GROUPS WITH METACYCLIC SYLOW 2-SUBGROUPS

A. R. CAMINA AND T. M. GAGEN

A group S is said to be metacyclic if it contains a normal cyclic subgroup N such that S/N is cyclic. In this note the following theorem is proved.

THEOREM. *Let G be a group, S a metacyclic Sylow 2-subgroup of G. If S has a cyclic normal subgroup N such that S/N is cyclic of order greater than 2, then G is soluble.*

Remark. We show that such a group G contains a 2-nilpotent normal subgroup of index a divisor of 6. The solubility of these groups requires the solubility of groups of odd order unavoidably.

Notation. All groups considered will be finite. Let G be a group, S a subset of G, A and B subgroups of G, N a normal subgroup of G.

$\langle S\rangle$: the subgroup of G generated by S.

$N_G(S)$: the normalizer of S in G.

$C_G(S)$: the centralizer of S in G.

When it is clear which group G we are considering we write $C(S)$, $N(S)$.

$[A, B]$: $\langle [a, b] = a^{-1}b^{-1}ab\colon a \in A, b \in B\rangle$.

$f(G \bmod N)$: the preimage in G of $f(G/N)$. Here f is a function from groups to subgroups.

$O_2(G)$: the maximal normal 2-subgroup of G.

$O_{2'}(G)$: the maximal normal odd order subgroup of G.

$O_{2',2}(G)$: $O_2(G \bmod O_{2'}(G))$.

S_p-subgroup: Sylow p-subgroup of G.

$Z(G)$: the centre of G.

$\Omega_1(S)$: If S is a p-group, then $\Omega_1(S)$ denotes the group generated by all elements of S of order p.

$\phi(S)$: The Frattini subgroup of S.

LEMMA 1. *Let S be a metacyclic 2-group and suppose that the automorphism group of S is not a 2-group. Then either S is abelian of type $(2^a, 2^a)$ or quaternion of order 8.*

We would like to thank the referee for pointing out to us that this lemma appears in (**12**, Lemma 5.27).

LEMMA 2. *Let G be a soluble group, S an S_2-subgroup of G. Suppose that $S = \langle x, y\colon x^{2^a} = 1, y^{2^b} = x^u, y^{-1}xy = x^r\rangle$ is metacyclic. Then $|G/O_{2',2}(G)| \leqq 6$.*

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Proof. Let $\bar{G} = G/O_{2',2}(G)$. By (**4**, Lemma 1.2.3), $\bar{G}$ is a subgroup of the group of automorphisms of $O_{2',2}(G)/O_{2'}(G)$, a metacyclic group. By Lemma 1, if $\bar{G} \neq 1$, an S_2-subgroup $O_{2',2}(G)$ is abelian of type $(2^c, 2^c)$ or quaternion of order 8. Thus $|\bar{G}| = 3 \cdot 2^s$, $s \geqq 0$. Since $O_2(\bar{G}) = 1$, $s \leqq 1$ and the lemma is proved.

LEMMA 3. *Let S be a metacyclic 2-group. Then S satisfies one of the following conditions:*

(i) *$Z(S)$ is non-cyclic. Then S has precisely three involutions;*
(ii) *S has just one involution and then S is cyclic or generalized quaternion, by* (**3**, *p.* 189)*;*
(iii) *S has just three involutions and $Z(S)$ is cyclic;*
(iv) *S has at least five involutions and $Z(S)$ is cyclic.*

Proof. Trivial. Note that if

$$S = \langle x, y \colon x^{2^a} = 1,\ y^{2^b} = x^u,\ y^{-1}xy = x^r \rangle,$$

then all the involutions in S are contained in $\langle x, y^{2^{b-1}} \rangle = T$, and T has a cyclic subgroup of index 2. Thus T is a very well-known group (see, for example, **3**, p. 187).

LEMMA 4. *Let G be a group, $S = \langle x, y \colon x^{2^a} = 1,\ y^{2^b} = x^u,\ y^{-1}xy = x^r \rangle$ an S_2-subgroup of G. Suppose that $|S/\langle x \rangle| \geqq 4$ and also that $Z(S)$ is non-cyclic. Then G is soluble.*

Proof. Let G be a minimal counterexample. We show that G has no subgroup G_1 of index 2. For such a group is either soluble, in which case we have G is soluble, or an S_2-subgroup T of G_1 has a cyclic subgroup of index 2 and no cyclic factor group of order greater than 2. Now S contains precisely three involutions by Lemma 3 and hence T has just one or three involutions. It follows that T is either generalized quaternion or abelian of type (2, 2). Since S has just three involutions, if T is generalized quaternion, $S = T \times \langle t_1 \rangle$, where t_1 is a central involution of S not in T. But then S is not 2-generated, a contradiction. If S has order 8, it is abelian since $S/\langle x \rangle$ is cyclic of order at least 4. But then G has a normal 2-complement, applying Burnside's theorem (**3**, p. 203). Hence G has no subgroup of index 2.

Consider $U = \Omega_1(Z(S))$ and $N(U)$. If $N(U) < G$, it is soluble by induction. If $N(U)$ has a subgroup of index 2, so does G, by (**3**, Theorem 14.4.2). This is not the case and thus $N(U)$ has no subgroup of index 2.

By Lemma 2, $N(U)$ contains a normal subgroup K of index 3 precisely. By Lemma 1, an S_2-subgroup of $N(U)$ and also of G is abelian of type $(2^a, 2^a)$ since it has a non-cyclic centre. By induction, $O_{2'}(G) = 1$ if G is non-soluble. Thus by (**1**, Theorem 1), G is soluble.

Thus we may assume that $N(U) = G$. Then either $C(U) = G$ or $[G\colon C(U)]$ is a divisor of 6. If $C(U) < G$, $[G\colon C(U)] = 3$ since G has no subgroup of index 2. But then $C(U)$ is soluble by induction. This contradiction shows that

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$G = C(U)$. Let t be the involution in $\langle x \rangle \cap U$. Then $\bar{S} = S/\langle t \rangle$ is a metacyclic group and $\bar{S}/\langle \bar{x} \rangle$ is cyclic of order at least 4, where $\langle \bar{x} \rangle = \langle x \rangle / \langle t \rangle$. Also $Z(\bar{S})$ is non-cyclic since $Z(\bar{S}) \cap \langle \bar{x} \rangle \neq 1$. By induction, $G/\langle t \rangle$ is soluble. This contradiction completes the proof.

LEMMA 5. *Let $S = \langle x, y \colon x^{2^a} = 1, y^{2^b} = x^u, y^{-1}xy = x^r \rangle$ be an S_2-subgroup of a group G and suppose that $|S/\langle x \rangle| \geqq 4$. Then G is soluble.*

Proof. Let G be a minimal counterexample. Then by Lemma 4 we may assume that $Z(S)$ is cyclic. Let $t \in Z(S)$ be an involution. We show that $t \in Z^*(G)$, where $Z^*(G) = Z(G \bmod O_{2'}(G))$. Since an S_2-subgroup of $\bar{G} = G/\langle t \rangle O_{2'}(G)$ has a cyclic factor group of order at least 4, $\bar{G}$ is soluble by induction. To show that $t \in Z^*(G)$ we use (**2**, Theorem 4). Thus there exists a subgroup U containing t, $U \leqq S$, and an element $g \in G$ of odd order such that $g^{-1}tg \neq t$ and $g \in N(U) \cap N(C_S(U))$, if $t \notin Z^*(G)$.

If S has precisely three involutions, then $N(U) \geqq S$. If $N(U) < G$, then $N(U)$ is soluble by induction. There are two cases. If $N(U)$ has no subgroup of index 2, then by Lemma 1 an S_2-subgroup S of G is abelian. This is not the case since $Z(S)$ is cyclic, as already remarked. Now $N(U)$ is not a 2-group, since $g \in N(U)$, and hence $N(U)$ contains a subgroup K of index 2 and $g \in K$. By Lemmas 1 and 2, an S_2-subgroup of K is abelian of type $(2^c, 2^c)$ since $U \leqq K$. For if $U \nleqq K$, then $[g, K \cap U] = 1$, a contradiction. But if S contains an abelian subgroup V of type $(2^c, 2^c)$ with index 2, $S = \langle x, y \rangle$ has order 2^{2c+1}. Either x has order 2^c when y has order precisely 2^{c+1}, since its order is no larger than this, or x has order 2^{c+1}, when we can choose $y \in V$ of order 2^c. In either case, the centre of S is non-cyclic since the automorphism group of a cyclic group of order 2^n has exponent 2^{n-2} if $n \geqq 3$, by (**7**, p. 146). Of course $c > 1$, since if $c = 1$, we have $|S| = 8$ and then S has more than three involutions. This is a contradiction.

Thus $N(U) = G$ and $C(U) < G$ since $C_S(U) < S$. By induction, an S_2-subgroup T of $C(U)$ has a cyclic subgroup of index 2 since $N(U)/C(U)$ is soluble. Also $T \leqq C(U)$ implies that $U \leqq Z(T)$. The only group T which has a non-cyclic centre and a cyclic subgroup of index 2 is the 4-group by Lemma 3. Note that T has no cyclic factor group of order greater than or equal to 4. But then $T = U$ and $C(U)$ is soluble as the S_2-subgroup of $C(U)$ is normal.

Thus we have that S has at least five involutions by Lemma 3 and, if $R = \langle x, y^{2^{b-1}} \rangle$, R contains all the involutions of S. Hence R is either dihedral of order greater than or equal to 8 or semi-dihedral of order greater than or equal to 16. Since $U \leqq R$, it follows that $C_R(U) = U$. Let $U = \langle t, t_1 \rangle$, where $t \in Z(S)$ is an involution and t_1 is a non-central involution. Then $C_S(U) = C_S(t_1)$. Now $C_S(t_1)/\langle t \rangle = C_S(t_1)/C_S(t_1) \cap \langle x \rangle \cong C_S(t_1)\langle x \rangle / \langle x \rangle$ is cyclic since $S/\langle x \rangle$ is. Now $t \in Z(C_S(t_1))$, since $t \in Z(S)$, and thus $C_S(t_1)$ is abelian. Now by (**1**, Lemma 4), $|C_S(t_1)| \neq 4$ since then S is dihedral or semi-dihedral and so has no cyclic factor group of order greater than 2. Thus

$C_S(t_1) = C_S(U)$ is abelian of type $(2^d, 2)$ and $d > 1$. This is impossible since then $g \in C(C_S(U))$ because g has odd order. But then $g \in C(U)$ and $g^{-1}tg = t$. This is a contradiction and the Theorem is proved.

Added in proof. It has been brought to our attention that P. L. Chabot has proved some similar and more general results in this direction in his thesis "Sylow 2-groups with cyclic commutator groups" (University of Notre Dame, Notre Dame, Indiana, September, 1969).

References

1. R. Brauer, *Some applications of the theory of blocks of characters of finite groups.* II, J. Algebra *1* (1964), 307–334.
2. G. Glauberman, *Central elements in core free groups*, J. Algebra *4* (1966), 403–420.
3. M. Hall, Jr., *The theory of groups* (Macmillan, New York, 1959).
4. P. Hall and G. Higman, *The p-length of a p-soluble group and reduction theorems for Burnside's problem*, Proc. London Math. Soc. *7* (1956), 1–42.
5. M. Suzuki, *A characterization of simple groups* LF(2, p), J. Fac. Sci. Univ. Tokyo (Sect. I) *6* (1951), 259–293.
6. J. G. Thompson, *Non-solvable finite groups all of whose local subgroups are solvable*, Bull. Amer. Math. Soc. *74* (1968), 383–437.
7. H. J. Zassenhaus, *The theory of groups* (Chelsea, New York, 1958).

University of East Anglia,
Norwich, England;
University of Sydney,
Sydney, Australia

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