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CENTRALIZERS IN THE SEMIGROUP OF INJECTIVE TRANSFORMATIONS ON AN INFINITE SET

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Abstract

For an infinite set X, denote by $\Gamma(X)$ the semigroup of all injective mappings from X to X. For $\alpha \in \Gamma(X)$, let $C(\alpha) = \{\beta \in \Gamma(X) : \alpha\beta = \beta\alpha\}$ be the centralizer of α in $\Gamma(X)$. For an arbitrary $\alpha \in \Gamma(X)$, we characterize the elements of $C(\alpha)$ and determine Green's relations in $C(\alpha)$, including the partial orders of $\mathcal{L}$-, $\mathcal{R}$-, and $\mathcal{J}$-classes.

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1. Introduction

For a semigroup S and an element a of S, the *centralizer* $C(a)$ of a in S is defined by $C(a) = \{x \in S : ax = xa\}$. It is clear that $C(a)$ is a subsemigroup of S.

A significant amount of research has been devoted to studying centralizers in semigroups of transformations on a finite set X. (For details and references concerning this research, see [8, Introduction].) These investigations have been motivated by the fact that, if S is a semigroup of transformations on X that contains the identity id_X, then for any $\alpha \in S$, the centralizer $C(\alpha)$ is a generalization of S in the sense that $S = C(\mathrm{id}_X)$. It is therefore of interest to find out which ideas, approaches, and techniques used to study S can be extended to the centralizers of its elements. Recent research indicates that centralizers of transformation semigroups are also important because they play a role in finding the group of automorphisms of a general semigroup [3, Theorem 2.23].

In contrast with the case of finite sets, little has been done regarding centralizers of transformations on an infinite set. The only exceptions, as far as this author was able to discover, are the studies of the centralizers of idempotent transformations in the full transformation semigroup in [1, 2, 20]. The present paper investigates the centralizers in the semigroup $\Gamma(X)$ of all injective transformations on an infinite set X. (When X is finite, $\Gamma(X)$ is of no interest as a semigroup since it is equal to the symmetric group $\mathrm{Sym}(X)$.) The semigroup $\Gamma(X)$ is a subsemigroup of the three very well-known semigroups of transformations: the semigroup $T(X)$ of full transformations, the semigroup $P(X)$ of partial transformations, and the symmetric inverse semigroup $I(X)$ of partial injective transformations on X. All four semigroups have the symmetric group $\mathrm{Sym}(X)$ of permutations on X as their group of units.

Numerous papers have been written about semigroups $T(X)$, $P(X)$, and $I(X)$. Much less research has been devoted to $\Gamma(X)$. One reason may be that $T(X)$, $P(X)$, and $I(X)$ are all regular semigroups, whereas $\Gamma(X)$ is highly nonregular since it contains only one idempotent (the identity id_X). Many problems, such as finding subsemigroups generated by idempotents, determining maximal inverse subsemigroups, and so on, therefore do not apply to $\Gamma(X)$. The semigroup $\Gamma(X)$ is universal for right cancelative semigroups with no idempotents (except possibly the identity): that is, any such semigroup can be embedded in $\Gamma(X)$ for some X [4, Lemma 1.0]. It has been studied mainly in the context of: ideals and congruences (see, for example, [12, 18]); $\mathcal{G}(X)$-normal semigroups (see [10, 11, 16]); Baer–Levi semigroups (see [13, 14]), that is, $\mathcal{J}$-classes of $\Gamma(X)$ with a prescribed infinite defect (see Remark 2.4 below); and BQ-semigroups (see [7, 17]).

Our objective is to study the centralizers in $\Gamma(X)$. In Section 2 we describe Green's relations in $\Gamma(X)$ to see how Green's relations in centralizers differ from those in $\Gamma(X)$. In Section 3 we describe the elements of $C(\alpha)$ in $\Gamma(X)$ (for an arbitrary element α of $\Gamma(X)$) using the unique decomposition of α into disjoint rays, double rays, and cycles. If X is finite, then this decomposition reduces to the usual decomposition of α into disjoint cycles. In Section 4 we determine Green's relations in any centralizer $C(\alpha)$, including the partial orders of $\mathcal{L}$-, $\mathcal{R}$-, and $\mathcal{J}$-classes.

Although our results are true for an arbitrary set X, they are only new for an infinite X. Suppose X is finite. Then $\Gamma(X) = \mathrm{Sym}(X)$, and the description of the elements of $C(\alpha)$ (Theorem 3.9) reduces to that obtained in [19, Section 2]. For every $\alpha \in \mathrm{Sym}(X)$, the centralizer $C(\alpha)$ is a group, so Green's relations in $C(\alpha)$ are all equal to the universal relation on $C(\alpha)$.

For the rest of this paper, we assume that X is an arbitrary infinite set.

2. The semigroup $\Gamma(X)$

In this section, we describe Green's relations in the semigroup $\Gamma(X)$. If S is a semigroup and $a, b \in S$, we say that $a\mathcal{L}b$ if $S^1a = S^1b$, $a\mathcal{R}b$ if $aS^1 = bS^1$, and $a\,\mathcal{J}\,b$ if $S^1aS^1 = S^1bS^1$, where S^1 is the semigroup S with an identity adjoined. We define $\mathcal{H}$ as the intersection of $\mathcal{L}$ and $\mathcal{R}$, and $\mathcal{D}$ as the join of $\mathcal{L}$ and $\mathcal{R}$, that is, the smallest equivalence relation on S containing both $\mathcal{L}$ and $\mathcal{R}$. These five equivalence relations are known as *Green's relations* [6, p. 45]. The relations $\mathcal{L}$ and $\mathcal{R}$ commute [6, Proposition 2.1.3], and consequently $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. Green's relations are one of the most important tools in studying semigroups.

If $\mathcal{T}$ is one of Green's relations and $a \in S$, we denote the equivalence class of a with respect to $\mathcal{T}$ by T_a. Since $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{J}$ are defined in terms of principal ideals in S, which are partially ordered by inclusion, we have the induced partial orders in the sets of the equivalence classes of $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{J}$: $L_a \le L_b$ if $S^1 a \subseteq S^1 b$, $R_a \le R_b$ if $aS^1 \subseteq bS^1$, and $J_a \le J_b$ if $S^1 a S^1 \subseteq S^1 b S^1$.

DEFINITION 2.1. Let $\alpha \in \Gamma(X)$. We denote the image of α by $\mathrm{im}(\alpha)$, the cardinality of $\mathrm{im}(\alpha)$, called the *rank* of α, by $\mathrm{rank}(\alpha)$, and the cardinality of $X \setminus \mathrm{im}(\alpha)$, called the *defect* of α, by $\mathrm{def}(\alpha)$. We will denote by $S(\alpha)$ the set of elements shifted by α and by $F(\alpha)$ the set of elements fixed by α, that is,

$$S(\alpha) = \{x \in X : x\alpha \ne x\} \quad \text{and} \quad F(\alpha) = \{x \in X : x\alpha = x\}.$$

(We will write mappings on the right and compose from left to right; that is, for functions $f : A \to B$ and $g : B \to C$, we will write xf, rather than $f(x)$, and $x(fg)$, rather than $g(f(x))$.)

PROPOSITION 2.2. *Let* $\alpha, \beta \in \Gamma(X)$. *Then:*
(1) $L_\alpha \le L_\beta \Leftrightarrow \mathrm{im}(\alpha) \subseteq \mathrm{im}(\beta)$*;*
(2) $R_\alpha \le R_\beta \Leftrightarrow \mathrm{def}(\alpha) \ge \mathrm{def}(\beta)$.

PROOF. Regarding (1), we can use the proof for $\mathcal{L}$ on $T(X)$ [4, Lemma 2.5], which carries over to $\Gamma(X)$. Statement (2) has been proved in [17, Lemma 2]. □

THEOREM 2.3. *Let* $\alpha, \beta \in \Gamma(X)$. *Then:*
(1) $\alpha \mathcal{L} \beta \Leftrightarrow \mathrm{im}(\alpha) = \mathrm{im}(\beta)$*;*
(2) $\alpha \mathcal{R} \beta \Leftrightarrow \mathrm{def}(\alpha) = \mathrm{def}(\beta)$*;*
(3) $\mathcal{H} = \mathcal{L}$ *and* $\mathcal{R} = \mathcal{D} = \mathcal{J}$*;*
(4) $J_\alpha \le J_\beta \Leftrightarrow \mathrm{def}(\alpha) \ge \mathrm{def}(\beta)$*;*
(5) *the* $\mathcal{J}$*-classes in* $\Gamma(X)$ *form a chain.*

PROOF. Statements (1) and (2) follow from Proposition 2.2.

Suppose $\alpha \mathcal{L} \beta$. Then, by (1), $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$, and so we have

$$\mathrm{def}(\alpha) = |X \setminus \mathrm{im}(\alpha)| = |X \setminus \mathrm{im}(\beta)| = \mathrm{def}(\beta).$$

Thus $\alpha \mathcal{R} \beta$ by (2), and so $\alpha \mathcal{H} \beta$ (since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$). Thus, we have proved that $\mathcal{L} \subseteq \mathcal{H}$, which implies that $\mathcal{H} = \mathcal{L}$ since $\mathcal{H} \subseteq \mathcal{L}$ in every semigroup. Let $\Gamma = \Gamma(X)$ and suppose $\alpha \mathcal{J} \beta$, that is, $\Gamma\alpha\Gamma = \Gamma\beta\Gamma$. By [18, Theorem 6], every right ideal of Γ is a left ideal. Applying this result to the right ideal $\beta\Gamma$, we obtain $\Gamma(\alpha\Gamma) = \Gamma(\beta\Gamma) \subseteq \beta\Gamma$. Thus $\alpha \in \beta\Gamma$ and similarly $\beta \in \alpha\Gamma$, so $\alpha \mathcal{R} \beta$. We have proved $\mathcal{J} \subseteq \mathcal{R}$, which implies that $\mathcal{R} = \mathcal{D} = \mathcal{J}$ since $\mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$ in every semigroup. We have proved (3).

Statement (4) follows from (3) and Proposition 2.2. Finally, (5) follows from (4). □

REMARK 2.4. Let $p = |X|$ and let q be a cardinal such that $0 \le q \le p$. By Theorem 2.3, the transformations $\alpha \in \Gamma(X)$ with defect q form a single $\mathcal{J}$-class. We will denote this $\mathcal{J}$-class by J_q. Moreover, by Theorem 2.3, for all cardinals q, r with $0 \le q, r \le p$,

$$J_q \le J_r \Leftrightarrow q \ge r. \tag{2.1}$$

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It follows from (2.1) that the chain of $\mathcal{J}$-classes in $\Gamma(X)$ is anti-isomorphic to the chain of cardinals $\{q : 0 \le q \le p\}$. Every $\mathcal{J}$-class of $\Gamma(X)$ of infinite defect q is a semigroup, known in the literature as the Baer–Levi semigroup of type (p, q) [4, Section 8.1].

EXAMPLE 2.5. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. The $\mathcal{J}$-classes of $\Gamma(\mathbb{N})$ are $J_0, J_1, J_2, J_3, \ldots, J_n, \ldots, J_{\aleph_0}$ with

$$J_0 > J_1 > J_2 > J_3 > \cdots > J_n > \cdots > J_{\aleph_0}.$$

The $\mathcal{J}$-class J_0 is the symmetric group $\mathrm{Sym}(\mathbb{N})$, which forms a single $\mathcal{H}$-class. For every $n \in \mathbb{N}$, the $\mathcal{J}$-class J_n is partitioned into countably many $\mathcal{L}$-classes. Each $\mathcal{L}$-class L of J_n can be labeled with $\{m_1, m_2, \ldots, m_n\}$, where $\mathbb{N} \setminus \{m_1, m_2, \ldots, m_n\}$ is the image of each element of L. The bottom $\mathcal{J}$-class $J_{\aleph_0}$ of $\Gamma(\mathbb{N})$ consists of all injective mappings $\alpha : \mathbb{N} \to \mathbb{N}$ such that $\mathbb{N} \setminus \mathrm{im}(\alpha)$ is infinite. This $\mathcal{J}$-class is the Baer–Levi semigroup of type $(\aleph_0, \aleph_0)$.

3. Elements of $C(\alpha)$

In this section we describe the elements of $C(\alpha)$ in $\Gamma(X)$ (for an arbitrary element α of $\Gamma(X)$) using the unique decomposition of α into disjoint rays, double rays, and cycles.

DEFINITION 3.1. Let $x_0, x_1, x_2, \ldots$ be pairwise distinct elements of X. We denote by $(x_0\, x_1\, x_2 \cdots\rangle$ the transformation $\eta \in \Gamma(X)$ such that $x_i\eta = x_{i+1}$ for $i = 0, 1, \ldots$ and $y\eta = y$ for all other $y \in X$. We call such an η a *ray*. Note that $\mathrm{im}(\eta) = X \setminus \{x_0\}$.

Let $\ldots, x_{-1}, x_0, x_1, \ldots$ be pairwise distinct elements of X. We denote by $\langle \cdots x_{-1}\, x_0\, x_1 \cdots\rangle$ the transformation $\omega \in \Gamma(X)$ such that $x_i\omega = x_{i+1}$ for all integers i and $y\omega = y$ for all other $y \in X$. We call such an ω a *double ray*.

Let $x_0, x_1, \ldots, x_{n-1}$ be pairwise distinct elements of X. We denote by $(x_0\, x_1 \cdots x_{n-1})$ the transformation $\lambda \in \Gamma(X)$ such that $x_i\lambda = x_{i+1}$ for $i = 0, 1, \ldots, n-2$, $x_{n-1}\lambda = x_0$, and $y\lambda = y$ for all other $y \in X$. We call such a λ an *n-cycle* or a *cycle*.

We adopted the names 'ray' and 'double ray' from graph theory [5, Section 8.1]. A ray $\eta = (x_0\, x_1\, x_2 \cdots\rangle$ and a double ray $\omega = \langle \cdots x_{-1}\, x_0\, x_1 \cdots\rangle$ can be represented by the following directed graphs:

$$x_0 \to x_1 \to x_2 \to \cdots \quad \text{and} \quad \cdots \to x_{-1} \to x_0 \to x_1 \to \cdots .$$

The following definition is given in [9, Definition 1.1].

DEFINITION 3.2. We say that $\alpha, \beta \in \Gamma(X)$ are *disjoint* if $S(\alpha) \cap S(\beta) = \emptyset$.

Let A be a set of pairwise disjoint transformations in $\Gamma(X)$. The *formal product* of elements of A, denoted by $\prod_{\alpha \in A} \alpha$, is a transformation in $\Gamma(X)$ defined by

$$x\Big(\prod_{\alpha \in A} \alpha\Big) = \begin{cases} x\alpha & \text{if } x \in S(\alpha) \text{ for some } \alpha \in A, \\ x & \text{otherwise.} \end{cases}$$

If $A = \emptyset$, we agree that $\prod_{\alpha \in A} \alpha = \mathrm{id}_X$, where id_X is the identity transformation on X.

We note that in [9] 'rays' are called 'chains' and 'double rays' are called 'infinite cycles'. The following result is proved in [9, Proposition 1.4].

PROPOSITION 3.3. *Let $\alpha \in \Gamma(X)$ with $\alpha \neq \mathrm{id}_X$. Then there exist unique sets A of rays, B of double rays, and C of cycles of length at least 2 such that the transformations in $A \cup B \cup C$ are pairwise disjoint and*

$$\alpha = \Big(\prod_{\eta \in A} \eta\Big)\Big(\prod_{\omega \in B} \omega\Big)\Big(\prod_{\lambda \in C} \lambda\Big). \tag{3.1}$$

We will call the product (3.1) the *ray–cycle decomposition* of α. If $\alpha \in \mathrm{Sym}(X)$, then $\alpha = (\prod_{\omega \in B} \omega)(\prod_{\lambda \in C} \lambda)$ (since $A = \emptyset$), which is the decomposition given in [15, Theorem 1.3.4].

REMARK 3.4. Let $\alpha \in \Gamma(X)$ with the ray–cycle decomposition as in (3.1) and let η, ω, and λ be a ray, a double ray, and a cycle in $\Gamma(X)$, respectively. Then:

$$\eta \in A \Leftrightarrow \eta = (x\ x\alpha\ x\alpha^2\ x\alpha^3 \cdots\rangle \quad \text{for some } x \in X \setminus \mathrm{im}(\alpha),$$
$$\omega \in B \Leftrightarrow \omega = \langle \cdots x\alpha^{-2}\ x\alpha^{-1}\ x\ x\alpha\ x\alpha^2 \cdots\rangle \quad \text{for some } x \in X,$$
$$\lambda \in C \Leftrightarrow \lambda = (x\ x\alpha \cdots x\alpha^{n-1}) \quad \text{for some } x \in X \text{ and some integer } n \geq 2.$$

DEFINITION 3.5. Let

$$\eta = (x_0\, x_1\, x_2 \cdots\rangle, \quad \omega = \langle \cdots x_{-1}\, x_0\, x_1 \cdots\rangle, \quad \text{and} \quad \lambda = (x_0\, x_1 \cdots x_{n-1})$$

be a ray, a double ray, and a cycle, respectively, in $\Gamma(X)$. For $\beta \in \Gamma(X)$, we define $\eta\beta^*$, $\omega\beta^*$, and $\lambda\beta^*$ as

$$\eta\beta^* = (x_0\beta\ x_1\beta\ x_2\beta \cdots\rangle,$$
$$\omega\beta^* = \langle \cdots x_{-1}\beta\ x_0\beta\ x_1\beta \cdots\rangle,$$
$$\lambda\beta^* = (x_0\beta\ x_1\beta \cdots x_{n-1}\beta).$$

Thus β^* maps rays to rays, double rays to double rays, and n-cycles to n-cycles.

DEFINITION 3.6. For $\alpha, \beta \in \Gamma(X)$, we will say that α is *contained* in β, and write $\alpha \sqsubset \beta$, if $x\alpha = x\beta$ for every $x \in S(\alpha)$.

Note that all rays, double rays, and cycles from the ray–cycle decomposition of α (see (3.1)) are contained in α.

NOTATION 3.7. For the rest of this paper we will fix the following notation. We denote by X an arbitrary infinite set. For $\alpha \in \Gamma(X)$, let A, B, and C be the sets that occur in the ray–cycle decomposition of α (see (3.1)). By A_α, B_α, and C_α we will mean the following sets:

$$A_\alpha = A, \quad B_\alpha = B, \quad C_\alpha = C \cup \{\{x\} : x \in F(\alpha)\}.$$

(Recall that $F(\alpha)$ is the set of fixed points of α.) For every integer $n \geq 2$, we denote by C_α^n the set of n-cycles that are contained in α, that is,

$$C_\alpha^n = \{\lambda \in C_\alpha : \lambda \text{ is a cycle of length } n\}.$$

We also define

$$C^1_\alpha = \{\{x\} : x \in F(\alpha)\}.$$

For $\beta \in \Gamma(X)$, we extend the definition of β^* by $\{x\}\beta^* = \{x\beta\}$ for every $\{x\} \in C^1_\alpha$. For $\lambda \in C^n_\alpha$, we will write $\lambda = (x_0\, x_1 \cdots x_{n-1})$. It should be understood that if $n = 1$, then we mean $\lambda = \{x_0\}$ and agree that $S(\{x_0\}) = \{x_0\}$.

Finally, for $x, y \in X$, we will write $x \xrightarrow{\alpha} y$ to mean $y = x\alpha$.

LEMMA 3.8. *Let $\alpha, \beta \in \Gamma(X)$. Then*

$$\beta \in C(\alpha) \Leftrightarrow \forall x, y \in X,\ \textit{if } x \xrightarrow{\alpha} y \textit{ then } x\beta \xrightarrow{\alpha} y\beta.$$

PROOF. Suppose $\beta \in C(\alpha)$. Let $x \xrightarrow{\alpha} y$, that is, $y = x\alpha$. Then, since $\alpha\beta = \beta\alpha$, we have $y\beta = (x\alpha)\beta = (x\beta)\alpha$, and so $x\beta \xrightarrow{\alpha} y\beta$.

Conversely, suppose that β satisfies the given condition. Let $x \in X$. Since $x \xrightarrow{\alpha} x\alpha$, we have $x\beta \xrightarrow{\alpha} (x\alpha)\beta$. But this means that $(x\alpha)\beta = (x\beta)\alpha$, which implies $\alpha\beta = \beta\alpha$. Hence $\beta \in C(\alpha)$. □

We can now characterize the elements of the centralizer $C(\alpha)$.

THEOREM 3.9. *Let $\alpha, \beta \in \Gamma(X)$. Then $\beta \in C(\alpha)$ if and only if for all $\eta \in A_\alpha$, $\omega \in B_\alpha$, and $\lambda \in C_\alpha$.*

(1) *Either there is a unique $\eta_1 \in A_\alpha$ such that $\eta\beta^* \sqsubset \eta_1$ or there is a unique $\omega_1 \in B_\alpha$ such that $\eta\beta^* \sqsubset \omega_1$, and*

(2) *$\omega\beta^* \in B_\alpha$ and $\lambda\beta^* \in C_\alpha$.*

PROOF. Suppose $\beta \in C(\alpha)$. Let $\eta = (x_0\, x_1\, x_2 \cdots\rangle \in A_\alpha$. Then

$$x_0 \xrightarrow{\alpha} x_1 \xrightarrow{\alpha} x_2 \xrightarrow{\alpha} \cdots,$$

and so, by Lemma 3.8,

$$x_0\beta \xrightarrow{\alpha} x_1\beta \xrightarrow{\alpha} x_2\beta \xrightarrow{\alpha} \cdots.$$

Suppose there exists $y_0 \in X \setminus \mathrm{im}(\alpha)$ such that $x_0\beta = y_0\alpha^k$ for some integer $k \geq 0$. Then

$$\eta_1 = (y_0\, y_0\alpha \cdots y_0\alpha^{k-1}\, y_0\alpha^k = x_0\beta\, x_1\beta\, x_2\beta \cdots\rangle \in A_\alpha$$

(by Remark 3.4) and $\eta\beta^* = (x_0\beta\, x_1\beta\, x_2\beta \cdots\rangle \sqsubset \eta_1$. Suppose such a y_0 does not exist. Then $x_0\beta \in \mathrm{im}(\alpha)$ since otherwise we could take $y_0 = x_0\beta$ and $k = 0$. Thus there exists $y_{-1} \in X$ such that $x_0\beta = y_{-1}\alpha$. The element y_{-1} is not in $\mathrm{im}(\alpha)$ since otherwise we could take $y_0 = y_{-1}$ and $k = 1$. Continuing by induction, we can construct an infinite sequence $\ldots, y_{-3}, y_{-2}, y_{-1}$ of elements of x such that

$$\cdots \xrightarrow{\alpha} y_{-3} \xrightarrow{\alpha} y_{-2} \xrightarrow{\alpha} y_{-1} \xrightarrow{\alpha} x_0\beta \xrightarrow{\alpha} x_1\beta \xrightarrow{\alpha} x_2\beta \xrightarrow{\alpha} \cdots.$$

Then $\omega = \langle \cdots y_{-3}\, y_{-2}\, y_{-1}\, x_0\beta\, x_1\beta\, x_2\beta \cdots\rangle \in B_\alpha$ (by Remark 3.4) and $\eta\beta^* \sqsubset \omega_1$. The uniqueness of η_1 and ω_1 follows from the fact that the rays and double rays that occur in the ray–cycle decomposition of α are pairwise disjoint. We have proved (1).

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Let $\omega = \langle \cdots x_{-1}\, x_0\, x_1 \cdots \rangle \in B_\alpha$. Then

$$\cdots x_{-1} \xrightarrow{\alpha} x_0 \xrightarrow{\alpha} x_1 \xrightarrow{\alpha} \cdots ,$$

and so, by Lemma 3.8,

$$\cdots x_{-1}\beta \xrightarrow{\alpha} x_0\beta \xrightarrow{\alpha} x_1\beta \xrightarrow{\alpha} \cdots .$$

Thus, by Remark 3.4, $\omega\beta^* = \langle \cdots x_{-1}\beta\, x_0\beta\, x_1\beta \cdots \rangle \in B_\alpha$. The proof that $\lambda\beta^* \in C_\alpha$ for every $\lambda \in C_\alpha$ is similar. We have proved (2).

Conversely, suppose that β satisfies (1) and (2). Then it follows immediately that for all $x, y \in X$, $x \xrightarrow{\alpha} y$ implies $x\beta \xrightarrow{\alpha} y\beta$, and so $\beta \in C(\alpha)$ by Lemma 3.8. □

4. Green's relations

In this section we determine Green's relations in $C(\alpha)$, for an arbitrary $\alpha \in \Gamma(X)$, including the partial orders of $\mathcal{L}$-, $\mathcal{R}$-, and $\mathcal{J}$-classes.

DEFINITION 4.1. Let $\alpha \in \Gamma(X)$. For $\beta \in C(\alpha)$, we define a mapping $h_\beta : A_\alpha \cup B_\alpha \cup C_\alpha \to A_\alpha \cup B_\alpha \cup C_\alpha$ by:

$$\delta h_\beta = \begin{cases} \eta & \text{if } \delta \in A_\alpha \text{ and } \delta\beta^* \sqsubset \eta \text{ for some } \eta \in A_\alpha, \\ \omega & \text{if } \delta \in A_\alpha \text{ and } \delta\beta^* \sqsubset \omega \text{ for some } \omega \in B_\alpha, \\ \delta\beta^* & \text{if } \delta \in B_\alpha \cup C_\alpha. \end{cases}$$

Note that h_β is well defined by Theorem 3.9.

We will frequently use the following lemma.

LEMMA 4.2. *Let $\alpha \in \Gamma(X)$, let $\beta, \gamma \in C(\alpha)$ and let $\eta \in A_\alpha$, $\omega \in B_\alpha$, and $\lambda \in C_\alpha$. Then:*

(1) *h_β is injective;*

(2) *$h_{\beta\gamma} = h_\beta h_\gamma$;*

(3) *if $\eta = (x_0\, x_1 \cdots \rangle$, then*

$$\eta h_\beta = (\cdots x_0\beta\, x_1\beta \cdots \rangle \in A_\alpha \quad \text{or} \quad \eta h_\beta = \langle \cdots x_0\beta\, x_1\beta \cdots \rangle \in B_\alpha;$$

(4) *if $\omega = \langle \cdots x_{-1}\, x_0\, x_1 \cdots \rangle$, then $\omega h_\beta = \langle \cdots x_{-1}\beta\, x_0\beta\, x_1\beta \cdots \rangle \in B_\alpha$;*

(5) *if $\lambda = (x_0 \cdots x_{n-1}) \in C_\alpha^n$, then $\lambda h_\beta = (x_0\beta \cdots x_{n-1}\beta) \in C_\alpha^n$;*

(6) *if B_α is finite, then $B_\alpha h_\beta = B_\alpha$, $B_\alpha \cap A_\alpha h_\beta = \emptyset$, and h_β restricted to A_α is a mapping from A_α to A_α.*

PROOF. This follows immediately from the definition of h_β and Theorem 3.9. □

4.1. Relation $\mathcal{L}$. Green's relation $\mathcal{L}$ in $C(\alpha)$ is simply the restriction of the relation $\mathcal{L}$ in $\Gamma(X)$ to $C(\alpha)$. This result will follow from the following proposition.

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PROPOSITION 4.3. *Let $\alpha \in \Gamma(X)$ and $\beta, \gamma \in C(\alpha)$. Then*

$$L_\gamma \leq L_\beta \Leftrightarrow \mathrm{im}(\gamma) \subseteq \mathrm{im}(\beta).$$

PROOF. Suppose that $L_\gamma \leq L_\beta$. Then $\gamma = \delta\beta$ for some $\delta \in C(\alpha)$, and so $\mathrm{im}(\gamma) = \mathrm{im}(\delta\beta) \subseteq \mathrm{im}(\beta)$. Conversely, suppose that $\mathrm{im}(\gamma) \subseteq \mathrm{im}(\beta)$. Then Proposition 2.2(1) implies that $\gamma = \delta\beta$ for some $\delta \in \Gamma(X)$. Then

$$\alpha\delta\beta = \alpha\gamma = \gamma\alpha = \delta\beta\alpha = \delta\alpha\beta,$$

and so $\alpha\delta = \delta\alpha$ since $\Gamma(X)$ is right cancelative. That is, $\delta \in C(\alpha)$ as required. □

THEOREM 4.4. *Let $\alpha \in \Gamma(X)$ and let $\beta, \gamma \in C(\alpha)$. Then $\beta\mathcal{L}\gamma$ in $C(\alpha)$ if and only if* $\mathrm{im}(\beta) = \mathrm{im}(\gamma)$.

PROOF. This follows immediately from Proposition 4.3. □

4.2. Relation $\mathcal{R}$. Unlike the relation $\mathcal{L}$, Green's relation $\mathcal{R}$ in $C(\alpha)$ is not the restriction of the relation $\mathcal{R}$ in $\Gamma(X)$ to $C(\alpha)$.

For a mapping $f : Y \to Z$ and $A \subseteq Y$, we denote by Af the image of A under f, that is, $Af = \{af : a \in A\}$.

LEMMA 4.5. *Let $\alpha \in \Gamma(X)$ and $\beta, \gamma, \delta \in C(\alpha)$ with $\gamma = \beta\delta$. Let $A = A_\alpha$, $B = B_\alpha$, and $C_n = C_\alpha^n$ ($n \geq 1$). Then:*
(1) $(A \setminus Ah_\beta)h_\delta \subseteq (A \setminus Ah_\gamma) \cup (B \setminus (Ah_\gamma \cup Bh_\gamma))$;
(2) $(B \setminus (Ah_\beta \cup Bh_\beta))h_\delta \subseteq B \setminus (Ah_\gamma \cup Bh_\gamma)$;
(3) *for every* $n \geq 1$, $(C_n \setminus C_nh_\beta)h_\delta \subseteq C_n \setminus C_nh_\gamma$.

PROOF. Let $\mu \in (A \setminus Ah_\beta)h_\delta$. Then there is $\eta \in A \setminus Ah_\beta$ such that $\mu = \eta h_\delta$. By Lemma 4.2, $\mu \in A \cup B$. Suppose $\mu \in A$. We claim that $\mu \in A \setminus Ah_\gamma$. Suppose to the contrary that $\mu \in Ah_\gamma$. Then $\mu = \eta_1 h_\gamma$ for some $\eta_1 \in A$. Thus

$$(\eta_1 h_\beta)h_\delta = \eta_1(h_\beta h_\delta) = \eta_1 h_{\beta\delta} = \eta_1 h_\gamma = \mu = \eta h_\delta,$$

which implies that $\eta_1 h_\beta = \eta$ (since h_δ is injective). But this is a contradiction since $\eta_1 h_\beta \in Ah_\beta$ and $\eta \notin Ah_\beta$. We have proved that if $\mu \in A$ then $\mu \in A \setminus Ah_\gamma$.

Suppose that $\mu \in B$. We claim that $\mu \in B \setminus (Ah_\gamma \cup Bh_\gamma)$. Suppose to the contrary that $\mu \in Ah_\gamma \cup Bh_\gamma$. Then either $\mu = \eta_2 h_\gamma$ for some $\eta_2 \in A$ or $\mu = \omega h_\gamma$ for some $\omega \in B$. Suppose that $\mu = \eta_2 h_\gamma$. Then

$$(\eta_2 h_\beta)h_\delta = \eta_2(h_\beta h_\delta) = \eta_2 h_{\beta\delta} = \eta_2 h_\gamma = \mu = \eta h_\delta,$$

which implies that $\eta_2 h_\beta = \eta$. But this is a contradiction since $\eta_2 h_\beta \in Ah_\beta$ and $\eta \notin Ah_\beta$. Suppose that $\mu = \omega h_\gamma$. Then

$$(\omega h_\beta)h_\delta = \omega(h_\beta h_\delta) = \omega h_{\beta\delta} = \omega h_\gamma = \mu = \eta h_\delta,$$

which implies that $\omega h_\beta = \eta$. But this is a contradiction since $\omega h_\beta \in B$ and $\eta \notin B$. We have proved that if $\mu \in B$ then $\mu \in B \setminus (Ah_\gamma \cup Bh_\gamma)$. It follows that $\mu \in (A \setminus Ah_\gamma) \cup (B \setminus (Ah_\gamma \cup Bh_\gamma))$, which proves (1). The proofs of (2) and (3) are similar. □

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We will also need the following lemma from set theory (whose proof is straightforward).

LEMMA 4.6. *Let A_1, B_1, A_2, and B_2 be sets such that*

$$A_1 \cap B_1 = \emptyset, \quad A_2 \cap B_2 = \emptyset, \quad |A_1| + |B_1| \geq |A_2| + |B_2|,$$

and $|B_1| \geq |B_2|$. Then there is an injective mapping $f : A_2 \cup B_2 \to A_1 \cup B_1$ such that $bf \in B_1$ for every $b \in B_2$.

Let $\alpha \in \Gamma(X)$ and $\beta \in C(\alpha)$. Let $\eta = (x_0\, x_1 \cdots\rangle \in A_\alpha$ and suppose $\eta_1 = \eta h_\beta = (y_0 \cdots y_{k-1}\, x_0\beta\, x_1\beta \cdots\rangle \in A_\alpha$. Then note that

$$\begin{aligned} S(\eta_1) \setminus \mathrm{im}(\beta) &= \{y_0, \ldots, y_{k-1}, x_0\beta, x_1\beta, \ldots\} \setminus \{x_0\beta, x_1\beta, \ldots\} \\ &= \{y_0, \ldots, y_{k-1}\}, \end{aligned}$$

and so $k = |S(\eta_1) \setminus \mathrm{im}(\beta)|$.

The following theorem characterizes the partial order of $\mathcal{R}$-classes in $C(\alpha)$.

THEOREM 4.7. *Let $\alpha \in \Gamma(X)$ and $\beta, \gamma \in C(\alpha)$. Let $A = A_\alpha$, $B = B_\alpha$, and $C_n = C_\alpha^n$ ($n \geq 1$). Then $R_\gamma \leq R_\beta$ if and only if the following conditions are satisfied.*

(1) *For every $\eta \in A$, if $\eta h_\gamma \in A$, then $\eta h_\beta \in A$ and $|S(\eta h_\gamma) \setminus \mathrm{im}(\gamma)| \geq |S(\eta h_\beta) \setminus \mathrm{im}(\beta)|$.*

(2) $|A \setminus Ah_\gamma| + |B \setminus (Ah_\gamma \cup Bh_\gamma)| \geq |A \setminus Ah_\beta| + |B \setminus (Ah_\beta \cup Bh_\beta)|$.

(3) $|B \setminus (Ah_\gamma \cup Bh_\gamma)| \geq |B \setminus (Ah_\beta \cup Bh_\beta)|$.

(4) $|C_n \setminus C_n h_\gamma| \geq |C_n \setminus C_n h_\beta|$ *for every $n \geq 1$.*

PROOF. Suppose $R_\gamma \leq R_\beta$, that is, $\gamma = \beta\delta$ for some $\delta \in C(\alpha)$. Let $\eta = (x \cdots\rangle \in A$ and suppose $\eta h_\gamma = (y_0 \cdots y_{k-1}\, x\gamma \cdots\rangle \in A$. Then $(\eta h_\beta)h_\delta = \eta(h_\beta h_\delta) = \eta h_{\beta\delta} = \eta h_\gamma \in A$, and so we must have $\eta h_\beta \in A$ (since $\omega h_\delta \in B$ for every $\omega \in B$). By Lemma 4.2, $\eta h_\beta = (z_0 \cdots z_{m-1}\, x\beta \cdots\rangle$. We have $(\eta h_\beta)h_\delta = \eta h_\gamma$ and $x\gamma = (x\beta)\delta$ (since $\gamma = \beta\delta$). Thus, by Lemma 4.2 again, we must have $z_{m-1}\delta = y_{k-1}$, $z_{m-2}\delta = y_{k-2}, \ldots$. But this is possible only if $k \geq m$. We have proved (1).

By Lemma 4.5,

$$(A \setminus Ah_\beta)h_\delta \cup (B \setminus (Ah_\beta \cup Bh_\beta))h_\delta \subseteq (A \setminus Ah_\gamma) \cup (B \setminus (Ah_\gamma \cup Bh_\gamma)),$$

and so

$$|(A \setminus Ah_\gamma) \cup (B \setminus (Ah_\gamma \cup Bh_\gamma))| \geq |(A \setminus Ah_\beta)h_\delta \cup (B \setminus (Ah_\beta \cup Bh_\beta))h_\delta|. \tag{4.1}$$

Since $(A \setminus Ah_\gamma) \cap (B \setminus (Ah_\gamma \cup Bh_\gamma)) = \emptyset$,

$$|(A \setminus Ah_\gamma) \cup (B \setminus (Ah_\gamma \cup Bh_\gamma))| = |A \setminus Ah_\gamma| + |B \setminus (Ah_\gamma \cup Bh_\gamma)|. \tag{4.2}$$

Since $(A \setminus Ah_\beta) \cap (B \setminus (Ah_\beta \cup Bh_\beta)) = \emptyset$ and h_δ is injective,

$$\begin{aligned} &|(A \setminus Ah_\beta)h_\delta \cup (B \setminus (Ah_\beta \cup Bh_\beta))h_\delta| \\ &\quad = |(A \setminus Ah_\beta)h_\delta| + |(B \setminus (Ah_\beta \cup Bh_\beta))h_\delta| \\ &\quad = |A \setminus Ah_\beta| + |B \setminus (Ah_\beta \cup Bh_\beta)|. \end{aligned} \tag{4.3}$$

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Now, (4.1)–(4.3) imply condition (2). Conditions (3) and (4) follow from Lemma 4.5 in a similar way.

Conversely, suppose that conditions (1)–(4) are satisfied. We will construct $\delta \in C(\alpha)$ such that $\gamma = \beta\delta$. We first define δ on $S(\mu)$ for every $\mu \in \mathrm{im}(h_\beta)$.

Let $\eta \in A \cap Ah_\beta$. Then there is a unique $\eta_1 = (x \cdots\rangle \in A$ such that

$$\eta = \eta_1 h_\beta = (z_0 \cdots z_{m-1}\, x\beta \cdots\rangle.$$

Let $\xi = \eta_1 h_\gamma$. Then either $\xi \in A$ or $\xi \in B$. Suppose that $\xi = (y_0 \cdots y_{k-1}\, x\gamma \cdots\rangle$ is in A. Then, by (1), $k \geq m$, and so we may define δ on $S(\eta)$ in such a way that $\eta\delta^* \sqsubset \xi$ and $(x\beta)\delta = x\gamma$. Suppose that $\xi = \langle \cdots x\gamma \cdots\rangle \in B$. In this case, we may certainly define δ on $S(\eta)$ in such a way that $\eta\delta^* \sqsubset \xi$ and $(x\beta)\delta = x\gamma$.

Let $\omega \in B \cap Ah_\beta$. Then there is a unique $\eta = (x \cdots\rangle \in A$ such that

$$\omega = \eta h_\beta = \langle \cdots x\beta \cdots\rangle.$$

Let $\omega_1 = \eta h_\gamma = \langle \cdots x\gamma \cdots\rangle$. (Note that, by (1), $\eta h_\beta \in B$ implies that $\eta h_\gamma \in B$.) We define δ on $S(\omega)$ in such a way that $\omega\delta^* = \omega_1$ and $(x\beta)\delta = x\gamma$.

Let $\omega \in Bh_\beta$. Then there is a unique $\omega_1 = \langle \cdots x_{-1}\, x_0\, x_1 \cdots\rangle \in B$ such that

$$\omega = \omega_1 h_\beta = \langle \cdots x_{-1}\beta\; x_0\beta\; x_1\beta \cdots\rangle.$$

Let

$$\omega_2 = \omega_1 h_\gamma = \langle \cdots x_{-1}\gamma\; x_0\gamma\; x_1\gamma \cdots\rangle.$$

We define δ on $S(\omega)$ in such a way that $\omega\delta^* = \omega_2$ and $(x_i\beta)\delta = x_i\gamma$ for every $i \in \mathbb{Z}$.

Let $\lambda \in C_n h_\beta$, where $n \geq 1$. Then there is a unique $\lambda_1 = (x_0 \cdots x_{n-1}) \in C_n$ such that $\lambda = \lambda_1 h_\beta = (x_0\beta \cdots x_{n-1}\beta)$. Let $\lambda_2 = \lambda_1 h_\gamma = (x_0\gamma \cdots x_{n-1}\gamma)$. We define δ on $S(\lambda)$ in such a way that $\lambda\delta^* = \lambda_2$ and $(x_i\beta)\delta = x_i\gamma$ for every $i \in \{0, \ldots, n-1\}$.

So far, we have defined δ on $S(\mu)$ for every $\mu \in \mathrm{im}(h_\beta)$. In particular, δ has been defined for every $x \in \mathrm{im}(\beta)$. Note that $\beta\delta = \gamma$ (regardless of how δ will be defined on the remaining elements of X) and that δ satisfies (1) and (2) of Theorem 3.9 for all $\eta \in A_\alpha \cap A_\alpha\beta$, $\omega \in B_\alpha \cap (A_\alpha\beta \cup B_\alpha\beta)$, and $\lambda \in C_\alpha \cap C_\alpha\beta$. It remains to complete the definition of δ in such a way that $\delta \in \Gamma(X)$ and $\delta \in C(\alpha)$.

By (2), (3), and Lemma 4.6, there is an injective mapping

$$k : (A \setminus Ah_\beta) \cup (B \setminus (Ah_\beta \cup Bh_\beta)) \to (A \setminus Ah_\gamma) \cup (B \setminus (Ah_\gamma \cup Bh_\gamma))$$

such that $\omega k \in B \setminus (Ah_\gamma \cup Bh_\gamma)$ for every $\omega \in B \setminus (Ah_\beta \cup Bh_\beta)$. By (4), we have that for every integer $n \geq 1$, there is an injective mapping

$$g_n : C_n \setminus C_n h_\beta \to C_n \setminus C_n h_\gamma.$$

If $\eta \in A \setminus Ah_\beta$, we define δ on $S(\eta)$ in such a way that $\eta\delta^* \sqsubset \eta k$. (If $\eta k \in A \setminus Ah_\gamma$, it is possible to define δ in such a way that $\eta\delta^* = \eta k$, but this does not matter.) If $\omega \in B \setminus (Ah_\beta \cup Bh_\beta)$, we define δ on $S(\omega)$ in such a way that $\omega\delta^* = \omega k$. (Note that $\omega k \in B \setminus (Ah_\gamma \cup Bh_\gamma)$.) Finally, if $\lambda \in C_n \setminus C_n h_\beta$ for some $n \geq 1$, we define δ on $S(\lambda)$ in such a way that $\lambda\delta^* = \lambda g_n$.

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The construction of δ is complete. By the definition of δ and Theorem 3.9, we have $\delta \in \Gamma(X)$, $\delta \in C(\alpha)$, and $\gamma = \beta\delta$. Hence $R_\gamma \leq R_\beta$, which completes the proof. □

By combining Theorem 4.7 and its dual, we immediately obtain a characterization of the $\mathcal{R}$ relation in $C(\alpha)$: namely, rewrite (1) as: 'for every $\eta \in A$,

$$\eta h_\beta \in A \text{ and } |S(\eta h_\beta) \setminus \operatorname{im}(\beta)| = k \Leftrightarrow \eta h_\gamma \in A \text{ and } |S(\eta h_\gamma) \setminus \operatorname{im}(\gamma)| = k\text{'},$$

and replace '$\geq$' with '$=$' in (2)–(4).

4.3. Relation $\mathcal{J}$. In the semigroup $\Gamma(X)$, we have $\mathcal{R} = \mathcal{D} = \mathcal{J}$. It will follow from this section that, in general, this is not true in the centralizer $C(\alpha)$.

The following theorem describes the partial order of the $\mathcal{J}$-classes in $C(\alpha)$.

THEOREM 4.8. *Let $\alpha \in \Gamma(X)$ and $\beta, \gamma \in C(\alpha)$. Let $A = A_\alpha$, $B = B_\alpha$, $C = C_\alpha$, and $C_n = C_\alpha^n$ $(n \geq 1)$. Then $J_\gamma \leq J_\beta$ if and only if the following conditions are satisfied.*

(1) *There are injective mappings $f : A \cap Ah_\gamma \to A \cap Ah_\beta$ and $g : B \cap Ah_\gamma \to (A \cup B)h_\beta$ such that*

$$|S(\eta) \setminus \operatorname{im}(\gamma)| \geq |S(\eta f) \setminus \operatorname{im}(\beta)|$$

for all $\eta \in A \cap Ah_\gamma$, $\operatorname{im}(f) \cap \operatorname{im}(g) = \emptyset$, and

$$\begin{aligned}&|A \setminus Ah_\gamma| + |B \setminus (Ah_\gamma \cup Bh_\gamma)|\\ &\quad \geq |A \setminus Ah_\beta| + |B \setminus (Ah_\beta \cup Bh_\beta)| + |Ah_\beta \setminus (\operatorname{im}(f) \cup \operatorname{im}(g))|.\end{aligned}$$

(2) $|B \setminus (Ah_\gamma \cup Bh_\gamma)| \geq |B \setminus (Ah_\beta \cup Bh_\beta)|$.

(3) $|C_n \setminus C_n h_\gamma| \geq |C_n \setminus C_n h_\beta|$ *for every $n \geq 1$.*

PROOF. Suppose that $J_\gamma \leq J_\beta$, that is, $\gamma = \varepsilon\beta\delta$ for some $\varepsilon, \delta \in C(\alpha)$. Let $\eta \in A \cap Ah_\gamma$. Then there is a unique $\eta_1 = (x \cdots\rangle \in A$ such that

$$\eta = \eta_1 h_\gamma = (y_0 \cdots y_{k-1}\, x\gamma \cdots\rangle \in A.$$

Note that $k = |S(\eta) \setminus \operatorname{im}(\gamma)|$. By Lemma 4.2,

$$\begin{aligned}\eta_1 h_\gamma \in A &\Rightarrow \eta_1 h_{\varepsilon\beta\delta} \in A\\ &\Rightarrow \eta_1(h_\varepsilon h_\beta h_\delta) \in A\\ &\Rightarrow ((\eta_1 h_\varepsilon)h_\beta)h_\delta \in A\\ &\Rightarrow (\eta_1 h_\varepsilon)h_\beta \in A\\ &\Rightarrow \eta_1 h_\varepsilon \in A.\end{aligned}$$

Let $\eta_1 h_\varepsilon = (w_0 \cdots w_{l-1}\, x\varepsilon \cdots\rangle$ and

$$(\eta_1 h_\varepsilon)h_\beta = (z_0 \cdots z_{m-1}\, w_0\beta \cdots w_{l-1}\beta\, (x\varepsilon)\beta \cdots\rangle.$$

Note that $m = |S((\eta_1 h_\varepsilon)h_\beta) \setminus \operatorname{im}(\beta)|$. We have

$$\begin{aligned}(z_0 \cdots z_{m-1}\, w_0\beta \cdots w_{l-1}\beta\, (x\varepsilon)\beta \cdots\rangle h_\delta &= ((\eta_1 h_\varepsilon)h_\beta)h_\delta = \eta_1 h_\gamma\\ &= (y_0 \cdots y_{k-1}\, x\gamma \cdots\rangle\end{aligned}$$

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and $x\gamma = ((x\varepsilon)\beta)\delta$ (since $\gamma = \varepsilon\beta\delta$). Thus we must have

$$(w_{l-1}\beta)\delta = y_{k-1}, \dots, (w_0\beta)\delta = y_{k-l}, z_{m-1}\delta = y_{k-l-1}, z_{m-2}\delta = y_{k-l-2}, \dots,$$

which implies that $k \geq m + l \geq m$. Recall that $k = |S(\eta) \setminus \mathrm{im}(\gamma)|$ and $m = |S((\eta_1 h_\varepsilon)h_\beta) \setminus \mathrm{im}(\beta)|$. Define $f : A \cap Ah_\gamma \to A \cap Ah_\beta$ by $\eta f = (\eta_1 h_\varepsilon)h_\beta$. Then f is injective (since h_ε and h_β are injective) and $|S(\eta) \setminus \mathrm{im}(\gamma)| \geq |S(\eta f) \setminus \mathrm{im}(\beta)|$ (since $k \geq m$).

Let $\omega \in B \cap Ah_\gamma$. Then there is a unique $\eta \in A$ such that $\omega = \eta h_\gamma$. Define $g : B \cap Ah_\gamma \to (A \cup B)h_\beta$ by $\omega g = (\eta h_\varepsilon)h_\beta$. Then $\omega g \in (A \cup B)h_\beta$ (since $\eta h_\varepsilon \in A \cup B$) and g is injective (since h_ε and h_β are injective).

Suppose that $\eta \in \mathrm{im}(f) \cap \mathrm{im}(g)$, that is, $\eta = \eta_1 f$ and $\eta = \omega g$ for some $\eta_1 \in A \cap Ah_\gamma$ and $\omega \in B \cap Ah_\gamma$. Thus $\eta_1 = \eta_2 h_\gamma$ and $\omega = \eta_3 h_\gamma$ for some $\eta_2, \eta_3 \in A$. By the definitions of f and g, we have $\eta = \eta_1 f = (\eta_2 h_\varepsilon)h_\beta$ and $\eta = \omega g = (\eta_3 h_\varepsilon)h_\beta$. But then, since $h_\gamma = h_\varepsilon h_\beta h_\delta$,

$$\eta h_\delta = ((\eta_2 h_\varepsilon)h_\beta)h_\delta = \eta_2 h_\gamma = \eta_1 \quad \text{and} \quad \eta h_\delta = ((\eta_3 h_\varepsilon)h_\beta)h_\delta = \eta_3 h_\gamma = \omega,$$

which is a contradiction since $\eta_1 \in A$ and $\omega \in B$. Hence $\mathrm{im}(f) \cap \mathrm{im}(g) = \emptyset$.

To prove the displayed inequality in (1), first note that, by the definitions of f and g, we have $\mathrm{im}(f) \cup \mathrm{im}(g) = (Ah_\varepsilon)h_\beta$, and so $Ah_\beta \setminus (\mathrm{im}(f) \cup \mathrm{im}(g)) = Ah_\beta \setminus (Ah_\varepsilon)h_\beta$. Define a mapping

$$\begin{aligned} j : (A \setminus (Ah_\varepsilon)h_\beta) \cup (B \setminus ((Ah_\varepsilon)h_\beta \ \cup \ (Bh_\varepsilon)h_\beta)) \cup (Ah_\beta \setminus (Ah_\varepsilon)h_\beta) \\ \to (A \setminus Ah_\gamma) \cup (B \setminus (Ah_\gamma \cup Bh_\gamma)) \end{aligned}$$

by $\mu j = \mu h_\delta$. Then j is injective (since h_δ is injective) but we must show that the codomain of j is as stated.

Let

$$\mu \in (A \setminus (Ah_\varepsilon)h_\beta) \cup (B \setminus ((Ah_\varepsilon)h_\beta \cup (Bh_\varepsilon)h_\beta)).$$

Then

$$\mu j = \mu h_\delta \in (A \setminus Ah_\gamma) \cup (B \setminus (Ah_\gamma \cup Bh_\gamma))$$

by Lemma 4.5 (since $\gamma = (\varepsilon\beta)\delta$ and $h_\varepsilon h_\beta = h_{\varepsilon\beta}$).

Let $\mu \in Ah_\beta \setminus (Ah_\varepsilon)h_\beta$, that is, $\mu = \eta h_\beta$ for some $\eta \in A$, and $\mu \notin (Ah_\varepsilon)h_\beta$. Then $\mu j = \mu h_\delta \in A \cup B$.

Suppose that $\mu h_\delta \in Ah_\gamma$, that is, $\mu h_\delta = \eta_1 h_\gamma$ for some $\eta_1 \in A$. Then $(\eta_1 h_\varepsilon)h_\beta h_\delta = \eta_1 h_\gamma = \mu h_\delta$, which implies that $(\eta_1 h_\varepsilon)h_\beta = \mu$ (since h_δ is injective). But this is a contradiction since $(\eta_1 h_\varepsilon)h_\beta \in (Ah_\varepsilon)h_\beta$ and $\mu \notin (Ah_\varepsilon)h_\beta$. Thus $\mu h_\delta \notin Ah_\gamma$.

Suppose that $\mu h_\delta \in Bh_\gamma$, that is, $\mu h_\delta = \omega h_\gamma$ for some $\omega \in B$. Then $(\omega h_\varepsilon)(h_\beta h_\delta) = \omega h_\gamma = \mu h_\delta = \eta(h_\beta h_\delta)$, which implies that $\omega h_\varepsilon = \eta$ (since $h_\beta h_\delta$ is injective). But this is a contradiction since $\omega h_\varepsilon \in B$ and $\eta \in A$. Thus $\mu h_\delta \notin Bh_\gamma$.

Hence $\mu j = \mu h_\delta \in (A \setminus Ah_\gamma) \cup (B \setminus (Ah_\gamma \cup Bh_\gamma))$, which concludes the proof that j is well defined. Since j is injective,

$$|(A \setminus Ah_\gamma) \cup (B \setminus (Ah_\gamma \cup Bh_\gamma))| \geq |\mathrm{dom}(j)|. \tag{4.4}$$

Since $(A \cap Ah_\varepsilon)h_\beta \subseteq Ah_\beta$ and $(Ah_\varepsilon)h_\beta \cup (Bh_\varepsilon)h_\beta \subseteq Ah_\beta \cup Bh_\beta$, we also have that

$$(A \setminus (Ah_\varepsilon)h_\beta) \cup (B \setminus ((Ah_\varepsilon)h_\beta \cup (Bh_\varepsilon)h_\beta)) \supseteq (A \setminus Ah_\beta) \cup (B \setminus (Ah_\beta \cup Bh_\beta)),$$

and so

$$|\mathrm{dom}(j)| \geq |(A \setminus Ah_\beta) \cup (B \setminus (Ah_\beta \cup Bh_\beta)) \cup (Ah_\beta \setminus (Ah_\varepsilon)h_\beta)|. \tag{4.5}$$

Since $(A \setminus Ah_\gamma) \cap (B \setminus (Ah_\gamma \cup Bh_\gamma)) = \emptyset$ and $A \setminus Ah_\beta$, $B \setminus (Ah_\beta \cup Bh_\beta)$, and $Ah_\beta \setminus (Ah_\varepsilon)h_\beta$ are pairwise disjoint, (4.4) and (4.5) imply the displayed inequality in (1).

Proofs of (2) and (3) are similar to (but easier than) the proof of the inequality in (1). For (2), we define an injection

$$k : B \setminus ((Ah_\varepsilon)h_\beta \cup (Bh_\varepsilon)h_\beta) \to B \setminus (Ah_\gamma \cup Bh_\gamma)$$

by $\omega k = \omega h_\delta$; and for (3), an injection $m : C_n \setminus (C_n h_\varepsilon)h_\beta \to C_n \setminus C_n h_\gamma$ by $\lambda m = \lambda h_\delta$. Then k and m are well defined by Lemma 4.5, and (2) and (3) easily follow.

Conversely, suppose that conditions (1)–(3) are satisfied. We will construct $\varepsilon, \delta \in C(\alpha)$ such that $\gamma = \varepsilon\beta\delta$. We first define ε on $S(\mu)$ for every $\mu \in A \cup B \cup C$, and δ on $S(\mu)$ for every $\mu \in \mathrm{im}(h_\beta) \setminus (Ah_\beta \setminus (\mathrm{im}(f) \cup \mathrm{im}(g)))$.

Let $\eta = (x \cdots\rangle \in A$ be such that $\eta_1 = \eta h_\gamma = (y_0 \cdots y_{k-1}\, x\gamma \cdots\rangle \in A$. Let $\eta_2 = \eta_1 f \in A \cap Ah_\beta$. Then there is a unique $\eta_3 = (w \cdots\rangle \in A$ such that $\eta_2 = \eta_3 h_\beta = (z_0 \cdots z_{m-1}\, w\beta \cdots\rangle$. Define ε on $S(\eta)$ and δ on $S(\eta_2)$ in such a way that $\eta\varepsilon^* = \eta_3$ and $\eta_2\delta \sqsubset \eta_1$ with $(w\beta)\delta = x\gamma$. (Note that this definition of δ is possible since $k = |S(\eta_1) \setminus \mathrm{im}(\gamma)| \geq |S(\eta_2) \setminus \mathrm{im}(\beta)| = m$ by (1), and that $x(\varepsilon\beta\delta) = ((x\varepsilon)\beta)\delta = (w\beta)\delta = x\gamma$.)

To proceed with the definitions of ε and δ, we need to prove the following:

$$|Bh_\gamma| + |\{\omega \in B \cap Ah_\gamma : \omega g \in Bh_\beta\}| = |B|. \tag{4.6}$$

We have $|Bh_\gamma| = |B|$ (since h_γ is injective) and $|\{\omega \in B \cap Ah_\gamma : \omega g \in Bh_\beta\}| \leq |B|$. Thus, if B is infinite, then $|Bh_\gamma| + |\{\omega \in B \cap Ah_\gamma : \omega g \in Bh_\beta\}| = |B|$. Suppose B is finite. Then $Bh_\gamma = B$ since $Bh_\gamma \subseteq B$ and $|Bh_\gamma| = |B|$. Hence

$$\{\omega \in B \cap Ah_\gamma : \omega g \in Bh_\beta\} = \{\omega \in Bh_\gamma \cap Ah_\gamma : \omega g \in Bh_\beta\} = \emptyset,$$

and so

$$|Bh_\gamma| + |\{\omega \in B \cap Ah_\gamma : \omega g \in Bh_\beta\}| = |Bh_\gamma| + 0 = |Bh_\gamma| = |B|.$$

We have proved (4.6).

Since $Bh_\gamma \cap \{\omega \in B \cap Ah_\gamma : \omega g \in Bh_\beta\} = \emptyset$, then

$$|Bh_\gamma \cup \{\omega \in B \cap Ah_\gamma : \omega g \in Bh_\beta\}| = |Bh_\gamma| + |\{\omega \in B \cap Ah_\gamma : \omega g \in Bh_\beta\}|.$$

Thus, by (4.6), $|Bh_\gamma \cup \{\omega \in B \cap Ah_\gamma : \omega g \in Bh_\beta\}| = |B|$. We also have that $|Bh_\beta| = |B|$ (since h_β is injective). Hence, there is a bijection

$$p : Bh_\gamma \cup \{\omega \in B \cap Ah_\gamma : \omega g \in Bh_\beta\} \to Bh_\beta.$$

Let $\eta = (x \cdots\rangle \in A$ be such that $\omega = \eta h_\gamma = \langle \cdots x\gamma \cdots\rangle \in B \cap Ah_\gamma$. Then $\mu = \omega g \in (A \cup B)h_\beta$.

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Suppose that $\mu \in Ah_\beta$. Then there is a unique $\eta_1 = (y\cdots\rangle \in A$ such that $\mu = \eta_1 h_\beta$. If $\mu = (z_0 \cdots z_{t-1}\ y\beta\cdots\rangle \in A$, then define ε on $S(\eta)$ and δ on $S(\mu)$ in such a way that $\eta\varepsilon^* = \eta_1$ and $\mu\delta^* \sqsubset \omega$ with $(y\beta)\delta = x\gamma$. If $\mu = \langle\cdots y\beta\cdots\rangle \in B$, then define ε on $S(\eta)$ and δ on $S(\mu)$ in such a way that $\eta\varepsilon^* = \eta_1$ and $\mu\delta^* = \omega$ with $(y\beta)\delta = x\gamma$. (Note that in both cases we have $x(\varepsilon\beta\delta) = ((x\varepsilon)\beta)\delta = (y\beta)\delta = x\gamma$.)

Suppose that $\mu \in Bh_\beta$. Then $\omega \in B \cap Ah_\gamma$ and $\omega g = \mu \in Bh_\beta$, that is, $\omega \in \mathrm{dom}(p)$. Let $\omega_1 = \omega p \in Bh_\beta$. Then there is a unique $\omega_2 = \langle\cdots y_{-1}\ y_0\ y_1\cdots\rangle \in B$ such that $\omega_1 = \omega_2 h_\beta = \langle\cdots y_{-1}\beta\ y_0\beta\ y_1\beta\cdots\rangle$. Define ε on $S(\eta)$ and δ on $S(\omega_1)$ in such a way that $\eta\varepsilon^* \sqsubset \omega_2$ with $x\varepsilon = y_0$ and $\omega_1\delta^* = \omega$ with $(y_0\beta)\delta = x\gamma$.

Let $\omega = \langle\cdots x_{-1}\ x_0\ x_1\cdots\rangle \in B$. Then

$$\omega_1 = \omega h_\gamma = \langle\cdots x_{-1}\gamma\ x_0\gamma\ x_1\gamma\cdots\rangle \in Bh_\gamma.$$

Let $\omega_2 = \omega_1 p \in Bh_\beta$. Then there is a unique $\omega_3 = \langle\cdots y_{-1}\ y_0\ y_1\cdots\rangle \in B$ such that $\omega_2 = \omega_3 h_\beta = \langle\cdots y_{-1}\beta\ y_0\beta\ y_1\beta\cdots\rangle$. Define ε on $S(\omega)$ and δ on $S(\omega_2)$ in such a way that $\omega\varepsilon^* = \omega_3$ with $x_i\varepsilon = y_i$ (for every $i \in \mathbb{Z}$) and $\omega_2\delta^* = \omega_1$ with $(y_i\beta)\delta = x_i\gamma$ (for every $i \in \mathbb{Z}$).

Let $\lambda = (x_0 \cdots x_{n-1}) \in C_n$, where $n \geq 1$. Then $\lambda_1 = \lambda h_\gamma = (x_0\gamma \cdots x_{n-1}\gamma) \in C_n h_\gamma$. Since $|C_n h_\gamma| = |C_n h_\beta|$, there is a bijection $k : C_n h_\gamma \to C_n h_\beta$. Let $\lambda_2 = \lambda_1 k \in C_n h_\beta$. Then there is a unique $\lambda_3 = (y_0 \cdots y_{n-1}) \in C_n$ such that $\lambda_2 = \lambda_3 h_\beta = (y_0\beta \cdots y_{n-1}\beta)$. Define ε on $S(\lambda)$ and δ on $S(\lambda_2)$ in such a way that $\lambda\varepsilon^* = \lambda_3$ with $x_i\varepsilon = y_i$ (for every $0 \leq i \leq n-1$) and $\lambda_2\delta^* = \lambda_1$ with $(y_i\beta)\delta = x_i\gamma$ (for every $0 \leq i \leq n-1$).

So far, we have defined ε on the whole of X and δ on $S(\mu)$ for every $\mu \in \mathrm{im}(h_\beta)$ except for those μ that lie in $Ah_\beta \setminus (\mathrm{im}(f) \cup \mathrm{im}(g))$. Also, by the construction of ε and δ, we already have $\varepsilon\beta\delta = \gamma$. It remains to define δ on $S(\mu)$ for every

$$\mu \in (A \setminus Ah_\beta) \cup (B \setminus (Ah_\beta \cup Bh_\beta)) \cup Ah_\beta \setminus (\mathrm{im}(f) \cup \mathrm{im}(g)).$$

We proceed as in the proof of Theorem 4.7.

By (1), (2), and Lemma 4.6, there is an injective mapping

$$t : (A \setminus Ah_\beta)\ \cup\ (B \setminus (Ah_\beta \cup Bh_\beta)) \cup (Ah_\beta \setminus (\mathrm{im}(f) \cup \mathrm{im}(g))) \\ \to (A \setminus Ah_\gamma) \cup (B \setminus (Ah_\gamma \cup Bh_\gamma))$$

such that $\omega t \in B \setminus (Ah_\gamma \cup Bh_\gamma)$ for every $\omega \in B \setminus (Ah_\beta \cup Bh_\beta)$. By (3), we have that for every integer $n \geq 1$, there is an injective mapping

$$q_n : C_n \setminus C_n h_\beta \to C_n \setminus C_n h_\gamma.$$

If $\eta \in A \setminus Ah_\beta$ or $\eta \in Ah_\beta \setminus (\mathrm{im}(f) \cup \mathrm{im}(g))$, we define δ on $S(\eta)$ in such a way that $\eta\delta^* \sqsubset \eta t$. If $\omega \in B \setminus (Ah_\beta \cup Bh_\beta)$, we define δ on $S(\omega)$ in such a way that $\omega\delta^* = \omega t$. Finally, if $\lambda \in C_n \setminus C_n h_\beta$ for some $n \geq 1$, we define δ on $S(\lambda)$ in such a way that $\lambda\delta^* = \lambda q_n$.

The construction of ε and δ is complete. By the definition of ε and δ and Theorem 3.9, we have $\varepsilon, \delta \in \Gamma(X)$, $\varepsilon, \delta \in C(\alpha)$, and $\gamma = \varepsilon\beta\delta$. Hence $J_\gamma \leq J_\beta$, which completes the proof. □

By combining Theorem 4.8 and its dual, we can easily obtain a characterization of the $\mathcal{J}$ relation in $C(\alpha)$: namely, rewrite (1) using two pairs of functions (f_1, g_1 and f_2, g_2) and two inequalities, and replace '$\geq$' with '$=$' in (2) and (3).

4.4. Relation $\mathcal{D}$. This section shows that, in general, the relation $\mathcal{D}$ in $C(\alpha)$ is strictly between the relations $\mathcal{R}$ and $\mathcal{J}$.

THEOREM 4.9. *Let $\alpha \in \Gamma(X)$ and $\beta, \gamma \in C(\alpha)$. Let $A = A_\alpha$, $B = B_\alpha$, and $C_n = C_\alpha^n$ ($n \geq 1$). Then $\beta\mathcal{D}\gamma$ in $C(\alpha)$ if and only if the following conditions are satisfied.*

(1) *There is a bijection $f : A \cap Ah_\beta \to A \cap Ah_\gamma$ such that for every $\eta \in A \cap Ah_\beta$,*

$$|S(\eta) \setminus \mathrm{im}(\beta)| = |S(\eta f) \setminus \mathrm{im}(\gamma)|.$$

(2) $|B \cap Ah_\beta| = |B \cap Ah_\gamma|$.

(3) $|A \setminus Ah_\beta| + |B \setminus (Ah_\beta \cup Bh_\beta)| = |A \setminus Ah_\gamma| + |B \setminus (Ah_\gamma \cup Bh_\gamma)|$.

(4) $|B \setminus (Ah_\beta \cup Bh_\beta)| = |B \setminus (Ah_\gamma \cup Bh_\gamma)|$.

(5) $|C_n \setminus C_n h_\beta| = |C_n \setminus C_n h_\gamma|$ *for every* $n \geq 1$.

PROOF. Suppose $\beta\mathcal{D}\gamma$. Then, since $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ in any semigroup [6, p. 46], there is $\delta \in C(\alpha)$ such that $\beta\mathcal{R}\delta$ and $\delta\mathcal{L}\gamma$. Let $\eta \in A \cap Ah_\delta$. Then there is a unique $\eta_1 = (x_0\, x_1 \cdots\rangle \in A$ such that $\eta = \eta_1\delta = (y_0 \cdots y_{k-1}\, x_0\delta\, x_1\delta \cdots\rangle$. Since $\delta\mathcal{L}\gamma$, we have $\mathrm{im}(\delta) = \mathrm{im}(\gamma)$ by Theorem 4.4. Thus there is a unique $\eta_2 = (z_0\, z_1 \cdots\rangle \in A$ such that $\eta = \eta_2\gamma = (y_0 \cdots y_{k-1}\, z_0\gamma\, z_1\gamma \cdots\rangle$.

We have proved that for every $\eta \in A \cap Ah_\delta$, $\eta \in A \cap Ah_\gamma$ and $S(\eta) \setminus \mathrm{im}(\delta) = S(\eta) \setminus \mathrm{im}(\gamma)$. By symmetry, the previous statement is also true when we switch δ and γ. It follows that

$$A \cap Ah_\delta = A \cap Ah_\gamma \quad \text{and} \quad (\forall \eta \in A \cap Ah_\delta)(S(\eta) \setminus \mathrm{im}(\delta) = S(\eta) \setminus \mathrm{im}(\gamma)). \tag{4.7}$$

It follows from (4.7) that

$$A \setminus Ah_\delta = A \setminus (A \cap Ah_\delta) = A \setminus (A \cap Ah_\gamma) = A \setminus Ah_\gamma. \tag{4.8}$$

Let $\eta \in A \cap Ah_\beta$. Then there is a unique $\eta_1 \in A$ such that $\eta = \eta_1 h_\beta$. Define a mapping $f : A \cap Ah_\beta \to A \cap Ah_\gamma$ by $\eta f = \eta_1 h_\delta$. Since $\beta\mathcal{R}\delta$, we have, by Theorem 4.7(1), that $\eta_1 h_\delta \in A$ and $|S(\eta_1 h_\beta) \setminus \mathrm{im}(\beta)| = |S(\eta_1 h_\delta) \setminus \mathrm{im}(\delta)|$. Thus, by (4.7), $\eta f \in A \cap Ah_\gamma$ and $|S(\eta) \setminus \mathrm{im}(\beta)| = |S(\eta f) \setminus \mathrm{im}(\gamma)|$. The mapping f is injective since h_δ is injective. Let $\mu \in A \cap Ah_\gamma$. Then, by (4.7), there is $\eta_1 \in A$ such that $\mu = \eta_1 h_\delta$. Since $\beta\mathcal{R}\delta$, $\eta_1 h_\delta \in A$ implies that $\eta_1 h_\beta \in A$. Thus $(\eta_1 h_\beta) f = \eta_1 h_\delta = \mu$, which shows that f is onto.

We have proved that (1) holds. Let $\omega \in B \cap Ah_\delta$. Then there is a unique $\eta = (x_0\, x_1 \cdots\rangle \in A$ such that $\omega = \eta h_\delta = \langle \cdots y_{-2}\, y_{-1}\, x_0\delta\, x_1\delta \cdots\rangle$. Since $\mathrm{im}(\delta) = \mathrm{im}(\gamma)$, there is a unique $\eta_1 = (z_0\, z_1 \cdots\rangle \in A$ such that

$$\omega = \eta_1 h_\gamma = \langle \cdots y_{-2}\, y_{-1}\, z_0\gamma\, z_1\gamma \cdots\rangle.$$

We have proved that $B \cap Ah_\delta \subseteq B \cap Ah_\gamma$. The reverse inclusion holds by symmetry, and so

$$B \cap Ah_\delta = B \cap Ah_\gamma. \tag{4.9}$$

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Since $\beta\mathcal{R}\delta$, we have, by Theorem 4.7(1), that for every $\eta \in A$, $\eta h_\beta \in B$ if and only if $\eta h_\delta \in B$. Hence $|B \cap Ah_\beta| = |B \cap Ah_\delta|$, and so (2) follows by (4.9).

Since $\mathcal{D} \subseteq \mathcal{J}$ in any semigroup, we have that $\beta\mathcal{J}\gamma$, and so (4) and (5) are satisfied by Theorem 4.8. Suppose that $\omega \in B \setminus (Ah_\delta \cup Bh_\delta)$. Then $S(\omega) \cap \mathrm{im}(\delta) = \emptyset$. Thus, since $\mathrm{im}(\delta) = \mathrm{im}(\gamma)$, we have $S(\omega) \cap \mathrm{im}(\gamma) = \emptyset$, and so $\omega \in B \setminus (Ah_\gamma \cup Bh_\gamma)$. We have proved that $B \setminus (Ah_\delta \cup Bh_\delta) \subseteq B \setminus (Ah_\gamma \cup Bh_\gamma)$. The reverse inclusion holds by a similar argument, and so

$$B \setminus (Ah_\delta \cup Bh_\delta) = B \setminus (Ah_\gamma \cup Bh_\gamma). \tag{4.10}$$

Since $\beta\mathcal{R}\delta$,

$$|A \setminus Ah_\beta| + |B \setminus (Ah_\beta \cup Bh_\beta)| = |A \setminus Ah_\delta| + |B \setminus (Ah_\delta \cup Bh_\delta)| \tag{4.11}$$

by Theorem 4.7. It is now clear that condition (3) is satisfied by (4.8), (4.10), and (4.11).

Conversely, suppose that β and γ satisfy (1)–(5). By (2), there is a bijection $g : B \cap Ah_\beta \to B \cap Ah_\gamma$. We will construct $\delta \in C(\alpha)$ such that $\beta\mathcal{R}(\beta\delta)$ and $(\beta\delta)\mathcal{L}\gamma$. We first define δ on $S(\mu)$ for every $\mu \in \mathrm{im}(h_\beta)$.

Let $\eta \in A \cap Ah_\beta$. Then there is a unique $\eta_1 = (x \cdots\rangle \in A$ such that $\eta = \eta_1 h_\beta = (z_0 \cdots z_{m-1}\, x\beta \cdots\rangle$. Let $\eta_2 = \eta f \in A \cap Ah_\gamma$. Then, by (1), there is a unique $\eta_3 = (y \cdots\rangle \in A$ such that $\eta_2 = \eta_3 h_\gamma = (w_0 \cdots w_{m-1}\, y\gamma \cdots\rangle$. Define δ on $S(\eta)$ in such a way that $\eta\delta^* \sqsubset \eta_2$ and $(x\beta)\delta = y\gamma$.

Let $\omega \in B \cap Ah_\beta$. Then there is a unique $\eta = (x \cdots\rangle \in A$ such that $\omega = \eta h_\beta = \langle \cdots x\beta \cdots\rangle$. Let $\omega_1 = \omega g \in B \cap Ah_\gamma$. Then there is a unique $\eta_2 = (y \cdots\rangle \in A$ such that $\omega_1 = \eta_2 h_\gamma = \langle \cdots y\gamma \cdots\rangle$. Define δ on $S(\omega)$ in such a way that $\omega\delta^* = \omega_1$ and $(x\beta)\delta = y\gamma$.

Let $\omega \in Bh_\beta$. Then there is a unique $\omega_1 = \langle \cdots x_{-1}\, x_0\, x_1 \cdots\rangle \in B$ such that $\omega = \omega_1 h_\beta = \langle \cdots x_{-1}\beta\, x_0\beta\, x_1\beta \cdots\rangle$. Let $\omega_2 = \omega_1 h_\gamma = \langle \cdots x_{-1}\gamma\, x_0\gamma\, x_1\gamma \cdots\rangle$. We define δ on $S(\omega)$ in such a way that $\omega\delta^* = \omega_2$ and $(x_i\beta)\delta = x_i\gamma$ for every $i \in \mathbb{Z}$.

Let $\lambda \in C_n h_\beta$, where $n \geq 1$. Then there is a unique $\lambda_1 = (x_0 \cdots x_{n-1}) \in C_n$ such that $\lambda = \lambda_1 h_\beta = (x_0\beta \cdots x_{n-1}\beta)$. Let $\lambda_2 = \lambda_1 h_\gamma = (x_0\gamma \cdots x_{n-1}\gamma)$. We define δ on $S(\lambda)$ in such a way that $\lambda\delta^* = \lambda_2$ and $(x_i\beta)\delta = x_i\gamma$ for every $i \in \{0, \ldots, n-1\}$.

So far, we have defined δ on $S(\mu)$ for every $\mu \in \mathrm{im}(h_\beta)$. In particular, δ has been defined for every $x \in \mathrm{im}(\beta)$. It remains to complete the definition of δ in such a way that $\delta \in \Gamma(X)$ and $\delta \in C(\alpha)$. This we do exactly as in the last part of the proof of Theorem 4.7 (the part that starts with the line preceding the displayed definition of the mapping k).

The construction of δ is complete. By the definition of δ, Theorems 3.9, 4.4, and 4.7, we have $\delta \in \Gamma(X)$, $\delta \in C(\alpha)$, $\beta\mathcal{R}(\beta\delta)$, and $(\beta\delta)\mathcal{L}\gamma$. Thus, $(\beta, \gamma) \in \mathcal{R} \circ \mathcal{L} = \mathcal{D}$, which completes the proof. □

In the semigroup $\Gamma(X)$, Green's relations $\mathcal{R}$, $\mathcal{D}$, and $\mathcal{J}$ coincide and the $\mathcal{J}$-classes form a chain (see Section 2). It is of interest to describe $\alpha \in \Gamma(X)$ for which Green's relations coincide in $C(\alpha)$, and $\alpha \in \Gamma(X)$ for which the $\mathcal{J}$-classes form a chain. These descriptions will be provided in a subsequent paper. In that paper, we will also find the structure of $C(\alpha)$ in terms of direct and wreath products of familiar semigroups in the case where α is a permutation.

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