}$$

where π_0 and π_1 are given by (4.8) and (4.9), respectively. Here, $\lambda(\pi_0 d_0 + \pi_1 d_1)$ is the effective arrival rate of the system. The social planner's objective is to optimize the expected social benefit, that is, a social planner is interested to find the socially optimal joining strategy (d_0^*, d_1^*) under which the social benefit of the whole system is maximum. The analytic expression for the socially optimal strategies is intractable, but its numerical computation is easier to compare with the observable cases.

4.2. Fully unobservable queue In the fully unobservable system, arriving customers do not know the state of the system before making their join or balk decision. However, they are aware of the model parameters. The customers' decision is to choose a joining probability d $(0 \le d \le 1)$, which will optimize their individual benefit. If all arriving customers join the system with probability d, then we have an M/M/1 vacation queueing system with synchronized abandonments, where the arrival rate is λd, the service time and vacation times are exponentially distributed with rate μ and ϕ, respectively. The stability condition for the queueing model is $\lambda d < \mu$. The state space Ω_{fu} for the Markov chain $\{(N(t), \zeta(t)), t \ge 0\}$ is same as Ω_{au}, and the transition rate diagram is illustrated in Figure 5. The stationary distributions $\{\pi_{k,i} \mid k \ge i\}$ of the

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system can be obtained from the balance equations, given below:

$$\begin{aligned}
\lambda d\pi_{0,0} &= \mu\pi_{1,1} + \phi\sum_{j=1}^{\infty} p^j \pi_{j,0},\\
(\lambda d + \phi)\pi_{n,0} &= \lambda d\pi_{n-1,0}, \quad n \geq 1,\\
(\lambda d + \mu)\pi_{1,1} &= \mu\pi_{2,1} + \phi q\sum_{j=1}^{\infty} j p^{j-1}\pi_{j,0},\\
(\lambda d + \mu)\pi_{n,1} &= \mu\pi_{n+1,1} + \lambda d\pi_{n-1,1} + \phi\sum_{j=n}^{\infty}\binom{j}{j-n} q^n p^{j-n}\pi_{j,0}, \quad n \geq 2.
\end{aligned}$$

For computation of the equilibrium mixed strategies, we obtain the stationary distribution from Lemma 4.1 by substituting the joining probabilities d_0 and d_1 by d, given that all customers follow the same mixed strategy d:

$$\begin{aligned}
\pi_{n,0} &= \frac{\phi(\mu - \lambda d)}{\lambda d(\mu - \lambda dp)}\Big(\frac{\lambda d}{\phi + \lambda d}\Big)^{n+1}, \quad n \geq 0,\\
\pi_{n,1} &= \frac{q\phi(\mu - \lambda d)}{(\phi + q(\lambda d - \mu))(\mu - \lambda dp)}\Big[\Big(\frac{\lambda d}{\mu}\Big)^n - \Big(\frac{\lambda dq}{\phi + \lambda dq}\Big)^n\Big], \quad n \geq 1.
\end{aligned}$$

The probability $\pi_0 = P(\zeta = 0)$ that the system is on a vacation mode and the probability $\pi_1 = P(\zeta = 1)$ that the system is on active mode are calculated as

$$\pi_0 = \sum_{n=1}^{\infty}\pi_{n,0} = \frac{\mu - \lambda d}{\mu - \lambda dp}, \quad \pi_1 = \sum_{n=1}^{\infty}\pi_{n,1} = \frac{\lambda dq}{\mu - \lambda dp}.$$

The average number of the customers in the system is

$$E(N) = \sum_{n=1}^{\infty} n(\pi_{n,0} + \pi_{n,1}) = \frac{\lambda d(\mu - \lambda d)}{\phi(\mu - \lambda dp)}\Big[1 + \frac{q(\mu\phi + \lambda dq(\mu - \lambda d))}{(\mu - \lambda d)^2}\Big].$$

Applying Little's law, the mean sojourn time of a customer who upon arrival decides to join the system is

$$W(d) = \frac{E(N)}{\lambda d} = \frac{\mu - \lambda d}{\phi(\mu - \lambda dp)}\Big[1 + \frac{q(\mu\phi + \lambda dq(\mu - \lambda d))}{(\mu - \lambda d)^2}\Big].$$

Differentiating the above expression with respect to d yields

$$W'(d) = \frac{\lambda\mu q[p(\mu - \lambda d)(\phi + \lambda d - \mu) + \phi(\mu - \lambda dp)]}{\phi(\mu - \lambda d)^2(\mu - \lambda dp)^2}.$$

By taking $\lambda < \mu$ and $d \in [0, 1]$, we get $W'(d) > 0$. Therefore, $W(d)$ is strictly increasing for $d \in [0, 1]$.

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4.2.1 *Equilibrium and socially optimal balking strategy.* We examine a fully unobservable queue with multiple vacations and synchronized abandonments, where arriving customers pursue a common strategy d such that the system is stable (that is, $\lambda d/\mu < 1$). The expected net benefit of a tagged customer who decides to enter is

$$\begin{aligned}\Delta_{fu}(d) &= R - C\frac{\mu-\lambda d}{\phi(\mu-\lambda dp)}\Big[1+\frac{q(\mu\phi+\lambda dq(\mu-\lambda d))}{(\mu-\lambda d)^2}\Big], \qquad (4.11)\\ \Delta_{fu}(0) &= R - C\Big(\frac{\mu+\phi q}{\mu\phi}\Big),\\ \Delta_{fu}(1) &= R - C\frac{\mu-\lambda}{\phi(\mu-\lambda p)}\Big[1+\frac{q\mu\phi+\lambda q^2(\mu-\lambda)}{(\mu-\lambda)^2}\Big].\end{aligned}$$

Since, the net benefit function $\Delta_{fu}(d)$ is strictly decreasing and changes sign in $(0,1)$ for

$$R \in \Big(\frac{C(\mu+\phi q)}{\mu\phi}, \frac{C(\mu-\lambda)}{\phi(\mu-\lambda p)}\Big(1+\frac{q\mu\phi+\lambda q^2(\mu-\lambda)}{(\mu-\lambda)^2}\Big)\Big),$$

we get a unique solution $d_e^* \in (0,1)$ to equation (4.11). Thus, the tagged customer's best response is to join the system with probability d_e^*. When

$$R \in \Big[\frac{C(\mu-\lambda)}{\phi(\mu-\lambda p)}\Big(1+\frac{q\mu\phi+\lambda q^2(\mu-\lambda)}{(\mu-\lambda)^2}\Big), \infty\Big),$$

the net benefit of a tagged customer is always positive for any $d \in (0,1)$. Thus, the tagged customer's best response is to join the system with probability 1. Note that $\Delta_{fu}(d) < 0$ for every d, if $0 < R < C(\mu+\phi q)/\mu\phi$. So, the best response of a customer upon arrival is to balk, and the unique equilibrium point is $d = 0$. Thus, there always exists a unique Nash equilibrium strategy d_e if the tagged customer decides to enter into the system, assuming R meets certain conditions. Consequently, the Nash equilibrium strategy is stated as follows:

$$d_e = \begin{cases} 0 & R \in \Big(0, \dfrac{C(\mu+\phi q)}{\mu\phi}\Big], \\ d_e^* & R \in \Big(\dfrac{C(\mu+\phi q)}{\mu\phi}, \dfrac{C(\mu-\lambda)}{\phi(\mu-\lambda p)}\Big(1+\dfrac{q\mu\phi+\lambda q^2(\mu-\lambda)}{(\mu-\lambda)^2}\Big)\Big), \\ 1 & R \in \Big[\dfrac{C(\mu-\lambda)}{\phi(\mu-\lambda p)}\Big(1+\dfrac{q\mu\phi+\lambda q^2(\mu-\lambda)}{(\mu-\lambda)^2}\Big), \infty\Big). \end{cases}$$

REMARK 4.4. If we assume $\lambda \geq \mu$, a unique Nash equilibrium mixed strategy d_e "join with probability d_e" exists, where $d_e = d_e^*$ for $R/C > (\mu+\phi q)/\mu\phi$; here d_e^* is the unique solution to (4.11).

When all customers pursue the above equilibrium mixed strategy d_e, the social benefit per time unit in equilibrium can be represented as $\Delta_s(d_e)$, and can be computed using (4.11). Next, we are interested in the joining strategy that optimizes the social

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benefit of the whole system. The social benefit per time unit when all customers follow a joining strategy d is

$$\Delta_s(d) = \lambda d(R - CW) = \lambda d\Big(R - \frac{C(\mu - \lambda d)}{\phi(\mu - \lambda dp)}\Big[1 + \frac{q(\mu\phi + \lambda dq(\mu - \lambda d))}{(\mu - \lambda d)^2}\Big]\Big).$$

Since $\Delta_s(d)$ is differentiable on [0, 1], the first two derivatives are

$$\Delta_s'(d) = \lambda(R - CW(d)) - \lambda dCW'(d), \quad \Delta_s''(d) = -\lambda C[W'(d) + dW''(d)],$$

where

$$W''(d) = \frac{q\lambda^2\mu}{\phi}\Big[\frac{2((\mu - \lambda dp) + p(\mu - \lambda d))(\phi\mu p + (\mu - \lambda d)(\phi + \lambda d - \mu))}{(\mu - \lambda d)^3(\mu - \lambda dp)^3} + \frac{(2(\mu - \lambda d) - \phi)}{(\mu - \lambda d)^2(\mu - \lambda dp)^2}\Big].$$

Since $W''(d) > 0$, so $\Delta_s''(d) < 0$, for all $d \in [0, 1]$, hence, $\Delta_s(d)$ is concave down in the closed interval [0, 1] and has a unique optimal point, say $d^* \in (0, 1)$ at which the social benefit is maximum. Thus, the socially optimal joining strategy of customers in the fully unobservable case is given by

$$d^* = \begin{cases} \dfrac{R/C - W(d)}{W'(d)} & \text{if } (R/C - W(d))/W'(d) \in (0, 1), \\ 1 & \text{if } (R/C - W(d))/W'(d) \in [1, \infty). \end{cases}$$

5. Numerical results

In this section, we discuss numerical results to show the effectiveness of the model studied here. The behaviour of the strategic customers under two distinct strategies is discussed with several model parameters. In particular, the equilibrium joining strategies and socially optimal behaviour for the observable models and the unobservable models are presented under different situations. The role of information on the social benefit of the system is also discussed. Maple 17 software is used to get the numerical data for the queueing systems with several model parameters. The numerical experiments are presented in three facets. In the first set of experiments (Figures 6–9), the equilibrium threshold strategies for the observable cases and in the second set of experiments (Figures 10–13) equilibrium joining probabilities for the unobservable cases are presented for different queueing parameters. In the third set of experiments (Figures 14–17), the social benefit of the system is presented for all the information cases.

In the first set of experiments, the effect of service, vacation, reward and abandonment on the strategic customer behaviour under controlled information is discussed. The behaviour of the equilibrium threshold of joining against service rate (Figure 6) in the observable vacation models with customer abandonment is similar to that of the thresholds in the observable vacation models without customer

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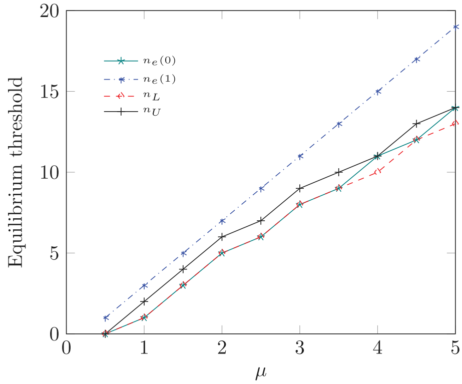

Figure 6. Equilibrium threshold strategies versus μ for an observable model with $\lambda = 1.0, \phi = 0.5$, $p = 0.2, R = 20, C = 5$.

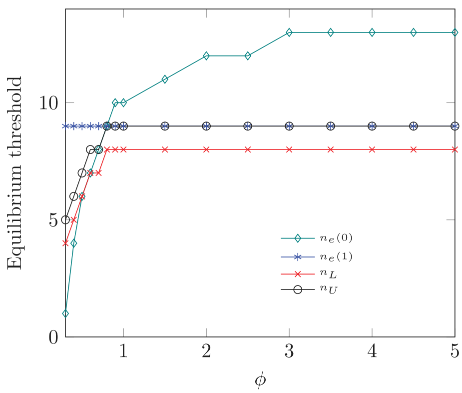

Figure 7. Equilibrium threshold strategies versus ϕ for an observable model with $\lambda = 1.0, \mu = 2.5$, $p = 0.2, R = 20, C = 5$.

abandonment. Customers not having the server state information follow a threshold strategy in between the strategies of customers that encounter the server on vacation and that encounter the server on regular service mode. The difference among equilibrium thresholds increases with increasing service rate. A similar behaviour of the equilibrium thresholds is observed in Figure 8 for smaller values of service reward. Here, the difference among the equilibrium thresholds decreases with increase in R up to 30, the thresholds coincide in the interval [30, 32] and after 32, $n_e(0)$ surpasses $n_e(1)$. This may be due to the increase in congestion level during vacation, which in turn aggravate the system resulting in customer abandonment. Figures 7 and 9 show the behaviour of impatient customers on varying vacation rate and abandonment probability, respectively. Note that when the vacation rate is smaller than one and the abandonment probability is at most 0.3, the relationship among the

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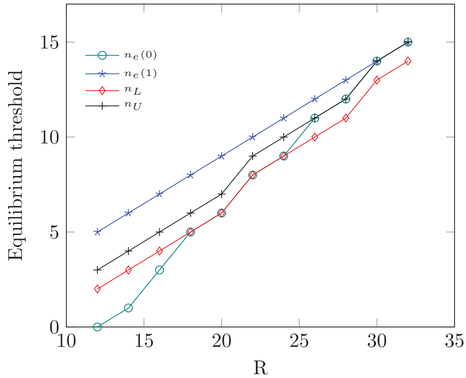

FIGURE 8. Equilibrium threshold strategies versus R for an observable model with $\lambda = 1.0, \mu = 2.5, \phi = 0.5, p = 0.2, C = 5$.

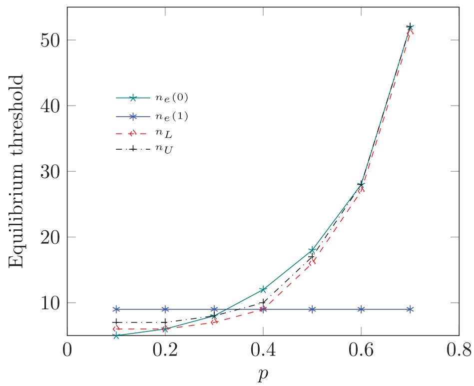

FIGURE 9. Equilibrium threshold strategies versus p for an observable model with $\lambda = 1.0, \mu = 2.5, \phi = 0.5, R = 20, C = 5$.

thresholds, $n_e(0) < n_e < n_e(1)$ is intuitive. As the vacation rate increases, the mean vacation time decreases, which results in shorter expected sojourn times for waiting customers. Thus, the waiting time of the rest customers decreases substantially due to the shorter expected sojourn time during vacation. This information about the smaller queue length attracts more customers to join the system, as a result, the threshold increases.

Secondly, the customers without any information on the number of customers ahead of the queue, follow a mixed strategy of joining or balking the queue. Customers with only the server state information, do not get any benefit in case of a system with the fast server, but have a visible effect when they join a slow server system. The effect of the server speed on the equilibrium joining probabilities is shown in Figure 10. The queue length information is more helpful than the server state information under a dynamic service rate. This is also true for other model parameters

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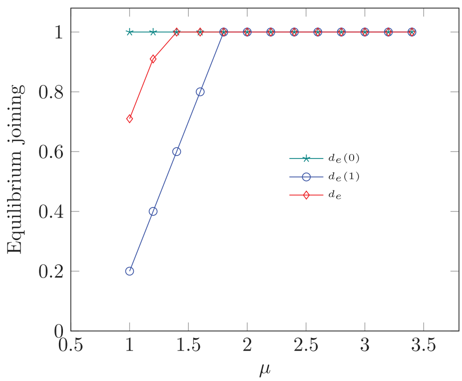

Figure 10. Equilibrium joining strategies versus μ for an unobservable model with $\lambda = 1.0, \phi = 0.5, p = 0.2, R = 3, C = 1$.

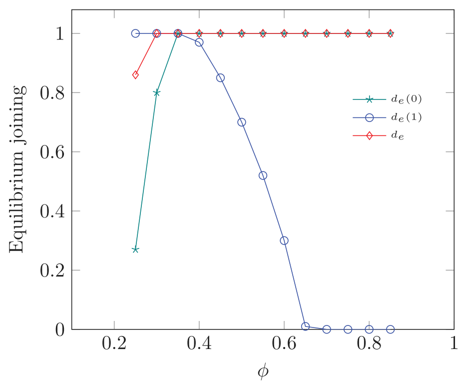

Figure 11. Equilibrium joining strategies versus ϕ for an unobservable model with $\lambda = 1.0, \mu = 1.5, p = 0.2, R = 3, C = 1$.

as seen from the numerical results (Figures 11, 13). The equilibrium thresholds are always nondecreasing functions of the parameters whereas the equilibrium mixed strategies are both nonincreasing for some values and nondecreasing for other values of the parameters. When customers are not aware of the system state, their equilibrium behaviour is to "follow the crowd" situation. There is no benefit of having the server state information in a system with slower arrivals as in Figure 12. But this has a positive effect in a system with more arrivals. In Figure 11, the equilibrium joining probabilities increases for $\phi \in [0.2, 0.35]$ and with vacation times becoming shorter, customers encountering busy server are less interested to join because of the presence of more impatient customers during the previous vacation period. When few customers leave the system due to impatience, new arrivals are interested to join the vacation state expecting shorter waiting times. When more impatient customers abandon the system, it is more beneficial for the arriving customers to join the vacation state. It

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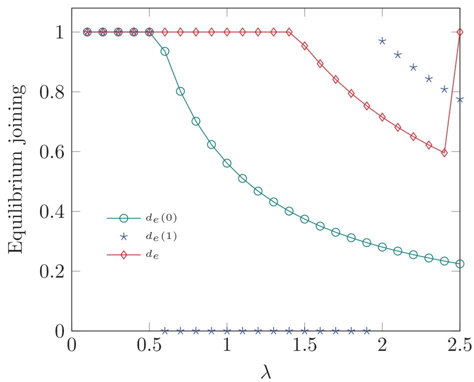

FIGURE 12. Equilibrium joining strategies versus λ for an unobservable model with $\mu = 2.5, \phi = 0.5, p = 0.3, R = 2.5, C = 1$.

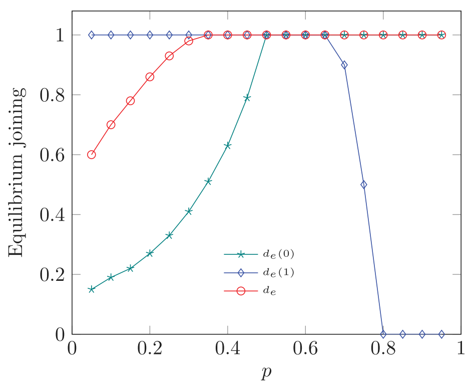

FIGURE 13. Equilibrium joining strategies versus p for an unobservable model with $\lambda = 1.0, \mu = 1.5, \phi = 0.5, R = 3, C = 1$.

adds negative externalities on the arriving customers who decide to enter the busy state, see Figure 13.

Finally, we discuss the social benefit under the four information settings, and present a comparative study among them. In Figure 14, the social benefit increases with increasing service rate, independent of the information policy. It is better to follow one information policy (either queue length or server state) than their combined one (fully observable case). For a fast server ($\mu > 2.4$) there are negligible differences in the social benefit under the full and no information policy. Similar behaviour of the social benefit function against the server vacation rates is seen in Figure 15. The social benefit is a concave function and attains maximum value 4.345098 unit at $\phi = 0.4$. In Figure 16, the social benefit is also a concave function under the observable, and no information case and constantly increases for the system revealing server state information only. Revealing more information to impatient customers helps in increasing the social

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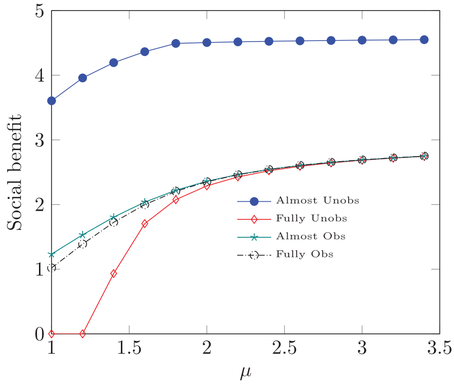

FIGURE 14. Social benefit versus μ for different information policies with $\lambda = 1.0, \phi = 0.5, p = 0.2,$ $R = 5, C = 1$.

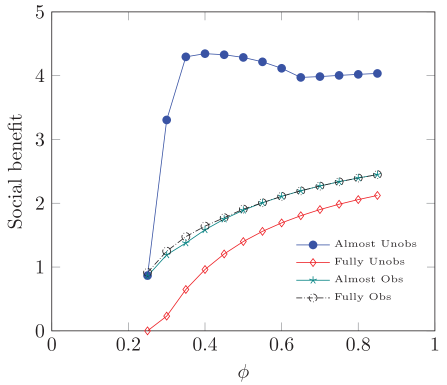

FIGURE 15. Social benefit versus ϕ for different information policies with $\lambda = 1.0, \mu = 1.5, p = 0.2,$ $R = 5, C = 1$.

benefit only when few customers leave the system, see Figure 17. As more customers leave the system, the server state information policy outperforms other information policies resulting in higher social benefit for the system. It is observed that disclosing server state information always helps to increase the social benefit in the unobservable case. In the observable case, it may be beneficial under some specific situations. For example, hiding the server state information increases the social benefit when the service rate is less than 2.5 unit (Figure 14) and also when customers are less impatient (Figure 17). On the other hand, hiding the server state information is not beneficial for a congested system (Figure 16) as well as for longer server vacations (Figure 15). In each experiment, the social benefits in the observable systems are bounded by the social benefits in the unobservable systems. The social benefit in the server state information (almost unobservable case) dominates that of the combined queue length and server state information (fully observable case). This confirms that revealing more

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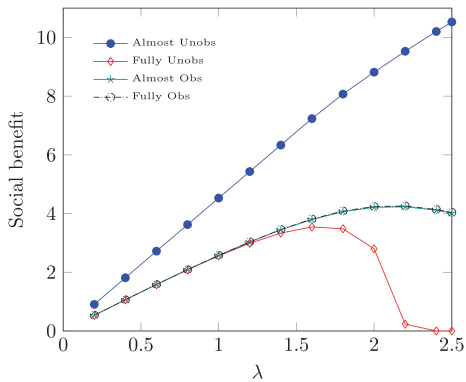

FIGURE 16. Social benefit versus λ for different information policies with $\mu = 2.5, \phi = 0.5, p = 0.2, R = 5, C = 1$.

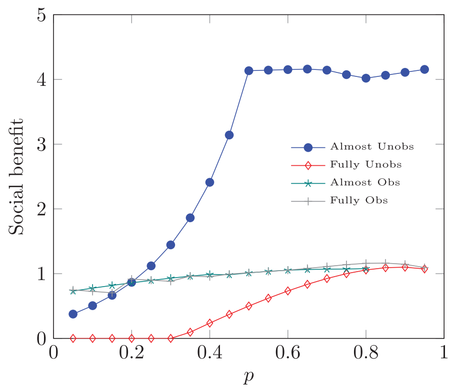

FIGURE 17. Social benefit versus p for different information policies with $\lambda = 1.0, \mu = 1.5, \phi = 0.25, R = 5, C = 1$.

information to the impatient customers is not beneficial for the social benefit of the system as a whole. A social planner has to play an important role in controlling the availability of appropriate information to the impatient customers.

6. Conclusion

In this paper, we analysed the customers' equilibrium and optimal social behaviour in Markovian queues with multiple vacations and synchronized reneging under fully/almost observable and almost/fully unobservable information cases. We have developed closed-form expressions of the stationary state probabilities using a recursive method. The social benefit based on various parameters of the information level has been examined under the corresponding strategies. The sensitivity analysis of the equilibrium thresholds in observable cases and equilibrium joining probabilities in

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unobservable cases are carried out by varying several model parameters. The effect of customers synchronized abandonments on the equilibrium strategies under the observable and unobservable models is presented. The dominance relation between the equilibrium thresholds is derived. The presented model, on one hand, can provide strategic customers with useful insights in decision making under a variety of information policies, and guide them whether to follow or avoid the crowd. On the other hand, it can provide useful information on the system manager to get the maximum benefit out of the impatient customers. A simultaneous study of the balking and reneging strategies in this problem is more challenging and is left to explore in future. It would be beneficial to consider enhancement of this methodology for the study of single/multiple working vacation models with this pattern of binomial transitions. This work may also be extended to incorporate arbitrarily distributed service demands or batch-arrival queue with synchronized abandonments.

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