ON THE RADICAL OF A CATEGORY

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1. Introduction

In [1] the concept of *completeness* of a functor was introduced and, in the case of additive * categories $\mathscr{C}$ and $\mathscr{D}$ and an additive functor T: $\mathscr{C} \to \mathscr{D}$, a criterion for T (supposed surjective) to be complete was given in terms of the kernel $\mathscr{K}$ of T: this was that for each object A of $\mathscr{C}$ the ideal $\mathscr{K}_A$ should be contained in the (Jacobson) radical of $\mathscr{C}_A$. (The meaning of this notation and nomenclature is recalled in § 2 below). The question arises whether in any additive category $\mathscr{C}$ there is a greatest ideal $\mathscr{K}$ with this property, so that the canonical functor $T: \mathscr{C} \to \mathscr{C}/\mathscr{K}$ is in some sense the coarsest that faithfully represents the objects (but not the maps) of $\mathscr{C}$.

This question is answered affirmatively in § 3 below; if $\mathscr{R}$ is the ideal in question, it turns out that $\mathscr{R}_A$ is not only contained in, but is in fact equal to, the radical of $\mathscr{C}_A$. The relation of $\mathscr{R}$ to $\mathscr{C}$ is entirely analogous to the relation of the radical of a ring to that ring, and we shall call $\mathscr{R}$ the *radical* of the category $\mathscr{C}$. On the one hand the existence and properties of $\mathscr{R}$ are but simple translations of well-known properties of the radical in a ring; on the other hand the "category" point of view, without adding anything essentially new to the theory of the radical in a ring, may be said to exhibit some of its properties in a new light.

The notion of completeness in no way requires the additivity of the categories and functors in question, and so we can ask similar questions for a general category. A functor can still be said to have a kernel, which is no longer an ideal but a *congruence*, that is, an equivalence relation compatible with the operation of composition of maps. We prove in § 4 that in any category $\mathscr{C}$ there is a greatest congruence $\mathfrak{r}$ such that $\mathscr{C} \to \mathscr{C}/\mathfrak{r}$

* What we call *additive* categories are commonly called *pre-additive*, the epithet *additive* being reserved for those categories which also admit finite direct sums. There is not yet a uniform, rational scheme for describing various types of categories, and we suggest tentatively that the description should first say what extra structure, if any, the sets of morphisms (= maps) possess — additive category, graded differential category, etc. — and then describe the existential hypotheses made — existence of direct sums, of kernels, etc. Thus what others call an *additive* category we shall call a *direct additive* category, shortened to *direct* category when none but additive categories are in question.

is complete. If $\mathscr{C}$ happens to be additive, it may happen that $\mathfrak{r}$ strictly exceeds the (congruence corresponding to the) radical $\mathscr{R}$ of $\mathscr{C}$, and so we need a name for $\mathfrak{r}$ different from "radical"; we shall call it the ***radix*** of $\mathscr{C}$. If the category $\mathscr{C}$ contains only a single object A, $\mathfrak{r}$ is a congruence in the monoid $\mathscr{C}_A$, and we shall also call $\mathfrak{r}$ the radix of this monoid, in analogy with the radical of a ring. But now, in distinction to the additive case, if $\mathscr{C}$ is a general category and $\mathfrak{r}$ its radix, $\mathfrak{r}_A$ may be strictly less than the radix of $\mathscr{C}_A$. We illustrate some of the possibilities by calculating the radices of various categories in § 5.

2. Definitions and general considerations

If A and B are objects in the category $\mathscr{C}$ we denote the set of maps (sometimes called morphisms) $f : A \to B$ in $\mathscr{C}$ by $\mathscr{C}(A, B)$, and we abbreviate $\mathscr{C}(A, A)$ to $\mathscr{C}(A)$ or $\mathscr{C}_A$. If $f : A \to B$ and $g : B \to C$ we write gf (and not fg) for the composed map $A \to C$, and we use 1 indifferently for the identity maps of various objects. We call $f \in \mathscr{C}(A, B)$ an *equivalence* if there is a $g \in \mathscr{C}(B, A)$ with $fg = 1$ and $gf = 1$; g is then unique and we write $g = f^{-1}$. We shall also call an equivalence $f \in \mathscr{C}_A$ a *unit* of the monoid $\mathscr{C}_A$.

By a *congruence* $\mathfrak{k}$ on the category $\mathscr{C}$ we mean the selection, for each pair of objects A, B in $\mathscr{C}$, of an equivalence relation $\mathfrak{k}(A, B)$ on the set $\mathscr{C}(A, B)$, subject to the requirement that, whenever $h \in \mathscr{C}(A, B)$, $f, f' \in \mathscr{C}(B, C)$, $g \in \mathscr{C}(C, D)$, and $f \equiv f'(\mathfrak{k}(B, C))$, then $gfh \equiv gf'h(\mathfrak{k}(A, D))$. We shall usually write $f \equiv f'(\mathfrak{k})$ rather than $f \equiv f'(\mathfrak{k}(B, C))$; we shall also write $\mathfrak{k}_A$ for $\mathfrak{k}(A, A)$. Congruences on $\mathscr{C}$ are ordered by: $\mathfrak{l} \geqq \mathfrak{k}$ if $f \equiv f'(\mathfrak{k})$ implies $f \equiv f'(\mathfrak{l})$.

The category $\mathscr{C}$ is said to be *additive* if each $\mathscr{C}(A, B)$ is an abelian group and composition of maps is bilinear. $\mathscr{C}_A$ is then a ring with identity. By an ideal $\mathscr{K}$ in $\mathscr{C}$ is meant the selection, for each pair of objects A, B in $\mathscr{C}$, of a subgroup $\mathscr{K}(A, B)$ of $\mathscr{C}(A, B)$, subject to the requirement that $gfh \in \mathscr{K}(A, D)$ whenever $g \in \mathscr{C}(C, D)$, $f \in \mathscr{K}(B, C)$, and $h \in \mathscr{C}(A, B)$. $\mathscr{K}_A = \mathscr{K}(A, A)$ is then an ideal of the ring $\mathscr{C}_A$. An ideal $\mathscr{K}$ determines a congruence by putting $f \equiv f'$ for $f - f' \in \mathscr{K}$; and the congruences $\mathfrak{k}$ that arise in this manner are those for which $f \equiv f'$ and $g \equiv g'$ imply $f - g \equiv f' - g'$. Where there is no danger of confusion, we shall use the same symbol for an ideal and the congruence it determines.

The additive category $\mathscr{C}$ is said to be *direct* if it admits finite direct sums (including the direct sum of *no* objects, that is, a null object). Any additive category $\mathscr{C}$ can be embedded as a full subcategory in a direct category $\mathscr{D}$ by taking as an object A of $\mathscr{D}$ a finite sequence $(A_1, A_2, \cdots, A_n)$ of objects of $\mathscr{C}$, and taking $\mathscr{D}(A, B)$, where A is as above and $B = (B_1, B_2, \cdots, B_m)$, to consist of the matrices $F = (f_{ij})$ where $f_{ij} \in \mathscr{C}(A_j, B_i)$.

A direct sum of A and B in $\mathscr{D}$ is then given by $(A_1, \cdots, A_n, B_1, \cdots, B_m)$.

If $\mathscr{C}$ and $\mathscr{D}$ are any categories and $T : \mathscr{C} \to \mathscr{D}$ a functor, we say that T is *surjective* if every object P of $\mathscr{D}$ is equivalent to an object of the form TA where A is an object of $\mathscr{C}$, and if moreover, for any two objects A, B of $\mathscr{C}$, any map $g \in \mathscr{D}(TA, TB)$ is Tf for some $f \in \mathscr{C}(A, B)$. We say T is *injective* if, $Tf = Tg$, for $f, g \in \mathscr{C}(A, B)$, implies $f = g$. If T is both surjective and injective we say it is *bijective*; if we admit the axiom of choice for a class that may not be a set, it comes to the same thing to say that T is an *isomorphism*, meaning that there is a functor $R : \mathscr{D} \to \mathscr{C}$ with each of TR and RT naturally equivalent to the appropriate identity functor. We say that T is *complete* if it is surjective and *equivalence-reflecting*: by which we mean that $f \in \mathscr{C}(A, B)$ is an equivalence whenever $Tf \in \mathscr{D}(TA, TB)$ is an equivalence. A bijective functor is easily seen to be complete. If T is complete then A and B are equivalent whenever TA and TB are; so that A is faithfully represented by TA, or TA is a "complete set of invariants" of A.

If $\mathfrak{k}$ is a congruence in $\mathscr{C}$ we can form a quotient category $\mathscr{C}/\mathfrak{k}$, with the same objects as $\mathscr{C}$, by defining $(\mathscr{C}/\mathfrak{k})(A, B)$ as $\mathscr{C}(A, B)/\mathfrak{k}(A, B)$, the quotient set of $\mathscr{C}(A, B)$ by the equivalence relation $\mathfrak{k}(A, B)$. There is a canonical surjective functor $S : \mathscr{C} \to \mathscr{C}/\mathfrak{k}$ which sends each map to its equivalence class (and is the identity on objects). If $\mathscr{C}$ is additive and $\mathfrak{k}$ is derived from an ideal $\mathscr{K}$, then $\mathscr{C}/\mathfrak{k} = \mathscr{C}/\mathscr{K}$ is additive, and S is an additive functor (that is, $S(f+g) = Sf+Sg$).

If $T : \mathscr{C} \to \mathscr{D}$ is any functor, a congruence $\mathfrak{k}$ on $\mathscr{C}$ called the *kernel* of T is defined by: $f \equiv f'(\mathfrak{k})$ if $Tf = Tf'$. T then factorizes as $\mathscr{C} \underset{S}{\to} \mathscr{C}/\mathfrak{k} \underset{U}{\to} \mathscr{D}$, where U is injective; and T is surjective if and only if U is bijective. Thus if T is surjective $\mathscr{D}$ is essentially determined by $\mathscr{C}$ and $\mathfrak{k}$. In particular, if T is surjective, it is complete if and only if S is so. If $\mathscr{C}$, $\mathscr{D}$, and T are additive, the kernel is an ideal and S and U are additive.

For a surjective functor T, completeness is equivalent to the apparently weaker condition that $f \in \mathscr{C}_A$ be a unit of $\mathscr{C}_A$ whenever $Tf = 1$. For if $f \in \mathscr{C}(A, B)$ and Tf is an equivalence, then because T is surjective there is a $g \in \mathscr{C}(B, A)$ with $Tg = (Tf)^{-1}$. Then $T(gf) = 1$ and $T(fg) = 1$, so that by hypothesis gf has an inverse h and fg has an inverse k. Since $hgf = 1$ and $fgk = 1$, f has a left inverse and a right inverse and so is an equivalence. This means, in terms of the kernel $\mathfrak{k}$ of T, that T is complete if and only if, for each object A of $\mathscr{C}$, the $\mathfrak{k}_A$-equivalence-class containing the identity element of $\mathscr{C}_A$ contains only units of $\mathscr{C}_A$. In the additive case, if the kernel is the ideal $\mathscr{K}$, this equivalence class is $1+\mathscr{K}_A$; and for this to consist only of units it is necessary and sufficient that $\mathscr{K}_A$ be contained in the radical of $\mathscr{C}_A$ (or again, that $\mathscr{K}_A$, considered in itself, should be a radical ring).

3. The radical of an additive category

If P is a direct sum of $A_1, \cdots, A_n$ in an additive category $\mathscr{C}$, we shall write $P = \oplus_1^n A_\alpha$ and shall denote the injection $A_\alpha \to P$ by i_α and the projection $P \to A_\alpha$ by p_α. Then we have

$$p_\alpha i_\beta = \delta_{\alpha\beta} \qquad \text{(the kronecker delta)},$$
$$\sum i_\alpha p_\alpha = 1.$$

It will be convenient to use $P = \oplus_1^m A_\alpha$ for a second direct sum, with maps i'_α, p'_α; and so on.

The maps $f \in \mathscr{C}(P, P')$ are in 1-1 correspondence with the matrices $F = (f_{\alpha\beta})$, where $f_{\alpha\beta} \in \mathscr{C}(A_\beta, A'_\alpha)$; f determines F by $f_{\alpha\beta} = p'_\alpha f i_\beta$, and F determines f by $f = \sum_{\alpha,\beta} i'_\alpha f_{\alpha\beta} p_\beta$. If similarly $g \in \mathscr{C}(P', P'')$ corresponds to the matrix G, then gf corresponds to the matrix product GF.

From the relations between f and F we deduce at once:

LEMMA 1. *If $\mathscr{I}$ is an ideal of $\mathscr{C}$, then $f \in \mathscr{I}(P, P')$ if and only if $f_{\alpha\beta} \in \mathscr{I}(A_\beta, A'_\alpha)$ for each α, β.*

Now let $\mathscr{D}$ be an additive category, of which $\mathscr{C}$ is a full subcategory. An ideal $\mathscr{J}$ of $\mathscr{D}$ determines by restriction to $\mathscr{C}$ an ideal $\mathscr{I}$ of $\mathscr{C}$: $\mathscr{I}(A, B) = \mathscr{J}(A, B)$ for A, $B \in \mathscr{C}$; we can call $\mathscr{I}$ the trace of $\mathscr{J}$ on $\mathscr{C}$.

LEMMA 2. *If every object of $\mathscr{D}$ can be expressed as a finite direct sum of objects of $\mathscr{C}$, then $\mathscr{J} \to \mathscr{I}$ is a one-to-one correspondence between the ideals of $\mathscr{D}$ and those of $\mathscr{C}$.*

By lemma 1, $\mathscr{J}$ is completely determined by $\mathscr{I}$, and it remains only to prove that any ideal $\mathscr{I}$ of $\mathscr{C}$ is the trace of such a $\mathscr{J}$. Given $\mathscr{I}$, define $\mathscr{J}$ thus: if $P = \oplus A_\alpha$, $P' = \oplus A'_\alpha$ are in $\mathscr{D}$, where A_α, $A'_\alpha \in \mathscr{C}$, then $f \in \mathscr{C}(P, P')$ is in $\mathscr{J}$ if and only if each $f_{\alpha\beta} = p'_\alpha f i_\beta$ is in $\mathscr{I}$. $\mathscr{J}(P, P')$ has to be proved independent of the direct decompositions of P and P' used; we prove this simultaneously with the fact that $\mathscr{J}$ is an ideal by verifying that $f \in \mathscr{J}$ implies $gfh \in \mathscr{J}$, where $h \in \mathscr{C}(P'', P)$ and $g \in \mathscr{C}(P', P''')$, $P'' = \oplus A''_\alpha$, $P''' = \oplus A'''_\alpha$: that $\mathscr{J}$ is well-defined follows by taking $g = 1$, $h = 1$. The verification is immediate: the elements of the matrix GFH are in $\mathscr{I}$ since those of F are and since $\mathscr{I}$ is an ideal.

It was remarked in [1] that an ideal $\mathscr{I}$ in an additive category is not determined by the knowledge of $\mathscr{I}_A$ for all objects A of $\mathscr{C}$. However:

LEMMA 3. *If the additive category $\mathscr{C}$ is direct, an ideal $\mathscr{I}$ of $\mathscr{C}$ is determined by the $\mathscr{I}_A$ alone.*

For, if A_1 and A_2 are objects of $\mathscr{C}$, let $P = A_1 \oplus A_2$ be a direct sum. Then, by lemma 1, $f \in \mathscr{C}(A_1, A_2)$ is in $\mathscr{I}(A_1, A_2)$ if and only if $i_2 f p_1$ is in $\mathscr{I}_P$.

Of course, the $\mathscr{I}_A$ cannot be chosen arbitrarily:

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LEMMA 4. *If in a direct category $\mathscr{C}$ we are given for each object A an ideal $\mathscr{I}_A$ of the ring $\mathscr{C}_A$, then $\mathscr{I}_A$ can be extended to an ideal $\mathscr{I}$ of $\mathscr{C}$ if and only if $gfh \in \mathscr{I}_B$ whenever $g \in \mathscr{C}(A, B)$, $f \in \mathscr{I}_A$, and $h \in \mathscr{C}(B, A)$.*

The condition is clearly necessary. If it is fulfilled define $\mathscr{J}(A_1, A_2)$ by $f \in \mathscr{J}(A_1, A_2)$ if and only if $i_2 f p_1 \in \mathscr{I}_P$, where $P = A_1 \oplus A_2$. Then $\mathscr{J}$ is an ideal; for if $g \in \mathscr{C}(A_2, A_2')$ and $h \in \mathscr{C}(A_1', A_1)$, let $P' = A_1' \oplus A_2'$. Then $f \in \mathscr{J}(A_1, A_2)$ implies $i_2' gfhp_1' = i_2' g p_2 \cdot i_2 f p_1 \cdot i_1 h p_1' \in \mathscr{I}_{P'}$, since $i_2 f p_1 \in \mathscr{I}_P$. Moreover $\mathscr{J}_A = \mathscr{I}_A$; for if $P = A_1 \oplus A_2$ where $A_1 = A_2 = A$, we have that $f \in \mathscr{I}_A$ implies $i_2 f p_1 \in \mathscr{I}_P$, so that $\mathscr{I}_A \leqq \mathscr{J}_A$; and $i_2 f p_1 \in \mathscr{I}_P$ implies $f = p_2 i_2 f p_1 i_1 \in \mathscr{I}_A$, so that $\mathscr{J}_A \leqq \mathscr{I}_A$.

LEMMA 5. *In a direct category $\mathscr{C}$, let $\mathscr{R}_A$ be the radical of $\mathscr{C}_A$ for each object A. Then $\mathscr{R}_A$ extends to a unique ideal $\mathscr{R}$ of $\mathscr{C}$.*

That the ideal is unique if it exists follows from lemma 3; we must verify that the $\mathscr{R}_A$ satisfy the condition of lemma 4.

Let $f \in \mathscr{R}_A$, $g \in \mathscr{C}(A, B)$, $h \in \mathscr{C}(B, A)$. Let $P = A \oplus B$, and let us represent elements of $\mathscr{C}_P$ by the corresponding matrices.

We use the fact that x is in the radical of a ring if and only if $1 - yx$ is a unit for all y in the ring. In this way we see that $\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{R}_P$; for $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1-af & 0 \\ -cf & 1 \end{pmatrix}$, and this has the inverse $\begin{pmatrix} u & 0 \\ cfu & 1 \end{pmatrix}$, where u is the inverse of $1-af$, which exists since $f \in \mathscr{R}_A$. Then for any $e \in \mathscr{C}_B$ we have, since $\mathscr{R}_P$ is an ideal, that $\mathscr{R}_P$ contains $\begin{pmatrix} 0 & 0 \\ eg & 0 \end{pmatrix}\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & egfh \end{pmatrix}$. Thus $\begin{pmatrix} 1 & 0 \\ 0 & 1-egfh \end{pmatrix}$ is a unit of $\mathscr{C}_P$, and hence $1-egfh$ is a unit of $\mathscr{C}_B$, for any $e \in \mathscr{C}_B$; we conclude that $gfh \in \mathscr{R}_B$, as required.

We are now in a position to define the radical of an additive category $\mathscr{C}$. We embed $\mathscr{C}$ in a direct category $\mathscr{D}$ as in § 2, and we take the unique ideal $\mathscr{R}$ of $\mathscr{D}$ such that $\mathscr{R}_P$ is the radical of $\mathscr{D}_P$ for each object P of $\mathscr{D}$. The radical of $\mathscr{C}$ is then the trace of $\mathscr{R}$ on $\mathscr{C}$, which we shall still call $\mathscr{R}$.

LEMMA 6. *If $\mathscr{R}$ is the radical of an additive category $\mathscr{C}$, $\mathscr{R}(A, B)$ depends only on the subcategory of $\mathscr{C}$ determined by A and B; in fact the necessary and sufficient condition for $f \in \mathscr{C}(A, B)$ to be in $\mathscr{R}(A, B)$ is that $1-gf$ be a unit of $\mathscr{C}_A$ for all $g \in \mathscr{C}(B, A)$.*

Let $P = A \oplus B$; then $f \in \mathscr{R}(A, B)$ if and only if $\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} \in \mathscr{R}_P$; that is, if and only if $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} = \begin{pmatrix} 1-bf & 0 \\ -df & 1 \end{pmatrix}$ is a unit for all b and d. But this has an inverse $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ if and only if $q = 0$, $s = 1$, p is an inverse of $1-bf$, and $r = dfp$; and so is a unit for all b and d if and only if $1-bf$ is a unit for all b.

THEOREM 1. *$\mathscr{R}$ is the greatest ideal of $\mathscr{C}$ for which $\mathscr{R}_A$ is contained in the radical of $\mathscr{C}_A$ for each A.*

For if $\mathscr{I}$ were any such ideal and $f \in \mathscr{I}(A, B)$, then for any $g \in \mathscr{C}(B, A)$

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we should have $gf \in \mathscr{I}_A$ and so $1-gf$ would be a unit of $\mathscr{C}_A$. Thus $\mathscr{I} \leqq \mathscr{R}$.

We remark that lemmas 1 and 6 now imply various known results on the radicals of certain rings. If $P = \oplus_1^n A_\alpha$ is a direct sum in $\mathscr{C}$, so that $\mathscr{C}_P$ consists of the matrices $F = (f_{\alpha\beta})$ with $f_{\alpha\beta} \in \mathscr{C}(A_\beta, A_\alpha)$, then, by lemma 1, $\mathscr{R}_P$, the radical of $\mathscr{C}_P$, consists of the matrices F with $f_{\alpha\beta} \in \mathscr{R}(A_\beta, A_\alpha)$. If all the A_α are identical with the object A, then $\mathscr{C}_P$ is the ring of $n \times n$ matrices with elements in $\mathscr{C}_A$, and its radical consists of the matrices with $f_{\alpha\beta} \in \mathscr{R}_A$ for all α, β. Again, if the A_α are indecomposable modules satisfying both chain conditions, we know that each $\mathscr{C}_{A_\alpha}$ is a local ring; then if $f \in \mathscr{C}(A_\alpha, A_\beta)$ and $g \in \mathscr{C}(A_\beta, A_\alpha)$, $1-gf$ is a unit whenever gf is a non-unit. If A_α and A_β are non-isomorphic, gf is of necessity a non-unit, otherwise f would map A_α isomorphically onto a direct summand of A_β; while if A_α and A_β are isomorphic, gf is a non-unit whenever f is a non-unit, and if f is a unit, $1-f^{-1}f = 0$ is not a unit. Thus in this case $\mathscr{R}_P$ consists of the matrices F in which no $f_{\alpha\beta}$ is an equivalence. Finally, we can easily calculate the radical of the category of finitely-generated abelian groups; it suffices to know $\mathscr{R}(A, B)$ when A and B are indecomposable. The indecomposables are $\boldsymbol{Z}$ and $\boldsymbol{Z}_{p^n}$; by the above, $\mathscr{R}(\boldsymbol{Z}_{p^n}, \boldsymbol{Z}_{q^m})$ consists of the non-isomorphisms; $\mathscr{C}(\boldsymbol{Z}_{p^n}, \boldsymbol{Z}) = 0$; $\mathscr{R}(\boldsymbol{Z}, \boldsymbol{Z}_{p^n}) = \mathscr{C}(\boldsymbol{Z}, \boldsymbol{Z}_{p^n}) = \boldsymbol{Z}_{p^n}$ since in this case $1-gf = 1$ because $g = 0$; and $\mathscr{R}(\boldsymbol{Z}) =$ radical of $\boldsymbol{Z} = 0$.

4. The radix of an arbitrary category

Let $\mathscr{C}$ be any category. For each triple A, B, C of objects of $\mathscr{C}$ we define an equivalence relation $\mathfrak{k}(A, B; C)$ on $\mathscr{C}(A, B)$ thus: if $f, g : A \to B$ are any maps in $\mathscr{C}$ then $f \equiv g(\mathfrak{k}(A, B; C))$ if and only if, for any $u \in \mathscr{C}(B, C)$ and any $v \in \mathscr{C}(C, A)$, ufv and ugv are either both units in $\mathscr{C}_C$ or both non-units in $\mathscr{C}_C$. We now define an equivalence relation $\mathfrak{r}(A, B)$ on $\mathscr{C}(A, B)$ by: $f \equiv g(\mathfrak{r}(A, B))$ if and only if $f \equiv g(\mathfrak{k}(A, B; C))$ for each object C of $\mathscr{C}$. (The objects of $\mathscr{C}$ in general form a class and not a set, but we can say that $\mathfrak{r}(A, B)$ is defined as the intersection of the *set* of those equivalence relations on $\mathscr{C}(A, B)$ which are of the form $\mathfrak{k}(A, B; C)$ for some object C of $\mathscr{C}$.)

THEOREM 2. *$\mathfrak{r}$, which we shall call the radix of $\mathscr{C}$, is a congruence on $\mathscr{C}$, and is the greatest one for which, for any A, $f \in \mathscr{C}_A$ is a unit whenever $f \equiv 1$.*

If $m \in \mathscr{C}(B, D)$, $n \in \mathscr{C}(E, A)$ and $f \equiv g(\mathfrak{r}(A, B))$ then $mfn \equiv mgn(\mathfrak{r}(E, D))$; for if C is any object of $\mathscr{C}$ and $u \in \mathscr{C}(D, C)$, $v \in \mathscr{C}(C, E)$, we have $umfnv$ and $umgnv$ are both units or both non-units of $\mathscr{C}_C$, since $f \equiv g(\mathfrak{k}(A, B; C))$ for all C; thus $\mathfrak{r}$ is indeed a congruence.

If $f \in \mathscr{C}_A$ and $f \equiv 1(\mathfrak{r}_A)$, we have $f \equiv 1(\mathfrak{k}(A, A; A))$ and so $1 \cdot f \cdot 1$ and $1 \cdot 1 \cdot 1$ are both units or both non-units; so that f is in fact a unit.

If $\mathfrak{s}$ is any congruence on $\mathscr{C}$ with the property that f is a unit whenever $f \equiv 1$, let $f \equiv g(\mathfrak{s}(A, B))$, $u \in \mathscr{C}(B, C)$, $v \in \mathscr{C}(C, A)$. Then since $\mathfrak{s}$ is a con-

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gruence $ufv \equiv ugv(\mathfrak{z}_C)$. If ufv is a unit of $\mathscr{C}_C$ we have, again since $\mathfrak{z}$ is a congruence, $1 = (ufv)(ufv)^{-1} \equiv (ugv)(ufv)^{-1}(\mathfrak{z}_C)$; thus by hypothesis $(ugv)(ufv)^{-1}$ is a unit, whence ugv is a unit. It follows that $f \equiv g(\mathfrak{k}(A, B; C))$, and since this is true for all C we have $f \equiv g(\mathfrak{r})$. Hence $\mathfrak{z} \leq \mathfrak{r}$.

If $\mathscr{C}$ is a category with a single object A, and thus in effect just a monoid $M = \mathscr{C}_A$, we shall also call the radix of $\mathscr{C}$ the radix of the monoid M. It is the equivalence relation on M given by: $f \equiv g$ if and only if ufv and ugv are simultaneously units or simultaneously non-units for all $u, v \in M$. It will appear in the examples below that, if $\mathfrak{r}$ is the radix of an arbitrary category $\mathscr{C}$, the radix of the monoid $\mathscr{C}_A$ may strictly exceed $\mathfrak{r}_A$.

5. Examples of radices

(a) We first consider the radix of a single monoid M. If all the elements of M are equivalent (i.e. modulo its radix $\mathfrak{r}$) they must all be units, and so M is a group. Conversely if M is a group it is clear that all its elements are equivalent. The simplest case apart from this trivial one is the case where there are just two equivalence classes in M. These classes must then consist of the units of M and the non-units of M; for whatever is equivalent to a unit is itself a unit. In this case the product of two non-units is always a non-unit, for if x and y are non-units we have $x \equiv y$ and so $x^2 \equiv xy$; x^2 is a non-unit since x is a non-unit, whence xy is a non-unit. It evidently comes to the same thing to say that if $xy = 1$ then x and y are units. If, conversely, M is such that the product of two non-units is always a non-unit, then the equivalence relation which partitions M into the units and the non-units is easily seen to be a congruence, and so must be the radix.

There are many classes of monoids that satisfy this condition; clearly any commutative monoid does so, and so does any finite monoid. For if M is finite and $xy = 1$, the map $z \to yz$ of M into itself is injective and so surjective, whence y has also a right inverse and so is a unit.

Again, if M happens to be a ring, the condition is satisfied if M is indecomposable (as a left M-module), for if $xy = 1$ and $yx \neq 1$ then yx is a non-trivial idempotent. The condition is also satisfied if M is a ring with the maximum condition for left ideals, as may be seen by a simple argument of the Fitting's-lemma type. Finally, if M is the ring of $n \times n$ matrices A over a commutative ring, the condition is again satisfied: for A is a unit if and only if $\det A$ is a unit.

(b) An example of a monoid not satisfying the condition above is given by the monoid M of all maps of N into itself, where N is the set of natural numbers. We shall calculate the radix of M.

Suppose that $f \in M$ and $f \equiv 1$; and suppose if possible that $f \neq 1$. We may suppose without loss of generality that $f(1) \neq 1$; let $f(1) = n$.

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Since $f \equiv 1$ it is a unit, and so bijective; thus $f(i) \neq n$ for any $i > 1$. Define, $v, u \in M$ by:

$$v(i) = i+1;$$
$$u(i) = i-1, \qquad i > 1;$$
$$u(1) = n.$$

Then $uv = 1$, and since $f \equiv 1$ we must have $ufv \equiv u1v = 1$, so that ufv must be a unit. But n is not in the image of fv, so that $n-1$ is not in the image of ufv, which therefore cannot be a unit. Hence the equivalence class of the identity map 1 reduces to 1 alone.

Now let f be any element of M with im f infinite, and suppose if possible that $g \equiv f$ but $g \neq f$. We may suppose $g(1) \neq f(1)$; and since whenever v is a unit $vg \equiv vf$ if and only if $g \equiv f$, we may further suppose that $f(1) = 1$. Define $u \in M$ by: u maps im f bijectively onto N, preserving the order; $u(i) = 2$ if $i \notin \text{im } f$. In particular $u(1) = 1$. Define $v \in M$ by choosing $v(i)$ to be any element of $(uf)^{-1}(i)$, taking $v(1)$ in particular to be 1. Then $ufv = 1$, so that $ugv = 1$ if $g \equiv f$. But $ugv(1) = ug(1) \neq 1$, so that $ugv \neq 1$. Hence the equivalence class of f reduces to f alone.

If im f is finite so is im ufv for any u and v; so the equivalence relation on M given by putting all f with im f finite into one class and letting any other f be the sole member of its class is in fact a congruence; clearly this congruence is the radix of M.

(c) We now consider the category $\mathscr{C}$ of all sets and all maps. Let X be a set with only two elements. Then if A, B are any sets and $f, g : A \to B$ any maps, a necessary condition for $f \equiv g$ (i.e. modulo the radix of $\mathscr{C}$) is that, for any $u : B \to X$ and any $v : X \to A$, ufv and ugv should be both units or both non-units of $\mathscr{C}_X$. Thus f and g are inequivalent if it is possible to find two elements x, y of A with $f(x)$ and $f(y)$ different and $f(x)$ equal to neither $g(x)$ nor $g(y)$; for then it is clear that u and v may be so chosen that ufv is the identity on X while ugv is constant.

If im f has at least three elements and $g \neq f$, it is always possible to choose x, y as above, and so $g \not\equiv f$; thus the equivalence class of f consists of f alone. If im f has exactly two elements a and b, it is still possible to find x, y as above unless either $g = f$ or g is related to f by: for each $z \in A$, $g(z) = a$ whenever $f(z) = b$ and $g(z) = b$ whenever $f(z) = a$.

If we define an equivalence relation $\mathfrak{r}(A, B)$ on $\mathscr{C}(A, B)$ by: all constant maps from one equivalence class; a map f with im f composed of exactly two elements is equivalent only to itself and to the map g related to it as above; any other map is the sole member of its equivalence class — then by what we have proved $\mathfrak{r}$ is certainly the radix of $\mathscr{C}$ if it is a congruence. However, $\mathfrak{r}$ is indeed a congruence, as may easily be verified, and so is the radix of $\mathscr{C}$.

(d) Now let $\mathscr{C}$ be the category of finite dimensional vector spaces over a (possibly skew) field k, and all linear maps. This is an additive category, and so has a radical: this radical is moreover 0 since every object of $\mathscr{C}$ is a direct sum of copies of k and the radical of $\mathscr{C}_k = k$ is 0. We now calculate the radix $\mathfrak{r}$ of $\mathscr{C}$.

In order that $f, g : V \to W$, where V and W are objects of $\mathscr{C}$, be equivalent, it is necessary that ufv and ugv be both units or both non-units for any maps $u : W \to k$ and $v : k \to V$. Since the only non-unit of k is 0, this means that ufv is to be 0 if and only if ugv is 0. It is easily seen that this is so if and only if $g = \lambda f$ for some $\lambda \neq 0$ in the centre of k.

Now the equivalence relation $\mathfrak{r}$, defined by "$f \equiv g$ if and only if $g = \lambda f$ for some $\lambda \neq 0$ in the centre of k", is clearly a congruence, which is therefore the radix of $\mathscr{C}$.

We observe that the radix of $\mathscr{C}$ properly exceeds its radical. Moreover the radical may be inferred from the radix in this case: since the radical is to be an ideal, and since 0 is equivalent only to itself in the radix, the radical can only be 0.

(e) If in the example of (d) we replace k by a local ring R, which we take commutative for simplicity, we can use the reasoning of (d) to infer the radix in this case. For let I be the ideal of non-units in R, let $k = R/I$, and let $\bar{x}$ be the image in k of $x \in R$. If we think of f and g as matrices with elements in R, let $\bar{f}, \bar{g}$ denote the matrices obtained from these by reducing each element modulo I.

To say that ufv and ugv are simultaneously non-units, is to say that $\bar{u}\bar{f}\bar{v}$ and $\bar{u}\bar{g}\bar{v}$ are simultaneously 0, or that $\bar{g} = \lambda\bar{f}$, $\lambda \neq 0$. This in turn gives $g = af+h$, where a is a unit of R and h is a matrix all of whose elements are non-units. This relation between f and g is obviously a congruence, and thus is the radix of $\mathscr{C}$. Once again the class of the zero matrix, consisting of those matrices whose elements are non-units, is the radical of $\mathscr{C}$.

(f) It need not be the case in an additive category, even if it is direct, that the class of 0 in the radix is the radical. Let A be a vector space of dimension $n > 1$ over a field k, and let $\mathscr{C}$ be the category whose objects are the finite direct sums of copies of A and whose maps are all the linear maps. If $\mathfrak{r}$ is the radix of $\mathscr{C}$, it is easy to verify, by the methods used above, that $\mathfrak{r}_A$ is just the partitioning of $\mathscr{C}_A$ into the units and the non-units, while $\mathscr{R}_A$ is of course 0.

Reference

[1] Kelly, G. M., Complete functors in homology. I., Proc. Camb. Phil. Soc. (to appear).

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