Can. J. Math. Vol. **44** (3), 1992 pp. 665–672

THE STRUCTURE OF SCHUR ALGEBRAS $S_k(n,p)$ FOR $n \geq p$

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ABSTRACT. By exploiting the known quasi-heredity of Schur algebras, the structure of basic algebras of the Schur algebras $S_k(n,p)$ for $n \geq p$ over an algebraically closed field k is completely determined.

Let k be an algebraically closed field and $n \geq r$ positive integers. Following J. A. Green [G, 2.6c], the Schur algebra $S(n,r)$ can be defined as the endomorphism ring of the r-fold tensor product of an n-dimensional k-space E considered as a right $kG(r)$-module; where $G(r)$ is the symmetric group of degree r (acting on the tensor product canonically).

It is known (see [G], 6.5g) that the Schur algebra $S(n,r)$ is Morita equivalent to $S(r,r)$. Moreover, if the characteristic p of the field k is zero or $r < p$, then the Schur algebra $S(r,r)$ is semisimple ([G], 2.6e). In this paper we consider the case $p > 0$ and $r = p$. The main result is the following theorem.

THEOREM. *Let k be an algebraically closed field with characteristic $p > 0$. Then each block of the Schur algebra $S(n,p)$ with $n \geq p$ is either simple or Morita equivalent to the path algebra P (over k) of*

$$\circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ \cdots \circ \underset{\beta_{m-1}}{\overset{\alpha_{m-1}}{\rightleftarrows}} \circ, \quad m \geq 1$$

modulo the ideal generated by

$$\alpha_i\alpha_{i+1},\ \beta_{i+1}\beta_i,\ \alpha_{i+1}\beta_{i+1} - \beta_i\alpha_i, \quad 1 \leq i \leq m-1; \alpha_1\beta_1,$$

where m depends only on p. Moreover, there is only one non-simple block. Thus, in particular, $S(n,p)$ is a quadratic algebra (i.e., the relations for the algebra are generated by elements of degree 2).

Note that the non-trivial block of $S(n,p)$ is Morita equivalent to the quotient of the quasi-Frobenius algebra $B = P/J$ with

$$J = \langle \alpha_i\alpha_{i+1}, \beta_{i+1}\beta_i, \alpha_{i+1}\beta_{i+1} - \beta_i\alpha_i, 1 \leq i \leq m-1; \alpha_1\beta_1\alpha_1, \beta_{m-1}\alpha_{m-1}\beta_{m-1} \rangle$$

by the simple ideal $\langle \alpha_1\beta_1 + J \rangle$, and that, in turn, B is a trivial extension of the path algebra of

$$\circ \xrightarrow{\alpha_1} \circ \cdots \circ \xrightarrow{\alpha_{m-1}} \circ, \quad m \geq 2$$

Received by the editors October 10, 1990.
AMS subject classification: 20C32, 16A46, 16A64.

modulo the ideal $\langle \alpha_i\alpha_{i+1} \mid 1 \le i \le m-2 \rangle$.

The proof of the theorem is based on the description of the quasi-hereditary endomorphism algebras of the form $\mathrm{End}_A({}_AA \oplus M)$, where A is a symmetric algebra (Section 2) and the following description of the Schur algebras.

PROPOSITION. *The Schur algebra $S(n,p)$ for $n \ge p$ is Morita equivalent to an algebra of the form* $\mathrm{End}_{kG(p)}\big(kG(p) \oplus M\big)$, *where M is a simple $kG(p)$-module.*

Indeed, since $S(n,p)$ is quasi-hereditary (see e.g. [P]), the basic algebra of $S(n,p)$ can be deduced from Theorem 2.8 of Section 2.

1. **Preliminaries: definitions and basic facts.** Throughout this paper, all algebras are finite-dimensional algebras with 1 over an algebraically closed field k, all modules are finitely generated left modules. For an algebra A we denote by N (or rad(A)) the Jacobson radical of A. For a module M, Soc(M) is the largest semisimple submodule of M, Top(M) is the largest semisimple factor module of M.

If an algebra A is given by a quiver $Q = (Q_0, Q_1)$ with relations, we denote by $P(x)$, $S(x)$ and e_x, for $x \in Q_0$, the indecomposable projective module, the simple module and the primitive idempotent corresponding to the vertex x, respectively. For the other notations about quivers in this paper we refer to [R, 2.2].

Let I be an ideal of an algebra A and $a \in A$ (or $J \subset A$). By $\bar{a}$ (or $\bar{J}$) we denote the image of a (or J) under the canonical map $A \longrightarrow A/I$.

Let us now recall some definitions.

(1) An ideal I of an algebra A is called a *heredity ideal* of A if

(i) $I = I^2$,

(ii) $INI = 0$,

(iii) ${}_AI$ is a projective A-module.

(2) An algebra A is called *quasi-hereditary* (see [CPS]) if there is a chain

$$0 = I_0 \subset I_1 \subset \cdots \subset I_n = A$$

of ideals of A for some natural number n such that I_i/I_{i-1} is a heredity ideal of A/I_{i-1}, for $i \in \{1,\dots,n\}$. In this case, the chain is called a *heredity chain*. If a heredity chain cannot be refined to another heredity chain, then it is called *maximal*.

Recall that quasi-hereditary algebras are introduced by Cline, Parshall and Scott in order to study highest weight categories arising in the representation theory of Lie algebras and algebraic groups. We shall frequently use the results on quasi-hereditary algebras in [DR].

(3) An algebra A is called a *quasi-Frobenius* algebra if the projective modules coincide with the injective modules.

(4) An algebra A is called *symmetric* if A admits a non-degenerate bilinear form $f : A \times A \longrightarrow k$ which is associative:

$$f(ab,c) = f(a,bc)$$

and symmetric:

$$f(a,b) = f(b,a)$$

Observe that symmetric algebras are quasi-Frobenius algebras and that connected non-simple quasi-Frobenius algebras have no heredity ideals (this follows easily from the part (1) of the following Proposition 1.1).

Concerning basic facts on symmetric algebras and quasi-Frobenius algebras we refer to [K] and [CR].

The following proposition is easy to prove:

PROPOSITION 1.1. *(1) Let e be a primitive idempotent of an algebra A such that AeA is a heredity ideal. Then every non-zero homomorphism from Ae to a projective A-module is a monomorphism.*

(2) Let M be an A-module such that $\operatorname{add}\operatorname{Soc}(M) = \operatorname{add}\operatorname{Top}(M)$, *and* $\operatorname{End}_A(M)$ *is quasi-hereditary. Then M has a simple direct summand.*

As a consequence of the above proposition we have the following fact.

THEOREM 1.2. *Let A be a connected non-simple quasi-Frobenius algebra and M a non-zero A-module. If the endomorphism algebra* $\operatorname{End}_A({}_AA \oplus M)$ *is quasi-hereditary, then M has a simple direct summand.*

This result reduces the investigation of whether there is an indecomposable module M over a given symmetric algebra A such that $\operatorname{End}_A({}_AA \oplus M)$ is quasi-hereditary to the case where M is a simple module. In the next section we shall deal with this question in detail.

COROLLARY 1.3. *Let G be a finite group. We denote by $b(G)$ the number of blocks of the group algebra $A := kG$ of G which has non-zero radicals. If there is a module M such that* $\operatorname{End}_A({}_AA \oplus M)$ *is quasi-hereditary, then the number of non-isomorphic simple summands of M is not smaller than $b(G)$.*

In general, we have the following fact.

REMARK 1.4. Let A be an algebra which has no heredity ideal and M be an indecomposable module such that $\operatorname{End}_A({}_AA \oplus M)$ is quasi-hereditary. Then M is a non-projective torsionless module (i.e. a submodule of a free A-module).

PROOF. As A has no heredity ideal, the one point extension of A by M can never become a quasi-hereditary algebra. By Proposition 1.1 (1), M is a torsionless module.

2. **Quasi-heredity of** $\operatorname{End}({}_AA \oplus M)$**.** In this section we shall study connected basic symmetric algebras A with the property that there is an indecomposable module M such that $\operatorname{End}_A({}_AA \oplus M)$ is quasi-hereditary. We may assume that $M = \operatorname{Soc}\big(P(1)\big)$ and ${}_AA = P(1) \oplus \cdots \oplus P(n)$. Put $E := \operatorname{End}_A({}_AA \oplus M)$. Throughout this section we shall keep these assumptions and notations.

https://doi.org/10.4153/CJM-1992-040-5 Published online by Cambridge University Press

In general, given a symmetric algebra B, the algebra $\mathrm{End}_B({}_BB \oplus M)$ need not be a quasi-hereditary algebra for any indecomposable module M. Consider, for example, the symmetric algebra given by the following quiver with the relation:

$$\circlearrowleft \alpha, \quad \alpha^3 = 0.$$

Assume that $\mathrm{End}_B(B \oplus M)$ is quasi-hereditary. Then M must be simple, but one can easily see that $\mathrm{End}_B({}_BB \oplus M)$ is not quasi-hereditary. This algebra is representation finite. Later we shall see that for representation infinite symmetric algebras B, $\mathrm{End}_B({}_BB \oplus M)$ is never quasi-hereditary.

Before we state our next result, we need the following well-known fact.

LEMMA 2.1. *Let B be a basic, quasi-Frobenius algebra and P an indecomposable projective left ideal of B. Then socle* $\mathrm{Soc}(P)$ *of P is an ideal of B.*

PROPOSITION 2.2. *Let A be a symmetric algebra. Then E is quasi-hereditary if and only if A/M is quasi-hereditary.*

PROOF. Let us denote by f_{n+1} the idempotent of E which projects ${}_AA \oplus M$ canonically onto M. Note that M is an ideal of A by Lemma 2.1 and that the algebra E is quasi-hereditary if and only if $E/Ef_{n+1}E$ is quasi-hereditary. Since $\mathrm{Top}\big(P(1)\big) \cong M = \mathrm{Soc}\big(P(1)\big)$ is a simple module, the latter is equivalent to that A/M is quasi-hereditary.

LEMMA 2.3. *Let A be a symmetric algebra such that A/M is quasi-hereditary. Then*
(1) $\dim \mathrm{Hom}\big(P(1), P(1)\big) = 2$.
(2) A/Ae_1A is quasi-hereditary.

PROOF. It is obvious that the ideal of $\bar{A} := A/M$ generated by $\bar{e}_i = e_i + M$ can not be a heredity ideal of $\bar{A}$ for $i \in \{2, \ldots, n\}$. Since A is symmetric and $\bar{e}_1\bar{N}\bar{e}_1 = 0$, we infer that $\dim \mathrm{Hom}\big(P(1), P(1)\big) = 2$. It is clear that A/Ae_1A is quasi-hereditary, because $A/Ae_1A \cong (A/M)\big/(Ae_1A/M)$ and Ae_1A/M is the first term of a heredity chain of $\bar{A}$.

REMARK 2.4. Let A and M be as in 2.3. Set $B = A/Ae_1A$. Suppose $B\bar{e}_2B$ is a minimal heredity ideal of B. Then $\dim \mathrm{Hom}\big(P(2), P(2)\big) = 2$ and $\dim \mathrm{Hom}\big(P(1), P(2)\big) = 1$.

PROOF. It follows from $\bar{e}_2\bar{N}\bar{e}_2 = 0$ and $\dim \mathrm{Hom}\big(P(1), P(1)\big) = 2$, together with $\mathrm{Hom}\big(P(1), P(2)\big) \neq 0$ that $\dim \mathrm{Hom}\big(P(2), P(2)\big) = 2$. Thus the last claim in the remark becomes now trivial, because the ideal $\bar{A}(e_1 \oplus M)\bar{A}$ is a minimal heredity ideal of A/M.

DEFINITION. A module M is called *serial* if the Loewy series

$$M \supseteq NM \supseteq N^2M \supseteq \cdots \supseteq N^rM = 0$$

is the unique composition series of M.

An algebra is called *serial* if for every primitive idempotent e, the module Ae and the right A-module eA are serial.

Using 2.2 and 2.3, we can easily verify that for a serial symmetric algebra with more than two simple modules (this should mean two isomorphism classes of simple modules), there is no indecomposable module M such that $\mathrm{End}_A({}_AA \oplus M)$ is quasi-hereditary.

As another consequence of 2.2, we have the following useful lemma.

LEMMA 2.5. *Let A^o denote the opposite algebra of A. Then for a symmetric algebra A, there is an indecomposable module M such that* $\mathrm{End}_A({}_AA \oplus M)$ *is quasi-hereditary if and only if this property holds for A^o.*

PROOF. This follows from 2.2 and the facts that an algebra R is quasi-hereditary if and only if R^o is quasi-hereditary, and that for any ideal I of R, it always holds that $(R/I)^o \cong R^o/I^o$.

PROPOSITION 2.6. *Let A be a symmetric algebra which possesses an indecomposable module M such that E is quasi-hereditary. Then there is an indecomposable projective module P which is serial and of Loewy length $L(P) \leq 3$.*

PROOF. In the following we want to show that $P := P(1)$ is a serial module. We may assume that $n > 2$.

Let $\bar{A} = A/\operatorname{Soc}(P)$ and

$$0 \subset \bar{A}\bar{e}_1\bar{A} \subset \bar{A}(\bar{e}_1 + \bar{e}_2)\bar{A} \subset \cdots \subset \bar{A}(\bar{e}_1 + \cdots + \bar{e}_n)\bar{A} = \bar{A}$$

be a heredity chain of $\bar{A}$. By Proposition 1.1, the $\bar{A}$-module $\bar{A}\bar{e}_1$ is a submodule of $\bar{A}\bar{e}_2 \cong Ae_2$. Thus the A-module $\bar{A}\bar{e}_1$ has a simple socle which is isomorphic to $S(2)$, that is, $\operatorname{Soc}^2(P)/\operatorname{Soc}(P) \cong S(2)$. Dually, we have that $\operatorname{rad}(P)/\operatorname{rad}^2(P) \cong S(2)$. Since $\dim \operatorname{Hom}\big(P(2), P\big) = 1$, the Loewy series of P must be of the following shape:

$$P(1) \supset \operatorname{rad}\big(P(1)\big) \supset \operatorname{Soc}\big(P(1)\big) \supset 0$$

with $P(1)/\operatorname{rad}\big(P(1)\big) \cong S(1)$, $\operatorname{rad}\big(P(1)\big)/\operatorname{Soc}\big(P(1)\big) \cong S(2)$ and $\operatorname{Soc}\big(P(1)\big) = S(1)$. This finishes the proof.

The following theorem establishes a complete description of a symmetric algebra A with the property that the endomorphism algebra $\mathrm{End}_A({}_AA \oplus M)$ of the module ${}_AA \oplus M$ is quasi-hereditary for every simple module M.

THEOREM 2.7. *Let A be a symmetric algebra with* $\operatorname{rad}(A) \neq 0$. *If for every simple module M, the algebra* $\mathrm{End}_A({}_AA \oplus M)$ *is quasi-hereditary, then A is one of the following algebras:*

(1) $\circ \circlearrowleft \alpha$, $\alpha^2 = 0$;

(2) $\circ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \circ$, $\alpha\beta\alpha = \beta\alpha\beta = 0$.

PROOF. We denote by $L(X)$ the Loewy length of a module X. By Proposition 2.6, $P(i)$ is a serial module of $L\big(P(i)\big) \leq 3$ for all i. Since $\operatorname{rad}(A) \neq 0$ and A is connected, we get $L\big(P(i)\big) \neq 1$. If there is one i such that $L\big(P(i)\big) = 2$, then A is of the form (1). Now we may assume that all $P(i)$, $i = 1, \ldots, n$, have Loewy length equal to 3. In this case, we get from the remark before Lemma 2.5 that A must be of the form (2).

THEOREM 2.8. *Let A be a connected basic symmetric algebra and M the socle of an indecomposable projective left ideal of A such that A/M is quasi-hereditary. Then A is a simple algebra, or isomorphic to $k[T]/(T^2)$, where $k[T]$ is the polynomial ring in one* *variable T, or there is a natural number $n \geq 2$, such that A is the path algebra of the following quivers*

$$\circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ \cdots \circ \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} \circ, \quad n \geq 2$$

modulo the ideal generated by

$$\alpha_{i-1}\alpha_i,\ \beta_i\beta_{i-1},\ \alpha_i\beta_i - \beta_{i-1}\alpha_{i-1}\ \textit{for}\ i = 2, \ldots, n-1;$$
$$\alpha_1\beta_1\alpha_1,\ \beta_{n-1}\alpha_{n-1}\beta_{n-1}.$$

PROOF. Assume that $\mathrm{rad}(A) \neq 0$. Let $\bar{A} = A/M$ and

$$0 \subset \bar{A}\bar{e}_1\bar{A} \subset \bar{A}(\bar{e}_1 + \bar{e}_2)\bar{A} \subset \cdots \subset \bar{A}(\bar{e}_1 + \cdots + \bar{e}_n)\bar{A}$$

be a heredity chain for $\bar{A}$. If $n = 1$, then the algebra must be simple or of the form $k[T]/(T^2)$. Suppose $n \geq 2$. We denote by c_{ij} the Cartan-invariants (i.e. $c_{ij} = \dim e_iAe_j$). If $P(2)$ is serial, then A must be the algebra above in the case $n = 2$, because we know from 2.6 that $c_{ij} = 0$ for all $j \notin \{1, 2\}$. Now suppose $P(2)$ is not serial. Then it follows from the heredity of the ideal $\bar{A}\bar{e}_1\bar{A}$ and 2.6 that $\mathrm{rad}\big(P(2)\big)/\,\mathrm{Soc}\big(P(2)\big) \cong S(1) \oplus K_2$. Let $I_1 = Ae_1A$ and $A_1 = A/I_1$. Since $c_{13} = 0$ and $A_1(e_2 + I_1)A_1$ is a heredity ideal of A_1, one gets that $c_{23} \neq 0$; thus $\dim \mathrm{Hom}\big(P(3), K_2\big) \neq 0$. Since A_1 is quasi-hereditary and the ideal $A_1\bar{e}_2A_1$ is a heredity ideal of A_1, we have, by Proposition 1.1, that $A_1\bar{e}_2$ is a submodule of $A_1\bar{e}_3 \cong Ae_3$. Thus K_2 has a simple socle which is isomorphic to $S(3)$. Dually, we can show that K_2 has a simple top which is isomorphic to $S(3)$. But from $c_{22} = 2$ and $c_{13} = 0$ it follows that $c_{23} = 1$ and $c_{33} = 2$. This means $K_2 \cong S(3)$. So the structure of $P(2)$ is completely determined. So if we repeat the above argument, then we get, after finitely many steps, the following Loewy structures of projective modules $P(i)$, $i = 1, \ldots, m$:

$$\begin{array}{ccccc}
P(1) & P(2) & & P(m) \\
 & & & \\
S(1) & S(2) & & S(m) \\
| & / \ \backslash & & | \\
S(2) & S(1)\ S(3) & ,\ldots, & S(m-1) \\
| & \backslash \ / & & | \\
S(1) & S(2) & & S(m).
\end{array}$$

Since $A/A(e_1 + \cdots + e_m)A$ is quasi-hereditary, we deduce that $m = n$. From the above structures it follows easily that the algebra A is given by the quiver with relations described in the theorem.

The converse of the above theorem holds.

LEMMA 2.9. *If A is given by the quiver with relations in 2.8, then there exists an indecomposable module M such that $\mathrm{End}_A({}_AA \oplus M)$ is quasi-hereditary.*

PROOF. Let $M = \mathrm{Soc}\big(P(1)\big)$ or $M = \mathrm{Soc}\big(P(n)\big)$. Then one can readily verify that the algebra $\mathrm{End}_A({}_AA \oplus M)$ is quasi-hereditary.

COROLLARY 2.10. *Let A be a symmetric algebra which has infinite representation type. Then there is no indecomposable module M such that $\mathrm{End}_A({}_AA \oplus M)$ is quasi-hereditary.*

PROOF. Note that the algebras given in 2.8 are representation-finite. Thus the corollary follows.

COROLLARY 2.11. *Let A be a non-simple, connected basic algebra. Then the following are equivalent:*

(i) A is symmetric and there is an indecomposable module M such that $\mathrm{End}_A({}_AA \oplus M)$ is quasi-hereditary,

(ii) The algebra A is a Brauer tree algebra with an open polygon graph having no exceptional vertex (for the definition see [A]).

PROOF OF THE MAIN THEOREM. We give first the proof of the proposition in the introduction. Let E be a k-space of dimension n and $E^{\oplus p}$ be the p-fold tensor power of E. Then $kG(p)$ is a direct summand of the $kG(p)$-module $E^{\oplus p}$. Thus $S(n,p)$ is of the form $\mathrm{End}_{kG(p)}\big(kG(p) \oplus M\big)$ for some $kG(p)$-module M. Since $S(p,p)$ has one simple module more than $kG(p)$ does (see [G, 6.4b]), the proposition follows now by [G, 6.5g] and Theorem 1.2.

Now we turn to the proof of the theorem. By the proposition, the algebra $S(n,p)$ is of the form $\mathrm{End}_A({}_AA \oplus M)$ with $A = kG(p)$ and M an indecomposable module. Since the Schur algebra is quasi-hereditary, the basic algebra of $S(n,p)$ must be of the desired form in the theorem by Theorem 2.8.

ACKNOWLEDGEMENT. The author is indebted to Claus Michael Ringel and Vlastimil Dlab for useful comments and suggestions.

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