THE LEBESGUE CONSTANTS FOR REGULAR HAUSDORFF METHODS

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1. Introduction. The unboundedness of the sequence of Lebesgue constants (norms), at a point, of certain transforms implies, as is well known, that there exist (i) a continuous function whose transform fails to converge to the function at the point in question (the du Bois–Reymond singularity), and (ii) another such function whose transform, while converging everywhere to the function, does not do so uniformly in any neighbourhood of the stipulated point (the Lebesgue singularity). The converses also hold in our case. The magnitude of such constants is, consequently, of some interest and has been calculated for many transforms.

Here we are concerned with the Lebesgue constants $L(n; g)$ arising from the application to Fourier series of the regular* Hausdorff summation method with weight function $g(t)$, $0 \leqslant t \leqslant 1$. The function $g(t)$ is of bounded variation,† continuous at the origin, with $g(0) = 0$ and $g(1) = 1$. The general properties of such methods are elaborated in **(3**, chapter XI**)**; specific applications to Fourier series are found in **(5)**. Among the important particular cases of Hausdorff methods are found the Cesàro, Hölder, and Euler means.

Our primary purpose here is to establish the following:

THEOREM 1. *Let $L(n; g)$ denote the* nth *Lebesgue constant for the regular Hausdorff method with weight function $g(t)$. Then*

$$L(n; g) = C(g) \log n + o(\log n) \qquad as\ddagger \qquad n \to \infty, \tag{1}$$

where

$$C(g) = (2/\pi^2)|g(1) - g(1-)| + (1/\pi)\mathscr{M}\left\{\left|\sum_k [g(\xi_k+) - g(\xi_k-)] \sin \xi_k x\right|\right\}. \tag{2}$$

Here ξ_k is the kth *discontinuity (jump) of $g(t)$ and the summation extends over all such (possibly countably infinite) values; $\mathscr{M}\{f(x)\}$ represents, as usual, the mean value of the almost-periodic function $f(x)$. Furthermore,*

$$0 \leqslant C(g) \leqslant (4/\pi^2)\, V(g), \tag{3}$$

where $V(g)$ is the total variation of $g(t)$, $0 \leqslant t \leqslant 1$, and

$$C(g) = 0 \quad \text{if and only if } g(t) \text{ is continuous.} \tag{4}$$

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*A summation method is "regular" if it sums the sequence $s_1, \ldots, s_n, \ldots$, to the (finite) value s whenever $s_n \to s$; "totally regular", if, in addition, this is the case when s is ∞.

†When the method is totally regular $g(t)$ is non-decreasing, and conversely.

‡Throughout this paper all o- and O- terms are taken as the parameter becomes infinite.

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If, in addition, the method is totally regular (*so that* $V(g) = 1$), *then also*

(5) $\quad C(g) = 4/\pi^2$ *if and only if the method is ordinary convergence.*

and

(6) $$C(g) \leqslant 2/\pi^2 \quad \text{when } g(1-) = g(1) \text{ and, in this case,}$$
$$C(g) = 2/\pi^2 \quad \text{if and only if the method is of Euler type.}$$

Thus, $C(g)$ is a constant depending only on the weight function $g(t)$.

Equation (4) shows that *any Hausdorff method with a discontinuous weight function exhibits the du Bois–Reymond singularity* (a result obtained originally by Hille and Tamarkin (**5**, Theorem 14.1), whose proof is along different lines) *and also the Lebesgue singularity.*

Moreover, (5) and (6) show, respectively, that, among all totally regular Hausdorff methods, ordinary convergence has the maximum principal term for the Lebesgue constants and that the Euler methods possess the same extremal property in the class of totally regular Hausdorff methods with weight functions continuous at $t = 1$.

Plainly, (4) does *not* imply that a Hausdorff method with a continuous, or even absolutely continuous, weight function sums the Fourier series of a continuous function everywhere, as the remainder in (1) can be unbounded. Indeed, Hille and Tamarkin (**5**, p. 534, Remark 2, also pp. 538 and 568) supplied examples showing that absolute continuity of the weight function is neither necessary nor sufficient for the effectiveness (for continuous functions) of the method.

That absolute continuity is not sufficient we prove anew here by showing that the error term $o(\log n)$ in (1) is "best possible" and cannot be improved even for the case of an increasing absolutely continuous $g(t)$. More precisely:

THEOREM 2. *Let* $\epsilon(n) \downarrow 0$ *as* $n \to \infty$. *There exists an increasing, absolutely continuous weight function* $g(t)$ *for which* $L(n; g) \neq o(\epsilon(n)\log n)$.

In addition, we consider also the special cases of Cesàro and Hölder means of positive fractional order. Here the weight functions are absolutely continuous. For (C, α) (**3**, p. 266),

(7) $$g_C(t) = 1 - (1 - t)^\alpha, \qquad \alpha > 0,$$

and for (H, α),

(8) $$g_H(t) = [1/\Gamma(\alpha)] \int_0^t (-\log x)^{\alpha-1}\, dx, \qquad \alpha > 0.$$

We shall supply a new proof of the result due to H. Cramér (**2**) arising from (7) and obtain the analogous statement concerning (8), together with a relation between the two:

THEOREM 3 (Cramér). *If* $g(t)$ *is given by* (7) *and* $0 < \alpha < 1$, *then* $\lim L(n; g)$ *exists* $(n \to \infty)$ *and equals*

(9) $$L(C_\alpha) = (2/\pi) \int_0^\infty x^{-2} \left| \alpha \int_0^x (1 - tx^{-1})^{\alpha-1} \sin t\, dt \right| dx.$$

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Moreover, $L(C_\alpha)$ *is a non-increasing function of* α *with* $L(C_\alpha) > 1$; *also,* $L(C_\alpha) \to 1$ *as* $\alpha \to 1-$, *and* $L(C_\alpha) \to +\infty$ *as* $\alpha \to 0+$.

THEOREM 4. *If* $g(t)$ *is given by* (8) *and* $0 < \alpha < 1$, *then* $\lim L(n; g)$ *exists* $(n \to \infty)$ *and equals*

$$L(H_\alpha) = (2/\pi) \int_0^\infty x^{-2} \left| [1/\Gamma(\alpha)] \int_0^x (\log xt^{-1})^{\alpha-1} \sin t \, dt \right| dx. \tag{10}$$

Moreover,

$$L(C_\alpha) \leqslant L(H_\alpha) \tag{10'}$$

and, for $0 < \alpha < \beta < 1$,

$$1 < L(C_\beta) \leqslant L(C_\alpha); \qquad 1 < L(H_\beta) \leqslant L(H_\alpha), \tag{10''}$$

with $L(H_\alpha) \to 1$ *as* $\alpha \to 1-$, *and* $L(H_\alpha) \to +\infty$ *as* $\alpha \to 0+$.

The methods (C, α) and (H, α), $\alpha > -1$, are well known to be "equivalent," as Hausdorff showed (**3**, p. 264), but not "totally equivalent" (I. Schur, cf. (**3**. p. 119), and Basu (**1**)).*

Basu (**1**) proved that, for $0 < \alpha < 1$, each sequence evaluable (H, α) to (finite or) infinite s is summable (C, α) to the same value and that the converse is not true for infinite s. Thus, (C, α) is slightly stronger than (H, α) for these α.

Inequality (10′) illustrates further this same imbalance, which is found also in the Gibbs phenomenon: In the (H, α) method the Gibbs phenomenon persists for larger values of α, as O. Szász (**8**) found, than in the (C, α) method, whose Gibbs phenomenon was discussed first by Cramér (**2**).

Our common point of departure for the proofs of all four theorems is a formula for $L(n; g)$ due to Livingston (**7**, p. 310 (3)), who used it to obtain a more precise version of (1) for methods of Euler type, which are Hausdorff means with one-jump step functions as their weight functions. His formula reads:

$$(11)\dagger \quad L(n; g) = (2/\pi) \int_0^{\frac{1}{2}\pi} \left| x^{-1} \int_0^{1-} [1 - 4t(1-t)\sin^2 x]^{\frac{1}{2}n} \sin 2nxt \, dg(t) \right.$$
$$\left. + [g(1) - g(1-)] \frac{\sin (2n+1)x}{\sin x} \right| dx + o(1).$$

2. Preliminary lemmas. We use a simpler version of (11), obtained at the sacrifice of some precision, having an error term which is $o(\log n)$ instead of $o(1)$ as above. Then we split $g(t)$ into its continuous and pure jump components. The continuous part will be shown to contribute $o(\log n)$, while

*Two methods are "equivalent" if each sums a sequence to the (finite) value s whenever the other does; "totally equivalent" if, in addition, the same is true for s infinite.

†It should be noted that the upper limits of the Stieltjes integrals in (11) and (12) are not the same, being $1-$ in the former and 1 in the latter.

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the pure jumps give rise to $C(g)\log n + o(\log n)$. This will complete the proof of the basic portions of Theorem 1.

The necessary lemmas form the content of this section.

LEMMA 1.

$$(12)^* \quad L(n;g) = (2/\pi)\int_1^{n^{1/2}} \left| x^{-1}\int_0^1 \sin xt\, dg(t)\right| dx$$
$$+ (2/\pi^2)|g(1) - g(1-)|\log n + o(\log n).$$

Proof. Replacing the factor $\{\sin(2n+1)x\}/\sin x$ by $\{\sin 2nx\}/x$ in (11) induces a bounded error, so that Livingston's formula, weakened slightly, can be written as

$$(13) \qquad L(n;g) = (2/\pi)\int_0^{\frac{1}{2}\pi} x^{-1}|K_n(x)|dx + O(1),$$

where

$$(14) \qquad K_n(x) = \int_0^1 [1 - 4t(1-t)\sin^2 x]^{\frac{1}{2}n} \sin 2nxt\, dg(t).$$

We decompose $L(n;g)$ and consider for fixed ϵ and A, $0 < \epsilon < 1 < A$,

$$I_1(n) = \int_0^{\epsilon/n^{1/2}} x^{-1}|K_n(x)|\, dx,$$
$$I_2(n) = \int_{\epsilon/n^{1/2}}^{A/n^{1/2}} x^{-1}|K_n(x)|\, dx,$$

and

$$I_3(n) = \int_{A/n^{1/2}}^{\frac{1}{2}\pi} x^{-1}|K_n(x)|\, dx.$$

As to $I_1(n)$: Here $0 \leqslant x \leqslant \epsilon/n^{\frac{1}{2}}$ and so

$$1 \geqslant [1 - 4t(1-t)\sin^2 x]^{\frac{1}{2}n} \geqslant [\cos^2 x]^{\frac{1}{2}n} = \cos^n x \geqslant 1 - \epsilon^2,$$

whence

$$\left| |K_n(x)| - \left|\int_0^1 \sin 2nxt\, dg(t)\right|\right| \leqslant \epsilon^2 V(g),$$

while, trivially,

$$\left| |K_n(x)| - \left|\int_0^1 \sin 2nxt\, dg(t)\right|\right| \leqslant (4nx)V(g).$$

Hence,

$$I_1(n) = \int_0^{\epsilon/n^{1/2}} x^{-1}\left|\int_0^1 \sin 2nxt\, dg(t)\right| dx + E_0,$$

*It should be noted that the upper limits of the Stieltjes integrals in (11) and (12) are not the same, being $1-$ in the former and 1 in the latter.

where

$$|E_0| \leqslant V(g) \int_0^{\epsilon/n} x^{-1}(4nx)dx + V(g) \int_{\epsilon/n}^{\epsilon/n^{1/2}} x^{-1}\epsilon^2 dx = (4\epsilon + \tfrac{1}{2}\epsilon^2 \log n) V(g).$$

Finally,

$$\int_{\epsilon/n^{1/2}}^{1/(2n^{1/2})} x^{-1} \left| \int_0^1 \sin 2nxt\, dg(t) \right| dx \leqslant [\log(1/\epsilon)] V(g),$$

so that, replacing $2nx$ by x,

$$I_1(n) = \int_1^{n^{1/2}} x^{-1} \left| \int_0^1 \sin xt\, dg(t) \right| dx + E_1, \tag{15}$$

with

$$|E_1| \leqslant [4\epsilon + \tfrac{1}{2}\epsilon^2 \log n + \log(1/\epsilon) + 1] V(g),$$

where the 1 has to be added because the portion of $I_1(n)$ going from 0 to 1 has been dropped.

As to $I_2(n)$: Since $|K_n(x)| \leqslant V(g)$, we have

$$0 \leqslant I_2(n) \leqslant [\log(A/\epsilon)] V(g). \tag{16}$$

As to $I_3(n)$: Here it is convenient to decompose $K_n(x)$ by writing

$$K_n(x) = [g(1) - g(1-)] \sin 2nx + \int_0^{1-} [1 - 4t(1-t)\sin^2 x]^{\frac{1}{2}n} \sin 2nxt\, dg(t). \tag{17}$$

For $\frac{1}{2}\pi \geqslant x \geqslant A/n^{\frac{1}{2}}$, $(\sin x)/x \geqslant 2/\pi$, so that

$$\begin{aligned}[1 - 4t(1-t)\sin^2 x]^{\frac{1}{2}n} &\leqslant [1 - 4t(1-t)(4A^2)/(\pi^2 n)]^{\frac{1}{2}n} \\ &= [1 - 16A^2\pi^{-2}t(1-t)n^{-1}]^{\frac{1}{2}n} \\ &< \exp\{-(8/\pi^2)(A^2 t)(1-t)\},\end{aligned}$$

since $(1 - k^{-1})^k \uparrow e^{-1}$.

Hence, the integral on the right in (17) is dominated in absolute value by

$$\phi(A) = \int_0^{1-} \exp\{-(8/\pi^2)(A^2 t)(1-t)\}\, d|g(t)|. \tag{18}$$

This approaches zero as $A \to \infty$, from the dominated convergence theorem, since $g(0+) = g(0) = 0$.

Thus,

$$I_3(n) = |g(1) - g(1-)| \int_{A/n^{1/2}}^{\frac{1}{2}\pi} x^{-1} |\sin 2nx| dx + E_3, \tag{19}$$

where, for all large n,

$$|E_3| \leqslant \phi(A) \int_{A/n^{1/2}}^{\frac{1}{2}\pi} x^{-1} dx \leqslant \phi(A) \log n.$$

Furthermore,

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$$\int_{A/n^{1/2}}^{\frac{1}{2}\pi} \frac{|\sin 2nx|}{x}\,dx = \int_{A/n^{1/2}}^{\frac{1}{2}\pi} \frac{|\sin 2nx| - (2/\pi)}{x}\,dx + \frac{2}{\pi}\log\frac{\pi n^{\frac{1}{2}}}{2A}$$
$$= (1/\pi)\log n + E_4,$$

where, since $A > 1$,

$$|E_4| \leqslant \log A + C,$$

with*

$$C = \sup_{V>U\geqslant 1}\left|\int_U^V \frac{|\sin t| - (2/\pi)}{t}\,dt\right|.$$

Now,

$$\frac{L(n;g)}{\log n} = \frac{2}{\pi}\frac{1}{\log n}[I_1(n) + I_2(n) + I_3(n)] + O\left(\frac{1}{\log n}\right),$$

and so, from (15), (16), and (19),

$$\limsup_{n\to\infty}\left|\frac{2}{\pi\log n}\int_1^{n^{1/2}}\left|\frac{1}{x}\int_0^1 \sin xt\,dg(t)\right|dx + (2/\pi^2)|g(1) - g(1-)| - \frac{L(n;g)}{\log n}\right|$$
$$\leqslant (2/\pi)\phi(A) + (1/\pi)\epsilon^2 V(g).$$

Letting $\epsilon\to 0$ and $A\to\infty$ completes the proof of Lemma 1, since $\phi(A)\to 0$ as $A\to\infty$.

Our next lemma is a direct generalization (even to the proof) of the corresponding theorem for Fourier series due to Wiener (**10**, p. 221**)**. It seems likely that it would be in the literature already, but, for lack of a reference, we include a proof.

LEMMA 2.† *If $h(t)$ is continuous and of bounded variation, $0 \leqslant t \leqslant 1$, then*

(20) $$\int_1^n x^{-1}\left|\int_0^1 \sin xt\,dh(t)\right|dx = o(\log n).$$

Proof. Without loss of generality, we assume that $h(0) = h(1) = 0$. Otherwise, we could subtract from $h(t)$ an appropriate linear function, and such a function contributes to our integral only

$$O(1)\int_1^n x^{-1}\left|\int_0^1 \sin xt\,dt\right|dx = O(1).$$

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*C is finite, since $\int_1^\infty \frac{|\sin t| - (2/\pi)}{t}\,dt$ converges.

†The converse also holds (as is the case in Wiener's theorem); that is, for $h(t)$ of bounded variation, (20) implies that $h(t)$ is continuous. This is an immediate consequence of (1) and (4), once Lemma 1 is taken into account.

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This assumption made, integration by parts shows that

$$\int_1^n x^{-1}\left|\int_0^1 \sin xt\, dh(t)\right| dx = \int_1^n \left|\int_0^1 (\cos xt)\, h(t)\, dt\right| dx.$$

Defining $h(t)$ to be zero for $t > 1$ and for $t < 0$, we have, from Parseval's theorem,

$$\int_{-\infty}^{\infty}\left\{\left[h\left(t+\frac{1}{k}\right) - h(t)\right]^2 + \ldots + \left[h(t+1) - h\left(t+\frac{k-1}{k}\right)\right]^2\right\} dt$$

$$= \{k/(2\pi)\}\int_{-\infty}^{\infty} 4\sin^2\{x/(2k)\}\left|\int_0^1 e^{ixt}h(t)\, dt\right|^2 dx$$

$$\geqslant pk\int_k^{2k}\left|\int_0^1 e^{ixt}h(t)\, dt\right|^2 dx \geqslant p\left\{\int_k^{2k}\left|\int_0^1 e^{ixt}h(t)\, dt\right| dx\right\}^2,$$

where $p > 0$, and the last inequality follows from that of Buniakowsky–Schwarz (**4**, p. 132).

Hence,

$$\int_k^{2k}\left|\int_0^1 (\cos xt)h(t)\, dt\right| dx$$

$$\leqslant \left[p^{-1}\int_{-1}^{1}\left\{\left[h\left(t+\frac{1}{k}\right)-h(t)\right]^2 + \ldots + \left[h(t+1) - h\left(t+\frac{k-1}{k}\right)\right]^2\right\}dt\right]^{\frac{1}{2}}$$

$$\leqslant p^{-\frac{1}{2}}[2V\omega(1/k)]^{\frac{1}{2}},$$

where ω is the modulus of continuity, and V the total variation, of $h(t)$.

The lemma follows by using the above estimate for $k = 1, 2, 4, 8, \ldots, 2^m$, where $m = [\log_2 n]$, the largest integer in $\log_2 n$, and adding, since $\omega(1/k) \to 0$ as $k \to \infty$.

3. Proof of Theorem 1. Now let $g(t) = h(t) + j(t)$, where $h(t)$ is continuous and

$$j(t) = \sum_{\xi_k \leqslant t} [g(\xi_k+) - g(\xi_k-)],$$

where the (possibly countably infinite) set $\{\xi_k\}$ consists of all the points of discontinuity (jumps) of $g(t)$.

By Lemma 1,

$$L(n; g) = (2/\pi)\int_1^{n^{1/2}} x^{-1}\left|\int_0^1 \sin xt\, dj(t)\right| dx + o(\log n)$$

$$+ O(1)\int_0^{n^{1/2}} x^{-1}\left|\int_0^1 \sin xt\, dh(t)\right| dx$$

$$+ (2/\pi^2)|g(1) - g(1-)|\log n,$$

and so, by Lemma 2,

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$$(21)\quad L(n;g) = (2/\pi)\int_0^{n^{1/2}} x^{-1}\left|\int_0^1 \sin xt\, dj(t)\right| dx + (2/\pi^2)|g(1) - g(1-)|\log n + o(\log n).$$

Now, if

$$F(T) = \int_0^T \left|\int_0^1 \sin xt\, dj(t)\right| dx,$$

then

$$F(T) = T\mathscr{M}\left\{\left|\sum_k [g(\xi_k+) - g(\xi_k-)]\sin \xi_k x\right|\right\} + o(T)$$
$$\equiv T A(g) + o(T),$$

say, while,

$$\int_1^{n^{1/2}} x^{-1}\left|\int_0^1 \sin xt\, dj(t)\right| dx = \int_1^{n^{1/2}} T^{-1}F'(T)\, dT$$
$$= n^{-\frac{1}{2}}F(n^{\frac{1}{2}}) - F(1) + \int_1^{n^{1/2}} T^{-2}F(T)\, dT$$
$$= O(1) + A(g)\int_1^{n^{1/2}} T^{-1}dT + o(1)\int_1^{n^{1/2}} T^{-1}\, dT$$
$$= \tfrac{1}{2}A(g)\log n + o(\log n).$$

Finally, then, substituting in (21), we obtain the desired conclusions (1) and (2).

The remaining conclusions (3), (4), (5), and (6) follow readily. (3) and (4) are obvious consequences of (2), since $g(t)$ is a function of bounded variation with $g(0) = 0$ and $g(1) = 1$ and $\mathscr{M}(|\sin x|) = 2/\pi$. The "if" part of (5) is plain, since ordinary convergence is the Hausdorff method with $g(t) = 0$, $0 \leqslant t < 1$, $g(1) = 1$. For the "only if" part, we note that for $C(g)$ to equal $4/\pi^2$ it is necessary that each term on the right of (2) be $2/\pi^2$, so that $g(1) - g(1-) = 1$, whence $g(1-) = 0$ (since $g(1) = 1$). The non-decreasing character of $g(t)$ then implies that $g(t) = 0$, $0 \leqslant t < 1$, so that $g(t)$ is the weight function for ordinary convergence.

The first part of (6) is obvious from (2) and (3). The "if" portion of the second part has been established by Livingston **(7)**. To establish the "only if" part we use the following lemma.

Lemma 3. *If $a_k \geqslant 0$ and if at least two of the a_k's are positive, then*

$$(22)\qquad \mathscr{M}\left\{\left|\sum_k a_k \sin \xi_k x\right|\right\} < (2/\pi)\sum_k a_k.$$

Proof. We consider

$$\sum_k a_k|\sin \xi_k x| - \left|\sum_k a_k \sin \xi_k x\right|.$$

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By hypothesis, this cannot be identically zero. It is, however, non-negative and almost-periodic, and so has positive mean. Thus, taking means,

$$\sum_k a_k(2/\pi) - \mathscr{M}\left\{\left|\sum_k a_k \sin \xi_k x\right|\right\} > 0,$$

and the lemma is verified.

Putting $a_k = g(\xi_k+) - g(\xi_k-)$ now shows, in view of the above lemma and (2), that $g(\xi_k+) - g(\xi_k-)$ can be different from zero for only one value of k if $C(g) = 2/\pi^2$, that is, that $g(t)$ can have at most one discontinuity. From (4) and the present assumption that $g(1) = g(1-)$, we know that it must have at least one jump in $0 < t < 1$. Thus, $g(t)$ is the one-jump step function which defines a method of Euler type, and the proof of (6) is complete.

4. Proof of Theorem 2. We show now that the error term in (1), $o(\log n)$, cannot be improved even for the class of increasing absolutely continuous weight functions.

By Livingston's formula (11) we have

$$L(n; g) \geqslant (2/\pi)\int_0^{\frac{1}{2}\pi n^{-1/2}} x^{-1}\left|\int_0^1 [1 - 4t(1-t)\sin^2 x]^{\frac{1}{2}n} \sin 2nx\, t\, dg(t)\right| dx + o(1),$$

which in turn yields

$$L(n; g) \geqslant (2/\pi)\int_0^{\pi n^{1/2}} x^{-1}\left|\int_0^1 \sin xt\, dg(t)\right| dx + O(1). \tag{23}$$

This latter inequality follows from its predecessor since

$$[1 - 4t(1-t)\sin^2 x]^{\frac{1}{2}n} = 1 + O(nx^2)$$

and

$$\int_0^{\frac{1}{2}\pi n^{-1/2}} x^{-1}(nx^2)\, dx = O(1).$$

We may assume, of course, that $\epsilon(n) \log n \to \infty$, since otherwise the theorem is trivial. This done, we proceed now with the construction of an L_1 function $q(t)$ and a $\delta > 0$ for which

$$\int_0^{\pi n^{1/2}} x^{-1}\left|\int_0^1 q(t) \sin xt\, dt\right| dx \geqslant \delta\epsilon(n) \log n, \qquad n = 1, 2, \ldots. \tag{24}$$

Now let $\{a_n\}$ be a convex sequence such that $a_n \to 0$ and $a_n \geqslant [\epsilon(n)]^{\frac{1}{2}}$ for all n. Then (**10**, p. 109; **11**, p. 183), $\frac{1}{2}a_0 + \Sigma a_n \cos nt$ is the Fourier series of some non-negative L_1 function, say $p(t)$. Define $q(t) = p\{\pi(t - \frac{1}{2})\} - p\{\pi (t + \frac{1}{2})\}$. Thus, $q(t)$ is in L_1 and has the Fourier series

$$2a_1 \sin \pi t - 2a_3 \sin 3\pi t + 2a_5 \sin 5\pi t - + \ldots.$$

Let

$$M = \int_0^1 |q(t)|\, dt$$

and recall that $M \geqslant 2a_1 \geqslant 2[\epsilon(n)]^{\frac{1}{2}}$ so that $[\epsilon(n)]^{\frac{1}{2}}/(2M) < \pi$.

Denote by I_k the interval

$$(2k+1)\,\pi - \frac{[\epsilon(n)]^{\frac{1}{2}}}{2M} \leqslant x \leqslant (2k+1)\,\pi$$

for $0 < 2k+1 \leqslant n^{\frac{1}{2}}$. These intervals are disjoint and lie in $(0, \pi n^{\frac{1}{2}})$. Throughout I_k we have

$$|\sin xt - \sin(2k+1)\,\pi t| \leqslant \frac{[\epsilon(n)]^{\frac{1}{2}}}{2M},$$

and so

$$\begin{aligned} &\left|\int_0^1 q(t)\sin xt\, dt\right| \\ &\qquad \geqslant \left|\int_0^1 q(t)\sin\{(2k+1)\,\pi t\}\, dt\right| - \frac{[\epsilon(n)]^{\frac{1}{2}}}{2M}\int_0^1 |q(t)|\, dt \\ &\qquad = |a_{2k+1}| - \tfrac{1}{2}[\epsilon(n)]^{\frac{1}{2}} \geqslant \tfrac{1}{2}[\epsilon(n)]^{\frac{1}{2}}. \end{aligned}$$

Thus, I_k having length $\frac{1}{2}[\epsilon(n)]^{\frac{1}{2}}/M$, we obtain

$$\int_{I_k} x^{-1}\left|\int_0^1 q(t)\sin xt\, dt\right| dx \geqslant \frac{\epsilon(n)}{4M\pi}\,\frac{1}{2k+1}.$$

Hence

$$\int_0^{\pi n^{1/2}} x^{-1}\left|\int_0^1 q(t)\sin xt\, dt\right| dx \geqslant \frac{\epsilon(n)}{4M\pi}\sum_{1\leqslant 2k+1\leqslant n^{1/2}} \frac{1}{2k+1} \geqslant \frac{\epsilon(n)\log n}{16M\pi}.$$

In view of (23), this proves (24) with $\delta = 1/(16M\pi)$.

Now, by the decomposition theorem, we can write

$$\int_0^t q(s)\, ds = c_1 g_1(t) - c_2 g_2(t),$$

where $g_i(t)$, $i = 1, 2$, are absolutely continuous, increasing and $g_i(0) = 0$, $g_i(1) = 1$.

If $g_1(t)$ and $g_2(t)$ both satisfied the relation $L(n; g) = o(\epsilon(n)\log n)$, then, from (23),

$$\int_0^{\pi n^{1/2}} x^{-1}\left|\int_0^1 q(t)\sin xt\, dt\right| dx = o(\epsilon(n)\log n),$$

which contradicts (24).

Thus, at least one of the pair $g_1(t)$, $g_2(t)$ must meet our requirements and the proof is complete.

5. Proof of Theorem 3. Here we revert to (11), which assumes now a simpler form since the weight function (7) is absolutely continuous. Disregarding the error term, which is $o(1)$, we denote the present case of the integral on the right of (11) by $L(n; C_\alpha)$, so that

$$L(n; C_\alpha) = (2\alpha/\pi)\int_0^{\frac{1}{2}\pi} x^{-1}\left|\int_0^1 [1 - 4t(1-t)\sin^2 x]^{\frac{1}{2}n} (1-t)^{\alpha-1}\sin 2nxt\, dt\right| dx. \tag{25}$$

The principle of stationary phase leads us to expect the chief contribution of the inner integrand to arise when the expression in brackets is virtually one. Accordingly, we set about replacing that expression by 1 and show that the resulting error is $o(1)$. To this end, we disregard the factor $(2\alpha/\pi)$ and decompose the integral as follows:

$$\int_0^{\frac{1}{2}\pi} x^{-1}\left|\int_0^1 [1 - 4t(1-t)\sin^2 x]^{\frac{1}{2}n} (1-t)^{\alpha-1}\sin 2nxt\, dt\right| dx = \int_0^{n^{\beta-1}} + \int_{n^{\beta-1}}^{\frac{1}{2}\pi}, \tag{26}$$

where β is a constant, $\frac{1}{2} < \beta < 1$.

We show first that the last integral is $o(1)$.

Denoting the absolute value of the inner integral by $V_n(x)$, we have

$$\int_{n^{\beta-1}}^{\frac{1}{2}\pi} x^{-1} V_n(x)\, dx < \{\max_x V_n(x)\}(1-\beta)\log n, \tag{27}$$

where the maximum is taken for $n^{\beta-1} \leqslant x \leqslant \frac{1}{2}\pi$.

Now, noting that $\sin x > (2/\pi)x$ for $0 < x < \frac{1}{2}\pi$, and that here x is between $n^{\beta-1}$ and $\frac{1}{2}\pi$,

$$\begin{aligned}&[1 - 4t(1-t)\sin^2 x]^{\frac{1}{2}n} < [1 - 16\pi^{-2}t(1-t)x^2]^{\frac{1}{2}n}\\ &< [1 - 16\pi^{-2}t(1-t)n^{2\beta-2}]^{\frac{1}{2}n} < \exp\{-8\pi^{-2}t(1-t)n^{2\beta-1}\},\end{aligned}$$

since $(1-k^{-1})^k$ increases to e^{-1} as k becomes infinite.

Thus,

$$\begin{aligned}0 < V_n(x) &< \int_0^1 \exp\{-8\pi^{-2}t(1-t)n^{2\beta-1}\}(1-t)^{\alpha-1}\, dt\\ &< 2\int_0^{\frac{1}{2}} t^{\alpha-1}\exp\{-8\pi^{-2}\, t(1-t)n^{2\beta-1}\}\, dt\\ &< 2\int_0^{\frac{1}{2}} t^{\alpha-1}\exp\{-4\pi^{-2}\, t\, n^{2\beta-1}\}\, dt\\ &< 2\int_0^{\infty} t^{\alpha-1}\exp\{-4\pi^{-2}\, t\, n^{2\beta-1}\}\, dt\\ &= 2\Gamma(\alpha)(\tfrac{1}{2}\pi)^{2\alpha}(n^{1-2\beta})^{\alpha}.\end{aligned}$$

In connection with (27), this establishes that

$$(28)\qquad \int_{n^{\beta-1}}^{\frac{1}{2}\pi} x^{-1} V_n(x)\,dx < \int_{n^{\beta-1}}^{\frac{1}{2}\pi} x^{-1} \left| \int_0^1 [1-4t(1-t)\sin^2 x]^{\frac{1}{2}n} (1-t)^{\alpha-1}\,dt \right| dx = o(1).$$

In the first integral on the right in (26) we replace the expression in brackets by 1 and show that the error committed is $o(1)$. To do so, we consider the difference $D_n(C_\alpha)$ of the two expressions:

$$(29)\qquad D_n(C_\alpha) = \int_0^{n^{\beta-1}} x^{-1} \left| \int_0^1 f_n(t) \sin 2nxt\,dt \right| dx,$$

where

$$(30)\qquad f_n(t) = \{1 - [1-4t(1-t)\sin^2 x]^{\frac{1}{2}n}\}(1-t)^{\alpha-1}.$$

We note that $f_n(0) = f_n(1-) = 0$ and integrate by parts the inner integral in (29), obtaining

$$D_n(C_\alpha) = \tfrac{1}{2}n^{-1} \int_0^{n^{\beta-1}} x^{-2} \left| \int_0^1 \{(1-\alpha)(1-t)^{-1} f_n(t) + (2n\sin^2 x)[1-4t(1-t)\sin^2 x]^{\frac{1}{2}n-1}(1-t)^{\alpha-1}(1-2t)\} \cos 2nxt\,dt \right| dx$$

$$< \tfrac{1}{2}n^{-1} \int_0^{n^{\beta-1}} x^{-2} \int_0^1 (1-t)^{-1} f_n(t)\,dt\,dx + \int_0^{n^{\beta-1}} x^{-2}\sin^2 x \int_0^1 (1-t)^{\alpha-1}\,dt\,dx$$

$$\equiv D_{n1}(C_\alpha) + D_{n2}(C_\alpha) = D_{n1}(C_\alpha) + o(1).$$

Now, (**4**, p. 40 (2.15.3))

$$(31)\qquad 0 \leqslant 1 - [1-4t(1-t)\sin^2 x]^{\frac{1}{2}n} \leqslant 2nt(1-t)\sin^2 x,$$

so that

$$(32)\qquad D_{n1}(C_\alpha) \leqslant \int_0^{n^{\beta-1}} x^{-2}\sin^2 x \int_0^1 (1-t)^{\alpha-1}\,dt\,dx = o(1).$$

Thus, $D_n(C_\alpha) = o(1)$ and we have

$$(33)\qquad L(n; C_\alpha) = (2\alpha/\pi) \int_0^{n^{\beta-1}} x^{-1} \left| \int_0^1 (1-t)^{\alpha-1} \sin 2nxt\,dt \right| dx + o(1)$$

$$= (2\alpha/\pi) \int_0^{2n^\beta} x^{-1} \left| \int_0^1 (1-t)^{\alpha-1} \sin xt\,dt \right| dx + o(1)$$

$$= (2\alpha/\pi) \int_0^{\infty} x^{-1} \left| \int_0^1 (1-t)^{\alpha-1} \sin xt\,dt \right| dx + o(1).$$

This completes the proof of (9), provided the infinite integral exists. That it does was shown in a few lines by Cramér (**2**, p. 10). Alternatively, this fact

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can be established by the method employed in the next section to prove the convergence of the integral in (10). The remaining parts of the theorem are either incorporated in (10″), proved in § 6, or are obvious.

Remark. Cramér based his proof of Theorem 3 on the equivalence of the Cesàro and Riesz means, rather than, as here, by regarding the (C, α) means as special Hausdorff methods.

6. Proof of Theorem 4. The proof of (10) follows the same lines as the proof of (9), and, in fact, utilizes some of the same calculations.

In analogy with the previous section, we define $L(n; H_\alpha)$ to be the integral on the right of (11), thereby committing an error of $o(1)$, with $g(t)$ now given by (8).

Thus,

$$(34)\quad L(n;H_\alpha) = \frac{2}{\pi\Gamma(\alpha)}\int_0^{\frac{1}{2}\pi}\frac{1}{x}\left|\int_0^1 [1 - 4t(1 - t)\sin^2 x]^{\frac{1}{2}n} (-\log t)^{\alpha-1}\sin 2nxt\, dt\right| dx.$$

As before, we consider first that portion of the integral from $x = n^{\beta-1}$ to $x = \frac{1}{2}\pi$, with $\frac{1}{2} < \beta < 1$.

Since $0 < \alpha < 1$, we have $(-\log t)^{\alpha-1} < (1 - t)^{\alpha-1}$, $0 < t < 1$, so that the portion under consideration is less than

$$(2/\pi)[1/\Gamma(\alpha)]\int_{n^{\beta-1}}^{\frac{1}{2}\pi} x^{-1}\left|\int_0^1 [1 - 4t(1 - t)\sin^2 x]^{\frac{1}{2}n}(1 - t)^{\alpha-1}dt\right| dx,$$

which, from (28), is $o(1)$.

Continuing, we define

$$D_n(H_\alpha) = \int_0^{n^{\beta-1}} x^{-1}\left|\int_0^1 \{1 - [1 - 4t(1 - t)\sin^2 x]^{\frac{1}{2}n}\}(-\log t)^{\alpha-1}\sin 2nxt\, dt\right| dx.$$

Integrating the inner integral by parts, this becomes

$$\begin{aligned} D_n(H_\alpha) = \tfrac{1}{2}n^{-1}\int_0^{n^{\beta-1}} x^{-2}\bigg|\int_0^1 &((1 - \alpha)(t^{-1})(-\log t)^{\alpha-2}\{1 - [1 - 4t(1 - t)\sin^2 x]^{\frac{1}{2}n}\} \\ &+ (2n\sin^2 x)(-\log t)^{\alpha-1}[1 - 4t(1 - t)\sin^2 x]^{\frac{1}{2}n-1}(1 - 2t)\}\cos 2nxt\, dt\bigg|\, dx \\ < \tfrac{1}{2}n^{-1}\int_0^{n^{\beta-1}} x^{-2}\int_0^1 &t^{-1}(1 - t)^{\alpha-2}\{1 - [1 - 4t(1 - t)\sin^2 x]^{\frac{1}{2}n}\}\, dt\, dx \\ &+ \int_0^{n^{\beta-1}} x^{-2}\sin^2 x\int_0^1 (1 - t)^{\alpha-1}\, dt\, dx. \end{aligned}$$

The last term is $D_{n2}(C_\alpha)$ which has been shown to be $o(1)$. Applying (31)

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to the preceding term shows it to be less than the integral in (32), which is also $o(1)$. Hence $D_n(H_\alpha) = o(1)$ and

$$(35)\quad L(n; H_\alpha)$$

$$= (2/\pi)[1/\Gamma(\alpha)] \int_0^{n^{\beta-1}} x^{-1} \left| \int_0^1 (-\log t)^{\alpha-1} \sin 2nxt\, dt \right| dx + o(1)$$

$$= (2/\pi)[1/\Gamma(\alpha)] \int_0^{\infty} x^{-1} \left| \int_0^1 (-\log t)^{\alpha-1} \sin xt\, dt \right| dx + o(1),$$

provided the infinite integral converges. If so, this completes the proof of (10).

That the integral converges is a consequence of Bromwich's Theorem, once we observe that there is no singularity at $x = 0$ in (35). We use that form of Bromwich's Theorem employed in (**9**, p. 230). For convenience we paraphrase its statement:

BROMWICH'S THEOREM. *Let $f(t)$ be of bounded variation for $t \geqslant 0$. Then, for* $0 < \alpha < 1$,

$$(36)\qquad x^\alpha \int_0^1 t^{\alpha-1} f(t)\theta(xt)\, dt = f(0+)\Gamma(\alpha)\theta(\tfrac{1}{2}\alpha\pi) + o(1),$$

where $\theta(t)$ denotes either of the functions $\cos t$ *or* $\sin t$.

Applying this to the inner integral in (35) yields

$$x^\alpha \int_0^1 (-\log t)^{\alpha-1} \sin xt\, dt = x^\alpha \int_0^1 t^{\alpha-1} f_\alpha(t) \sin\{x(1-t)\}\, dt$$

$$= x^\alpha \sin x \int_0^1 t^{\alpha-1} f_\alpha(t) \cos xt\, dt - x^\alpha \cos x \int_0^1 t^{\alpha-1} f_\alpha(t) \sin xt\, dt$$

$$= (\sin x)\{\Gamma(\alpha) f_\alpha(0+) \cos \tfrac{1}{2}\alpha\pi + o(1)\} - (\cos x)\{\Gamma(\alpha) f_\alpha(0+) \sin \tfrac{1}{2}\alpha\pi + o(1)\}$$

$$= \Gamma(\alpha) f_\alpha(0+) \sin(x - \tfrac{1}{2}\alpha\pi) + o(1) = \Gamma(\alpha) \sin(x - \tfrac{1}{2}\alpha\pi) + o(1),$$

where $f_\alpha(t) = [-t^{-1}\log(1-t)]^{\alpha-1}$, so that $f_\alpha(0+) = 1$.

Thus, the inner integral is $O(x^{-\alpha})$, making the integrand of the infinite integral $O(x^{-\alpha-1})$, establishing its convergence (in view of the regularity at $x = 0$, already pointed out).

Remark 1. An even easier application of Bromwich's Theorem establishes

$$(37)\qquad x^\alpha \int_0^1 (1-t)^{\alpha-1} \sin xt\, dt = \Gamma(\alpha) \sin(x - \tfrac{1}{2}\alpha\pi) + o(1).$$

This result demonstrates the convergence of the integral in (9).

It also shows that *the requirement of Bromwich's Theorem that $f(t)$ be of bounded variation cannot be relaxed to the slightly weaker assumption that $f(t)$ be monotonic, positive, and integrable.* For, could this be done, we would have

$$x^\alpha \int_0^1 (1-t)^{\alpha-1} \sin xt\, dt = x^\alpha \int_0^1 t^{\alpha-1}(t^{-1}-1)^{\alpha-1} \sin xt\, dt$$

$$= \Gamma(\alpha)(\sin \tfrac{1}{2}\alpha\pi) f(0+) + o(1) = o(1),$$

since $f(t) = (t^{-1}-1)^{\alpha-1}$ and $0 < \alpha < 1$, and this contradicts (37).

The same point can be made by considering similarly

$$x^\alpha \int_0^1 (-\log t)^{\alpha-1} \sin xt\, dt.$$

Remark 2. Information similar to Bromwich's Theorem is collected **(11**, chapter v, § 2, and p. 379**)**, where references to further literature are found.

Reverting now to the proof of Theorem 4, we require a lemma in order to establish (10′) and (10″):

LEMMA 4. *Given regular Hausdorff methods T_1 and T_2 with associated Lebesgue constants $L_1(n)$ and $L_2(n)$, respectively. Suppose that there is a "totally regular" Hausdorff method U such that $T_2 = UT_1$. Then, if* $\lim L_i(n)$ *exists* $(n \to \infty)$ *and equals L_i, $i = 1, 2$, we have $L_2 \leqslant L_1$.*

Proof. The matrix of U has exclusively non-negative elements, as shown (together with the converse) by Hurwitz **(6**, p. 243**)**, so that

$$D_n^{(2)}(t) = \sum_{m=0}^{n} \gamma_{nm} D_m^{(1)}(t), \qquad \gamma_{nm} \geqslant 0,$$

where $D_k^{(i)}(t)$ denotes the T_i transform of the Dirichlet kernel, $i = 1, 2$. Hence

$$|D_n^{(2)}(t)| \leqslant \sum_{m=0}^{n} \gamma_{nm} |D_m^{(1)}(t)|,$$

and, integrating,

$$L_2(n) \leqslant \sum_{m=0}^{n} \gamma_{nm} L_1(m).$$

The conclusion now follows from the regularity of the method U and the existence of the limits L_1 and L_2.

The lemma established, the proofs of (10′) and (10″) are immediate. For (10′), we identify T_1 with (H, α) and T_2 with (C, α), and note that Basu **(1**, pp. 453–454**)** has shown that the corresponding U is totally regular.

As to (10″): first let T_1 be (C, α) and T_2 be (C, β). The corresponding U is totally regular **(6**, p. 245**)** and so $L(C_\beta) \leqslant L(C_\alpha)$ for $0 < \alpha < \beta$. Next, let T_1 be (H, α) and T_2 be (H, β). Again, the corresponding U is totally regular **(1**, pp. 459–460**)** so that $L(H_\beta) \leqslant L(H_\alpha)$. Finally, we note that $L(C_\alpha)$ and $L(H_\alpha)$ each exceed the integrals obtained, respectively, by deleting the absolute value signs about the respective inner integrals, and that the resulting values are both 1.

The remaining assertions of Theorem 3 are obvious.

Remark. That the Gibbs phenomenon is found for (H, α) for at least as large α as for (C, α) also follows from the positivity of the matrix $(H, \alpha)/(C, \alpha)$, since this implies that the oscillation of the (C, α) means cannot exceed that of the (H, α) means (**3**, p. 52, Theorem 9).

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