Canad. Math. Bull. Vol. **53** (2), 2010 pp. 340–346
doi:10.4153/CMB-2010-025-0

Regular Points of a Subcartesian Space

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Abstract. We discuss properties of the regular part S_{reg} of a subcartesian space S. We show that S_{reg} is open and dense in S and the restriction to S_{reg} of the tangent bundle space of S is locally trivial.

1 Introduction

In 1967, Sikorski began to study smooth structures on topological spaces in terms of their corresponding rings of smooth functions [3]. He introduced the concept of a *differential space* which is a generalization of the notion of a smooth manifold. This concept has the advantage that the category of differential spaces is closed under the operation of taking subsets. In other words, every subset of a differential space inherits a structure of a differential space such that the inclusion map is smooth, [4,5]. The theory of differential spaces has been further developed by several authors.

Also in 1967, Aronszajn introduced the notion of a *subcartesian space* [1], which can be described as a Hausdorff differential space that is locally diffeomorphic to a differential subspace of a Euclidean space $\mathbb{R}^n$. The original definition of Aronszajn uses a singular atlas rather than the differential structure provided by the ring of smooth functions. In the literature on differential spaces, subcartesian spaces as introduced by Aronszajn are called *differential spaces of class* D_0, [7,8]. We use here the term *subcartesian (differential) spaces* because it is more descriptive.

For a differential space S, with the ring $C^\infty(S)$ of smooth functions on S, vectors tangent to S at $x \in S$ are defined as derivations of $C^\infty(S)$ at x. They form a vector space denoted by T_xS. The tangent bundle space of S is the set $TS = \bigcup_{x\in S} T_xS$ with an induced structure of a differential space such that the map $\tau\colon TS \to S$, defined by $\tau^{-1}(x) = T_xS$ for every $x \in S$, is smooth. It should be mentioned that the dimension of T_xS may depend on $x \in S$. In the literature, TS is also called the tangent pseudobundle of S, or the Zariski tangent bundle space of S and is denoted by T^ZS.

A point $x \in S$ is said to be *regular* if there exists a neighbourhood U of x in S such that $\dim T_yS = \dim T_xS$ for all $y \in U$. Instead of using the dimension of the tangent space T_xS at x, we could use the *structural dimension* of S at x, which is defined as the minimum of all natural numbers n for which there exists a diffeomorphism of a neighbourhood of x in S onto a subset of $\mathbb{R}^n$ [2]. We will show that these two notions of dimension are equivalent.

The regular component of a subcartesian space S is the set S_{reg} consisting of regular points of S. The aim of this note is to show that for a subcartesian space S the regular

Received by the editors April 22, 2007.
Published electronically December 4, 2009.
AMS subject classification: **58A40**.
Keywords: differential structures, singular and regular points.

component S_{reg} is an open and dense subset of S and that the restriction of TS to S_{reg} is a locally trivial fibration. It should be noted that S_{reg} need not be a manifold. For example, for commonly discussed fractals, like the Koch curve or the Sierpinski gasket, all points are regular. Throughout this paper we follow the terminology and notations from [6, 9].

2 Preliminaries

Let S be a subcartesian space, *i.e.*, a Hausdorff differential space S such that for every point $p \in S$, there exists $n \in \mathbb{N}$ and a neighbourhood of p diffeomorphic to a differential subspace of $\mathbb{R}^n$ which need not be open. For a subcartesian space S, local analysis in a sufficiently small open subset U of S can be performed in terms of its diffeomorphic image embedded in $\mathbb{R}^n$. Hence, most of our analysis will be done in terms of differential subspaces of $\mathbb{R}^n$.

Let S be a differential subspace of $\mathbb{R}^n$. A function $f\colon S \to \mathbb{R}$ is smooth if, for every $x \in S$, there exists a neighbourhood U of x in $\mathbb{R}^n$ and a function $f_x \in C^\infty(\mathbb{R}^n)$ such that

$$f|_{U\cap S} = f_x|_{U\cap S}.$$

Thus, the differential structure of S is determined by the ring

$$R(S) = \{f|_S : f \in C^\infty(\mathbb{R}^n)\}$$

consisting of restrictions to S of smooth functions on $\mathbb{R}^n$. Let $N(S)$ denote the ideal of functions in $C^\infty(\mathbb{R}^n)$ which identically vanish on S:

$$N(S) = \{f \in C^\infty(\mathbb{R}^n) : f|_S = 0\}.$$

We can identify $R(S)$ with the quotient $C^\infty(\mathbb{R}^n)/N(S)$.

Let S be a differential space and $C^\infty(S)$ the ring of smooth functions on S. For $x \in S$, a derivation of $C^\infty(S)$ at x is a linear map $u\colon C^\infty(S) \to \mathbb{R}$, such that $f \mapsto u \cdot f$ satisfying Leibniz' rule

$$u \cdot (fh) = (u \cdot f)h(x) + (u \cdot h)f(x)$$

for every $f, h \in C^\infty(S)$. Derivations at x of $C^\infty(S)$ form the tangent space of S at x denoted by T_xS. The union of the tangent spaces T_xS, as x varies over S, is the tangent bundle of S and is denoted by TS. We denote by $\tau_S\colon TS \to S$ the tangent bundle projection defined such that $\tau_S(u) = x$ if $u \in T_xS$. The differential structure of the tangent bundle space of a differential space and smoothness of the tangent bundle projection have been discussed in [4]. For the sake of completeness, we describe the differential structure of TS for a subcartesian space S.

Consider first a differential subspace S of $\mathbb{R}^n$. We denote by $q_1, \dots, q_n$ the restrictions to S of the canonical coordinate functions $(x_1, \dots, x_n)$ on $\mathbb{R}^n$. For every function $f \in C^\infty(S)$ and $x \in S$, there exists a neighbourhood U of x in $\mathbb{R}^n$ and $F \in C^\infty(\mathbb{R}^n)$ such that

$$f|_{U\cap S} = F(q_1, \dots, q_n)|_{U\cap S}. \tag{2.1}$$

Consider $v \in T_xS$, and let $v_i = v \cdot q_i$ for $i = 1, \ldots, n$. Equation (2.1) yields

$$v \cdot f = (\partial_1|_x F)(v \cdot q_1) + \cdots + (\partial_n|_x F)(v \cdot q_n) = v_1 \partial_1|_x F + \cdots + v_n \partial_n|_x F. \tag{2.2}$$

Equation (2.2) shows that $v \in T_xS$ can be identified with a vector $(v_1, \ldots, v_n) \in \mathbb{R}^n$. Since T_xS has the structure of a vector space, the set

$$V_x = \{(v_1, \ldots, v_n) \in \mathbb{R}^n \mid v \in T_xS\}$$

is a vector subspace of $\mathbb{R}^n$. The tangent bundle TS can be presented as a subset of $\mathbb{R}^{2n}$ as follows:

$$TS = \{(x, v) = (q_1, \ldots, q_n, v_1, \ldots, v_n) \in \mathbb{R}^{2n} \mid x \in S \text{ and } v \in V_x\}.$$

We denote by $\tau_S \colon TS \to S$ the tangent bundle projection given by $\tau_S(x, v) = x$, for every $(x, v) \in TS$.

For every $f \in C^\infty(S)$, the differential of f is a function $df \colon TS \to \mathbb{R}$ given by

$$df(v) = v \cdot f$$

for every $v \in TS$. The differential structure of TS is generated by the family of functions $\{q_1 \circ \tau_S, \ldots, q_n \circ \tau_S, dq_1, \ldots, dq_n\}$. In other words, a function $h \colon TS \to \mathbb{R}$ is smooth if, for every $v \in TS$, there is a neighbourhood W of v in $\mathbb{R}^{2n}$ and $H \in C^\infty(\mathbb{R}^{2n})$ such that

$$h|_{W \cap TS} = H(q_1 \circ \tau_S, \ldots, q_n \circ \tau_S, dq_1, \ldots, dq_n)|_{W \cap TS}.$$

For $f \in C^\infty(S)$ satisfying equation (2.1), we have

$$f \circ \tau_S|_{\tau_S^{-1}(U)} = F(q_1 \circ \tau_S, \ldots, q_n \circ \tau_S)|_{\tau_S^{-1}(U)},$$

which implies that $\tau_S^* f = f \circ \tau_S \in C^\infty(TS)$. Thus, the tangent bundle projection τ_S is smooth.

As before, let S be a differential subspace of $\mathbb{R}^n$. A derivation v of $C^\infty(S)$ at $x \in S$ restricts to a derivation of $R(S)$ at x.

Proposition 2.1 *Every derivation of $R(S)$ at x extends to a unique derivation of $C^\infty(S)$ at x.*

Proof Let w be a derivation of $R(S)$ at $x \in S$. Consider $f \in C^\infty(S)$. There exist an open neighbourhood U of x in $\mathbb{R}^n$ and a function $f_x \in C^\infty(\mathbb{R}^n)$ such that $f|_{U \cap S} = f_x|_{U \cap S}$. Set $\widetilde{w}(f) = w(f_x|_S)$. Let V be another open neighbourhood of x in $\mathbb{R}^n$ and $g_x \in C^\infty(\mathbb{R}^n)$ a function such that $f|_{V \cap S} = g_x|_{V \cap S}$. We have that $U \cap V \cap S$ is an open subset of S and $f_x|_{U \cap V \cap S} = g_x|_{U \cap V \cap S}$. Therefore $(f_x - g_x)|_{U \cap V \cap S} = 0$, *i.e.*, $(f_x - g_x)|_S \in R(S) \subset C^\infty(\mathbb{R}^n)$ vanishes identically on the open subset $U \cap V \cap S$ of S. Hence, $w(f_x|_S - g_x|_S) = 0$. This proves that the extension $\widetilde{w}$ is a well-defined derivation of $C^\infty(S)$ extending the derivation w of $R(S)$ at x. Finally, it is clear that such an extension $\widetilde{w}$ of w is uniquely defined. ■

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Remark 2.2 Equation (2.2) shows that every derivation of $C^\infty(S)$ at $x \in S \subseteq \mathbb{R}^n$ can be extended to a derivation of $C^\infty(\mathbb{R}^n)$. We can ask the question under what conditions a derivation w of $C^\infty(\mathbb{R}^n)$ at $x \in S \subseteq \mathbb{R}^n$ defines a derivation of $C^\infty(S)$ at x.

Proposition 2.3 *A derivation w of $C^\infty(\mathbb{R}^n)$ at $x \in S \subseteq \mathbb{R}^n$ defines a derivation of $C^\infty(S)$ at x if and only if w annihilates $N(S)$,* i.e., *$w(f) = 0$ for all $f \in N(S)$.*

Proof It follows from Proposition 2.1 and Remark 2.2 that derivations at x of $C^\infty(S)$ can be identified with derivations at x of $R(S)$. Now one uses the identification

$$R(S) \equiv \frac{C^\infty(\mathbb{R}^n)}{N(S)} = \frac{C^\infty(\mathbb{R}^n)}{\sim},$$

where $f \sim g$ in $C^\infty(\mathbb{R}^n)$ if and only if $f - g \in N(S)$. For a derivation w at x of $C^\infty(\mathbb{R}^n)$, one defines $w([f]) = w(f)$. It is clear that this defines a derivation of $R(S)$ if and only if $w(f) = 0$ for all $f \in N(S)$. ∎

3 The Regular Component of a Subcartesian Space

We now discuss the notion of structural dimension introduced by Marshall [2].

Definition 3.1 Let S be a subcartesian space. The structural dimension of a point $x \in S$ is the smallest integer, denoted by n_x, such that for some open neighbourhood $U \subseteq S$ of x, there is a diffeomorphism $\varphi\colon U \to V$ for some arbitrary subset $V \subseteq \mathbb{R}^n$.

A real-valued function $f\colon D \to \mathbb{R}$ is upper semi-continuous if the subset of D determined by $\{x \in D : f(x) < a\}$, for any $a \in \mathbb{R}$, is open.

Lemma 3.2 *The function $N\colon S \to \mathbb{N}\colon x \mapsto n_x$ is upper semi-continuous.*

Proof Let $S_i = \{x \in S : n_x \leq i\}$. Assume that S_i is not open. Then there exists a point $z \in S_i$ such that there is no open neighbourhood $U \subseteq S_i$ of z. But then, there is no open neighbourhood $V \subseteq S$ of z diffeomorphic to an arbitrary subset of $\mathbb{R}^j$ for any $j \leq i$. Hence, $n_z > i$, and so z is not in S_i. Thus, S_i is open, and so the structural dimension serves as an upper semi-continuous function on S. ∎

Definition 3.3 A point $x \in S$ is called a structurally regular point if there is a neighbourhood U of x in S such that $n_y = n_x$ for all $y \in U$. A point that is not structurally regular is called structurally singular.

The regular component S_{reg} of a subcartesian space S is the set of all structurally regular points of S.

Lemma 3.4 *For every point x of a subcartesian space S, the structural dimension of S at x is equal to* $\dim T_xS$.

Proof Let $n = n_x$. So there is a neighbourhood $U \subseteq S$ of x diffeomorphic to a differential subspace of $\mathbb{R}^n$. Since any derivation of $C^\infty(S)$ can be extended to a derivation of $C^\infty(\mathbb{R}^n)$, we have $\dim T_xS \leq \dim \mathbb{R}^n = n$.

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Now assume that $\dim T_xS < n$. Then there exists a derivation $u \in T_x\mathbb{R}^n$ that is not an extension of a derivation of $C^\infty(S)$. This implies by Proposition 2.3 that there is a function $f \in N(U)$ such that $u(f) \neq 0$. In this case, if $p^1, \dots, p^n$ are the canonical coordinate functions on $\mathbb{R}^n$, then $\partial_{p^j}|_x f \neq 0$, for some $j \in \{1, \dots, n\}$. Hence, there is a neighbourhood $V \subseteq f^{-1}(0)$ of x that is a submanifold of $\mathbb{R}^n$. It is clear that the structural dimension of S at points in V is $m < n$ (m being the dimension of V as a manifold). There exists an open neighbourhood $\tilde{V} \subseteq V$ of x diffeomorphic to an open subset of $\mathbb{R}^m$. Since $f \in N(U)$, there exists a neighbourhood $W \subset U \subset f^{-1}(0)$ of x. So $\tilde{V} \cap W$ is a neighbourhood of x in $\mathbb{R}^m$. This is a contradiction as the structural dimension $n_x = n > m$. Therefore, $\dim T_xS = n_x$. ∎

Lemma 3.5 *Let n be the maximum of the structural dimensions of S at points of an open subset $V \subset S$. If every open subset contained in V has a point at which the structural dimension is n, then V consists of regular points.*

Proof The assumption implies that the subset $W = \{x \in V : n_x = n\}$ is dense in V. For each $x \in V$, let O_x be an open neighbourhood of x in V diffeomorphic to a subset of $\mathbb{R}^n$. Take $y \in V \setminus W$. Then $n_y < n$ (by the definition of n). Let O_y be an open neighbourhood of y in V diffeomorphic to a subset of $\mathbb{R}^{n_y}$. Since W is dense in V, there exists $x \in W \cap O_y$. So $O_x \cap O_y$ is diffeomorphic to a subset of $\mathbb{R}^{n_y}$. But n is the minimum of all m such that a neighbourhood of x is diffeomorphic to a subset of $\mathbb{R}^m$. Since $O_x \cap O_y$ is a neighbourhood of x diffeomorphic to a subset of $\mathbb{R}^{n_y}$, we have $n \leq n_y$. But $n_y < n$ by assumption. Therefore, $V \setminus W$ is empty, *i.e.,* the dimension of S at a point of the open subset V is n. This implies that every point in V is structurally regular. ∎

Theorem 3.6 *The set S_{reg} of all structurally regular points of a subcartesian space S is open and dense in S.*

Proof Let $x \in S_{\text{reg}}$. Since x is a structurally regular point, there exists an open neighbourhood $U \subseteq S$ of x such that for every $y \in U$, $n_y = n_x$. This implies that every point of U is structurally regular. Hence, $U \subseteq S_{\text{reg}}$. Therefore, S_{reg} is an open subset of S.

Now suppose that the subset S_{reg} of structural regular points is not dense in S. In this case, there exists a non-empty open subset $U \subseteq S$ such that U contains no structurally regular points, *i.e.,* every point in U is a structurally singular point. Without loss of generality, we assume that U is diffeomorphic to a differential subspace of $\mathbb{R}^n$ for some $n > 0$. In fact, n cannot be 0, otherwise U would be a set of isolated points which are regular by the induced topology. Define $S_i = \{x \in S : n_x \leq i\}$. Assume that $U \subset S_k$ (for some $k > 0$). It follows that if $V_1 \subset U$ is an open subset, then V_1 contains infinitely many points where the structural dimensions are at least two different numbers from 0 to k. Let n_1 be the maximum of these structural dimensions at points in V_1. By Lemma 3.5, there exists an open subset $V_2 \subset V_1$ such that the maximum of structural dimensions of S at points in V_2 is $n_2 < n_1$. Similarly, there exists an open subset $V_3 \subset V_2$ with a maximum of structural dimensions at its points $n_3 < n_2$. Thus, continuing this process, we have the decreasing sequence $n_1 > n_2 > n_3 > \cdots > n_i$, stopping at some $n_i \geq 0$. We reach some open subset

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$V_i \subset U$ such that the structural dimension at all points of V_i is $n_i \geq 0$. Hence, all points of V_i are regular points. As a consequence, since U contains no regular points, U is not a subspace of S_k for any $k \geq 0$. But we are dealing only with finite structural dimensions, and U was chosen to be diffeomorphic to a differential subspace of $\mathbb{R}^n$ for some n, so we have $U \subset S_n$, which is a contradiction. Therefore, a non-empty open subset $U \subset S$ containing no structurally regular points does not exist. This completes the proof that the set S_{reg} of all structurally regular points of a subcartesian space S is dense in S. ■

Theorem 3.7 *Let S be a subcartesian space. Then the restriction of the tangent bundle projection $\tau\colon TS \to S$ to $T(S_{\text{reg}})$ is a locally trivial fibration over S_{reg}. For each $x \in S_{\text{reg}}$ with structural dimension n, there is a neighbourhood W of x in S and a family $X_1, \dots, X_n$ of global derivations of $C^\infty(S)$ such that $T_W S = \tau^{-1}(W)$ is spanned by the restrictions $X_1, \dots, X_n$ to V.*

Proof Let $x \in S_{\text{reg}}$ with $n_x = n$. Since S_{reg} is open, there exists a neighbourhood $V \subset S_{\text{reg}}$ of x such that $n_y = n$ for all $y \in V$. As S is a subcartesian space, we may assume without loss of generality that there is an embedding φ of V into $\mathbb{R}^n$. We first prove that TV the set of all pointwise derivations of $C^\infty(V)$ is a trivial bundle.

Let $R(V)$ consist of restrictions to V of all smooth functions on $\mathbb{R}^n$, and $N(V)$ be the space of functions on $\mathbb{R}^n$ which vanish on V. We identify $R(V)$ with $C^\infty(\mathbb{R}^n)$ modulo $N(V)$. It follows that $\partial_i|_y(f|_V) = 0$ for every $i = 1, \dots, n$, each $f \in N(V)$ and $y \in V \subset \mathbb{R}^n$. By Proposition 2.3, we have that $\partial_1|_y, \dots, \partial_n|_y$ define derivations of $C^\infty(V)$ at each $y \in V$. Hence, there are n sections $X_1, \dots, X_n$ of the tangent bundle projection $\tau_V\colon TV \to V$ such that $X_i|_y(h \mod N(V)) = (\partial_i|_y h)$ for every $i = 1, \dots, n$, $h \in R(V)$ and $y \in V$. Now we need to prove that the sections $X_1, \dots, X_n$ are smooth. Let $q_1, \dots, q_n$ be restrictions to V of the coordinate functions on $\mathbb{R}^n$. For $i = 1, \dots, n$, we denote by dq_i the function on TV such that $dq_i(w) = w(q_i)$ for every $w \in TV$. The differential structure of TV is generated by the functions $(\tau_V^* q_1, \dots, \tau_V^* q_n, dq_1, \dots, dq_n)$ in the sense that every function $f \in C^\infty(TV)$ is of the form $f = F(\tau_V^* q_1, \dots, \tau_V^* q_n, dq_1, \dots, dq_n)$ for some $F \in C^\infty(\mathbb{R}^{2n})$. In order to show that $X_i\colon V \to TV$ is smooth, it suffices to show that for every $f \in C^\infty(TV)$ the pull-back $X_i^* f$ is in $C^\infty(V)$. Since

$$dq_i \circ X_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

it follows that

$$\begin{aligned} X_i^* f = f \circ X_i &= F(\tau_V^* q_1, \dots, \tau_V^* q_n, dq_1, \dots, dq_n) \circ X_i \\ &= F(\tau_V^* q_1 \circ X_i, \dots, \tau_V^* q_n \circ X_i, dq_1 \circ X_i, \dots, dq_n \circ X_i) \\ &= F(q_1 \circ \tau_V \circ X_i, \dots, q_n \circ \tau_V \circ X_i, \delta_{1i}, \dots, \delta_{ni}) \\ &= F(q_1, \dots, q_n, \delta_{1i}, \dots, \delta_{ni}). \end{aligned}$$

Hence $X_i^* f$ is in $C^\infty(V)$. This implies that the tangent bundle space TV is globally spanned by n linearly independent smooth sections $X_1, \dots, X_n$. Thus, TV is a trivial

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bundle. We can choose an open neighbourhood W of x contained in V such that its closure $\bar{W}$ is also in V. Using bump functions that are equal to 1 on W and 0 outside of V, we can construct derivations of $C^\infty(S)$ that extend restrictions of $X_1, \dots, X_n$ to W. Hence TW is spanned by the restrictions to W of global derivations of $C^\infty(S)$. ■

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