A NOTE ON BAKER'S METHOD

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1. Introduction

Let $\alpha_1, \cdots, \alpha_n$ $(n \geqq 2)$ be (fixed) multiplicatively independent non zero algebraic numbers and set $M(H) = \min |\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n|$ the minimum taken over all algebraic numbers $\beta_1, \cdots, \beta_n$ not all equal to zero, of degrees not exceeding a fixed natural number d_0, and heights not exceeding an arbitrary natural number H. Then an important result [1] of Baker states that $M(H) > Ae^{-(\log H)^{n+1+\varepsilon}}$ for every fixed $\varepsilon > 0$ and an explicit constant $A = A(\alpha_1, \cdots, \alpha_n, d_0, \varepsilon)$. It may be remarked that Baker deduces his general result from the special case where β_n is fixed to be -1. The following straight forward generalization might be of some interest since it shows that the exponent $n+1+\varepsilon$ need not be the best, and that the best exponent obtainable by his method has some chance of being $1+\varepsilon$ (see the corollary to the Theorem).

THEOREM. *Let $\alpha_1, \cdots, \alpha_{n+f}$ $(n \geqq 1, f \geqq 1)$ be (fixed) multiplicatively independent non-zero algebraic numbers and set*

$$M(H, \alpha_{n+i}) = \min |\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \log \alpha_{n+i}|, \qquad (i = 1, \cdots, f),$$

the minimum taken over all algebraic numbers $\beta_1, \cdots, \beta_n$ of degrees not exceeding a fixed natural number d_0, and heights not exceeding an arbitrary natural number H. Then there holds

$$\sum_{i=1}^{f} M(H, \alpha_{n+i}) > Ae^{-(\log H)^{(n+1)/f+1+\varepsilon}}$$

with an explicit positive constant $A = A(\alpha_1, \cdots, \alpha_{n+f}, d_0, \varepsilon)$.

COROLLARY. *Given $\alpha_1, \cdots, \alpha_n$ as in the theorem and an $\varepsilon > 0$, there exists an algebraic number α_{n+1} for which $M(H, \alpha_{n+1})$ exceeds $Ae^{-(\log H)^{1+\varepsilon}}$ for infinitely many H, with an explicit positive constant*

$$A = A(\alpha_1, \cdots, \alpha_{n+1}, d_0, \varepsilon).$$

The corollary follows from the theorem on making f large and using Dirichlet's box principle. The author is thankful to Professors A. Baker, K. Mahler, K. G. Ramanathan and C. L. Siegel for encouragement.

2. Proof of the theorem

We begin by stating

LEMMA (Generalization of Siegel's, [2] pp. 35—38). *Suppose that the coefficients of the linear forms* $y_k = a_{k1}x_1 + \cdots + a_{kq}x_q$ ($k = 1$ to p, $p < q$) *are integers in an algebraic number field K of degree d and* $\overline{|a_{kl}|} \leqq A (\geqq 1)$. *Then there exist rational integers* $x_1, \cdots, x_q$ *not all zero, satisfying* $y_1 = 0, \cdots, y_p = 0$ *such that*

$$|x_k| < 1 + (2qA)^{(pd(d+1))/(2q-pd(d+1))} \qquad k = 1, \cdots, q$$

provided $2q > pd(d+1)$.

The proof of the lemma consists of an estimation of the number of algebraic numbers of bounded height and degree and will be left as an exercise to the reader.

Let us write $\alpha_{n+i} = \alpha_1^{\beta_{i1}} \cdots \alpha_n^{\beta_{in}} e^{\beta_i}$ ($i = 1, \cdots, f$) where $||\beta_{ij}||$ is an $f \times n$ matrix of algebraic numbers and $|\beta_i|$ are all small say all $\leqq 1/e$ and so $e^{\beta_i} = 1 + \beta_i \gamma_i$ where

$$|\gamma_i| = \left| 1 + \frac{\beta_i}{2!} + \frac{\beta_i^2}{3!} + \cdots \right| \leqq e.$$

For a natural number N we have also $e^{\beta_i N} = 1 + \beta_i \gamma_i'$ where

$$|\gamma_i'| = \left| \frac{1}{\beta_i} \{(1+\beta_i\gamma_i)^N - 1\} \right| = |\gamma_i| \left| \sum_{\nu=1}^{N} \binom{N}{\nu} (\beta_i \gamma_i)^{\nu-1} \right| \leqq 2^N e.$$

We shall fix $|\beta_i|$ to be much smaller later. We shall write $S_i = \max_j \operatorname{size}(\beta_{ji})$, $S = \max_i S_i$ (by size (α) for any algebraic number α we mean $d(\alpha) + \overline{|\alpha|}$ where $d(\alpha)$ is the least natural number for which $\alpha d(\alpha)$ is an algebraic integer. The notation $\overline{|\alpha|}$ denotes maximum of the absolute values of the conjugates of α). We introduce the fundamental function

$$\Phi(z_1, \cdots, z_n) = \sum_{\underline{\lambda}=0}^{L} p(\lambda) \alpha_1^{\gamma_1 z_1} \cdots \alpha_n^{\gamma_n z_n}\ {}^{1} \quad \text{where}$$

(1)

$$\gamma_i = \lambda_i + \lambda_{n+1}\beta_{1i} + \cdots + \lambda_{n+f}\beta_{fi} \qquad (i = 1, \cdots, n),$$

with an estimate for the rational integers $p(\underline{\lambda}) = p(\lambda_1, \cdots, \lambda_{n+f})$ not all zero, to be specified immediately. We shall see that

$$\text{(2)} \qquad \sum_{\underline{\lambda}=0}^{L} p(\underline{\lambda}) \alpha_1^{\lambda_1 l} \cdots \alpha_{n+f}^{\lambda_{n+f} l} \gamma_1^{m_1} \cdots \gamma_n^{m_n} = 0, \quad \begin{pmatrix} 1 \leqq l \leqq h \\ 0 \leqq m_1 + \cdots + m_n \leqq k \end{pmatrix}$$

with h and k large and $|p(\underline{\lambda})|$ small. This will be done by the lemma above. We have i) number of equations $\leqq h(k+1)^n$; ii) number of unknowns

[1] The sum is over all $\lambda_1, \cdots, \lambda_{n+f}$ running independently from 0 to L.

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$= (L+1)^{n+f} > L^{n+f}$ and iii) estimate for the size of the coefficients $\leqq C_1^{Lh}(C_2SL)^{fk}$ where $C_1 = (\max_{i=1,\cdots,n+f} \text{size}\ (\alpha_i))^{n+f}$ and $C_2 = 2$ (because size $\gamma_i = d(\gamma_i-\lambda_i)+\overline{|\gamma_i-\lambda_i|}+\lambda_i \leqq \text{size}\ (\gamma_i-\lambda_i)+L \leqq (SL)^f+L \leqq (2SL)^f$). So under the condition

$$L^{n+f} \geqq h(k+1)^n d(d+1) \tag{3}$$

(where d is the degree of the number field obtained by adjoining $\alpha_1, \cdots, \alpha_{n+f}$ and β_{ji} ($j = 1, \cdots, n$; $i = 1, \cdots, f$) to the rationals), the conditions (2) are all satisfied with

$$|p(\underline{\lambda})| \leqq 2(2(L+1)^{n+f}C_1^{2Lh}(C_2SL)^{2fk}) < C_3^{Lh+k}(SL)^{2fk} \tag{4}$$

where since

$$4(L+1)^{n+f}C_1^{2Lh}C_2^{2fk} \leqq (C_1^2 8^{n+f})^{Lh}(C_2^{2f})^k, \quad C_3 = \max\ (C_1^2 8^{n+f}, C_2^{2f}).$$

The rest of the argument is by induction. Suppose now that

$$\sum_{\underline{\lambda}=0}^{L} p(\underline{\lambda})\alpha_1^{\lambda_1 l}\cdots\alpha_{n+f}^{\lambda_{n+f} l}\gamma_1^{m_1}\cdots\gamma_n^{m_n} = 0 \quad \begin{pmatrix} 1 \leqq l \leqq h_1(\geqq h) \\ 0 \leqq m_1+\cdots+m_n \leqq k_1(\leqq k) \end{pmatrix}. \tag{5}$$

Then we shall require β_i to be so small that

$$\sum_{\underline{\lambda}=0}^{L} p(\underline{\lambda})\alpha_1^{\lambda_1 l}\cdots\alpha_{n+f}^{\lambda_{n+f} l}\gamma_1^{m_1}\cdots\gamma_n^{m_n} = 0 \quad \begin{pmatrix} h_1 < l \leqq h_2 \\ 0 \leqq m_1+\cdots+m_n \leqq k_2(< k_1) \end{pmatrix}. \tag{6}$$

We now set

$$f(z) = \Phi_{m_1,\cdots,m_n}(z_1,\cdots,z_n)_{z_1=\cdots=z_n=z} \qquad (0 \leqq m_1+\cdots+m_n \leqq k_2), \tag{7}$$

and we have for $h_1 < l \leqq h_2$

$$\begin{aligned} f(l) &= \frac{1}{2\pi i}\int_{|z|=4h_2} \frac{f(z)}{z-l}\prod_{r=1}^{h_1}\left(\frac{l-r}{z-r}\right)^{k_1-k_2+1} dz \\ &\quad - \frac{1}{2\pi i}\sum_{r=1}^{h_1}\sum_{m=0}^{k_1-k_2}\frac{f^{(m)}(r)}{m!}\int_{|z-r|=\frac{1}{2}}\frac{(z-r)^m}{z-l}\prod_{r=1}^{h_1}\left(\frac{l-r}{z-r}\right)^{k_1-k_2+1} dz. \end{aligned} \tag{8}$$

This formula shows that $|f(l)|$ is considerably small, while $f(l)$ being apart from a power product of $\log \alpha_1, \cdots, \log \alpha_n$, very close to a non zero algebraic number, the assumption that $f(l) \neq 0$ leads to a lower bound for $|f(l)|$ which when $\beta = \max_i |\beta_i|$ is small, leads to a contradiction. We note the relation

$$\begin{aligned} f^{(m)}(r) &= \left(\frac{\partial}{\partial z_1}+\cdots+\frac{\partial}{\partial z_n}\right)^m \Phi_{m_1,\cdots,m_n}(z_1,\cdots,z_n)_{z_1=\cdots=z_n=r} \\ &= \sum_{j_1+\cdots+j_n=m}\frac{m!}{j_1!\cdots j_n!}\Phi_{m_1+j_1,\cdots,m_n+j_n}(r,\cdots,r) \end{aligned}$$

and the fact that $(m_1+j_1)+\cdots+(m_n+j_n) \leqq k_2+k_1-k_2 = k_1$.

We now split up the proof into five parts ($C_1, C_2, \cdots$ will denote constants $\geqq 2$).

1. *Upper estimate for* $f^{(m)}(r)$. Now for all l and $m_1, \cdots, m_n$

$$\Phi_{m_1,\cdots,m_n}(l,\cdots,l) = (\log \alpha_1)^{m_1}\cdots(\log \alpha_n)^{m_n}\sum_{\underline{\lambda}=0}^{L} p(\underline{\lambda})\alpha_1^{\lambda_1 l}\cdots\alpha_{n+f}^{\lambda_{n+f} l}$$
$$\times e^{-\beta_1\lambda_{n+1} l}\cdots e^{-\beta_f\lambda_{n+f} l}\gamma_1^{m_1}\cdots\gamma_n^{m_n}.$$

We use $e^{-\beta_i\lambda_{n+i} l} = 1+\gamma_i'\beta_i$ with $|\gamma_i'| \leqq e2^{\lambda_{n+i} l}$ and so since $|\beta_i| \leqq (1/e)$, we have for natural numbers $N_1, \cdots, N_f$

$$|e^{\beta_1 N_1+\cdots+\beta_f N_f}-1| = |\prod_{i=1}^{k}|((e^{\beta_i N_i}-1)+1)-1|$$
$$= \sum_{\substack{e_1,\cdots,e_f=0,1\\ \text{not all the } e\text{'s}=0}} |\prod_{i=1}^{f}(e^{\beta_i N_i}-1)|^{e_i} \leqq \sum_{\substack{e_1,\cdots,e_f=0,1\\ \text{not all the } e\text{'s}=0}} \prod_{i=1}^{f}(|\beta_i|e2^{N_i})^{e_i}$$
$$\leqq \beta\prod_{i=1}^{f}(1+e2^{N_i}) \leqq \beta\times 7^{N_1+N_2+\cdots+N_f}$$

where $\beta = \max_{i=1 \text{ to } f}|\beta_i|$.

Hence (5) gives for $1 \leqq r \leqq h_1$ and $0 \leqq j_1+\cdots j_n \leqq k_1-k_2$,

$$|\Phi_{m_1+j_1,\cdots,m_n+j_n}(r,\cdots,r)|$$
$$= |(\log \alpha_1)^{m_1+j_1}\cdots(\log \alpha_n)^{m_n+j_n}\sum_{\lambda=0}^{L} p(\lambda)\alpha_1^{\lambda_1 r}\cdots\alpha_{n+f}^{\lambda_{n+f} r}$$
$$\times(e^{-\beta_1\lambda_{n+1}r-\cdots-\beta_f\lambda_{n+f}r}-1)\gamma_1^{m_1+j_1}\cdots\gamma_n^{m_n+j_n}|$$
$$\leqq \beta(L+1)^{n+f}C_3^{Lh+k}(SL)^{2fk}C_1^{Lh_1}7^{fLh_1}(C_2SL)^{fk_1}C_4^{k_1}$$

where $\quad C_4 = \max_{i=1,\cdots,m}(1+|\log\alpha_i|)$,

$$\leqq \beta C_6^{Lh_1+k}(SL)^{fk_1+2fk} \quad\text{where}\quad C_6^{Lh_1+k} \geqq C_2^{fk_1}C_3^{Lh_1+k}C_4^kC_5^{Lh_1}2^{L(n+f)},$$

$C_5 = 7^f C_1$ i.e. we may set $C_6 = 2^{n+f}C_2^fC_3C_4C_5$.

So we have

$$|f^{(m)}(r)| \leqq n^m\beta C_6^{Lh_1+k}(SL)^{f(k_1+2k)}$$
$$\leqq \beta C_7^{Lh_1+k}(SL)^{3fk} \quad\text{with}\quad C_7 = nC_6. \tag{9}$$

2. *Estimate for max* $|f(z)|$ *on* $|z| = 4h_2$

$$f(z) = (\log \alpha_1)^{m_1}\cdots(\log \alpha_n)^{m_n}\sum_{\underline{\lambda}=0}^{L} p(\underline{\lambda})\alpha_1^{\lambda_1 z}\cdots\alpha_{n+f}^{\lambda_{n+f} z}$$
$$\times e^{-\beta_1\lambda_{n+1} z}\cdots e^{-\beta_f\lambda_{n+f} z}\gamma_1^{m_1}\cdots\gamma_n^{m_n}$$

and so

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$$\max_{|z|=4h_2} |f(z)| \leqq (L+1)^{n+f} C_3^{Lh+k} (SL)^{2fk} C_8^{Lh_2} (C_2 SL)^{fk_2} C_4^{k_2} \quad \text{with} \quad C_8 = C_1^8,$$

$$(10) \qquad \leqq C_9^{Lh_2+k} (SL)^{3fk} \quad \text{where}$$

$$C_9^{Lh_2+k} \geqq (C_3 C_8)^{Lh_2+k} (2^{n+f} C_2^f C_4)^{Lh_2+k}$$

We fix $C_9 = 2^{n+f} C_2^f C_3 C_4 C_8$.

3. *Lower estimate for $|f(l)|$ on the assumption that*

$$\theta = \sum_{\underline{\lambda}=0}^{L} p(\underline{\lambda}) \alpha_1^{\lambda_1 l} \cdots \alpha_{n+f}^{\lambda_{n+f} l} \gamma_1^{m_1} \cdots \gamma_n^{m_n} \neq 0.$$

we have

$$\Big| (\log \alpha_1)^{-m_n} \cdots (\log \alpha_n)^{-m_n} f(l) - \sum_{\underline{\lambda}=0}^{L} p(\underline{\lambda}) \alpha_1^{\lambda_1 l} \cdots \alpha_{n+f}^{\lambda_{n+f} l} \gamma_1^{m_1} \cdots \gamma_n^{m_n} \Big|$$

$$\leqq \sum_{\underline{\lambda}=0}^{L} |p(\underline{\lambda}) \alpha_1^{\lambda_1 l} \cdots \alpha_{n+f}^{\lambda_{n+f} l} 7^{\lambda_{n+1} l + \cdots + \lambda_{n+f} l} \gamma_1^{m_1} \cdots \gamma_n^{m_n}| \beta$$

$$\leqq \beta (L+1)^{n+f} C_3^{Lh+k} (SL)^{2fk} C_1^{Lh_2} (7^f)^{Lh_2} (C_2 SL)^{fk_2}$$

$$\leqq \beta C_6^{Lh_2+k} (SL)^{3fk}$$

since

$$h_1 < l \leqq h_2 \text{ and } m_1 + \cdots + m_n \leqq k_2 \ (< k_1 \leqq k).$$

Also $A\theta \neq 0$ is an algebraic integer for some natural number A,

$$A \leqq C_{10}^{Lh_2} S^{fk_2} \text{ with } C_{10} = (\text{size } \alpha_1) \cdots (\text{size } \alpha_{n+f})$$

and

$$\boxed{\theta} \leqq (L+1)^{n+f} C_3^{Lh+k} (SL)^{2fk} C_{10}^{Lh_2} (C_2 SL)^{fk_2}.$$

Thus we have as usual (here d denotes the degree of θ)

$$|\theta| \geqq A^{-d} \boxed{\theta}^{-(d-1)} \geqq A^{-d} \{(L+1)^{n+f} C_3^{Lh+k} (SL)^{2fk} C_{10}^{Lh_2} (C_2 SL)^{fk_2}\}^{-(d-1)}$$

and so

$$|\theta| \geqq \{C_{10}^{Lh_2} S^{k_2 f} (L+1)^{n+f} C_3^{Lh+k} (SL)^{2fk} C_{10}^{Lh_2} (C_2 SL)^{fk_2}\}^{-d}$$

$$\geqq \{D_{11}^{Lh_2+k}\}^{-1} (SL)^{-4kfd} \text{ where } D_{11}^{Lh_2+k} \geqq (C_3 C_{10}^2 2^{n+f} C_2^f)^{d(Lh_2+k)}$$

and we may set $D_{11} = (C_2^f C_3 C_{10}^2 2^{n+f})^d$.

So we have the inequality

$$|(\log \alpha_1)^{-m_1} \cdots (\log \alpha_n)^{-m_n} f(l)| > \{D_{11}^{Lh_2+k}\}^{-1} (SL)^{-4kfd} - \beta C_6^{Lh_2+k} (SL)^{3fk}$$

and here the L.H.S. being $\leqq f(l)$ $(1 + \max_{i=1 \text{ to } n} |\log \alpha_i|^{-1})^k$ we can wiite $C_{11} = D_{11}$ $(1 + \max_{i=1 \text{ to } n} |\log \alpha_i|^{-1}|)$ and thus

$$(11) \qquad |f(l)| > \{C_{11}^{Lh_2+k}\}^{-1} (SL)^{-4kfd} - \beta C_6^{Lh_2+k} (SL)^{3fk}.$$

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4. *An upper estimate for* $|f(l)|$ *in* (8)

$$|f(l)| < \frac{1}{2\pi}\,\frac{2\pi\cdot 4h_2}{4h_2-2h_2}\,C_9^{Lh_2+k}(SL)^{3fk}\left(\frac{h_2}{4h_2-2h_2}\right)^{(k_1-k_2+1)h_1}$$
$$+\frac{1}{2\pi}\,h_1(k_1-k_2+1)\beta C_7^{Lh_1+k}(SL)^{3fk}2\pi\cdot\tfrac{1}{2}(2h_2)^{h_1(k_1-k_2+1)}\cdot 2$$

and so

$$(12)\qquad |f(l)| < 2C_9^{Lh_2+k}(SL)^{3fk}2^{-h_1(k_1-k_2+1)}+h_1(k_1-k_2)\beta C_7^{Lh_1+k}(SL)^{3fk}\times(2h_2)^{h_1(k_1-k_2+1)}.$$

We now set $k_2=[k_1/2]$ and so $2^{-h_1(k_1-k_2+1)}<2^{-\frac{1}{2}h_1k_1}$,

$$(2h_2)^{h_1(k_1-k_2+1)}<(2h_2)^{h_1((k_1/2)+2)}\text{ and } k_1-k_2<\frac{k_1}{2}+1.$$

5. *Final step.* In order to ensure that (6) holds we have to see that (11) and (12) contradict, i.e.

$$\{C_{11}^{Lh_2+k}\}^{-1}(SL)^{-4fkd}\geqq 2C_9^{Lh_2+k}(SL)^{3fk}2^{-\frac{1}{2}h_1k_1}+\beta C_6^{Lh_2+k}(SL)^{3fk}$$
$$+h_1\left(\frac{k_1}{2}+1\right)\beta C_7^{Lh_1+k}(SL)^{3fk}(2h_2)^{h_1((k_1/2)+2)}$$

i.e.

$$2^{\frac{1}{2}h_1k_1}\geqq(2C_9C_{11})^{Lh_2+k}(SL)^{(4d+3)fk}$$
$$+\beta(C_6C_{11})^{Lh_2+k}(SL)^{(4d+3)fk}2^{\frac{1}{2}h_1k_1}$$
$$+\beta h_1\left(\frac{k_1}{2}+1\right)(C_7C_{11})^{Lh_2+k}(SL)^{(4d+3)}(2h_2)^{h_1((k_1/2)+2)}2^{\frac{1}{2}h_1k_1}$$

i.e. we have to satisfy some thing like

$$(13)\qquad 2^{\frac{1}{2}h_1k_1}\geqq C_{12}^{Lh_2+k\log(SL)}\{1+\beta C_{13}^{h_1k_1\log h_2}\}$$

where

$$C_{12}^{Lh_2+k\log(SL)}\geqq(C_{11}\max(3C_9,C_6,C_7))^{Lh_2+k+(4d+3)fk\log(SL)}$$

and since if

$$L\geqq 2,\ k+(4d+3)kf\log(SL)\leqq 2(4d+3)kf(SL)$$

we can take for example

$$C_{12}=(C_{11}(3C_9+C_6+C_7))^{(4d+3)(2f)}.$$

Now C_{13} has to satisfy

$$C_{13}^{h_1k_1\log h_2}\geqq 2^{\frac{1}{2}h_1k_1}+h_1\left(\frac{k_1}{2}+1\right)2^{\frac{1}{2}h_1k_1}(2h_2)^{h_1((k_1/2)+2)}$$

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i.e. $\quad C_{13}^{h_1k_1\log h_2} \geqq 2^{3h_1k_1} h_2^{h_1(k_1+4)}$ if $h_1 \geqq 2$ and $k_1 \geqq 2$

i.e. $\quad C_{13}^{h_1k_1\log h_2} \geqq h_2^{16h_1(k_1+1)}$ i.e. $C_{13}^{h_1k_1\log h_2} \geqq h_2^{32h_1k_1}$.

Hence we can take $C_{13} = e^{32}$.

Let C_{14} and C_{15} be large constants and E a small positive constant. We set

$$(14) \quad h = [C_{14}\log(3S)],\ L = [h^{(1+nE)/f}],\ k = \left[\frac{1}{D} h^{E+(1+nE)/f}\right],\ D = 3^{n+1}d^2,$$

$$(15) \quad \begin{cases} h_1 = h,\ k_1 = k;\ h_2 = \left[\dfrac{h_1k_1}{C_{15}L}\right],\ k_2 = \left[\dfrac{k_1}{2}\right], \cdots \\ h_r = \left[\dfrac{h_{r-1}k_{r-1}}{C_{15}L}\right],\ k_r = \left[\dfrac{k_{r-1}}{2}\right], \cdots \qquad (r = 2, \cdots, \tilde{r}) \qquad \text{where} \\ \tilde{r} = \left[\dfrac{(n+f)((1+nE)/f)+E-1}{E}\right] + 2. \end{cases}$$

We choose C_{14}, C_{15} to satisfy

$$(16) \qquad 2^{\frac{1}{2}h_rk_r} \geqq C_{12}^{Lh_{r+1}+k\log SL}\{1+\beta C_{13}^{h_rk_r\log h_{r+1}}\} \qquad (r = 1, 2, \cdots, \tilde{r}-1)$$

where we fix β to be so small as

$$(17) \qquad \beta \leqq C_{13}^{-h_{\tilde{r}-1}k_{\tilde{r}-1}\log h_{\tilde{r}}}.$$

We have also to satisfy (3). It is easy to see that all these are satisfied by making C_{14}, C_{15} large in (14) and (15). Also we fix $\tilde{r}$ and see that $h_{r-1}k_{r-1}\log h_r$ is an increasing function of r by making C_{14} and C_{15} large. It is also easy to see that $h_{\tilde{r}-1}k_{\tilde{r}-1}\log h_{\tilde{r}}$ lies between two constant multiples of $(\log(3S))^{1+\varepsilon}$. Thus we see that (17) is false, i.e. the theorem is proved with S in place of H. The passage to H is trivial.

Added in proof. When this note was in the course oȷ publication Prof. N. I. Fieldman has proved much more than what is conjectured in the introduction.

References

[1] A. Baker, 'Linear forms in the logarithms of algebraic numbers', *Mathematika*, Vol. 13 (1966), pp. 204–216.

[2] C. L. Siegel, *Transcendental numbers* (Annals of Maths., Studies Number 16, Princeton University Press, 1949).

[3] N. I. Feldman, 'Улучщение оценки линеинои формы от логарифмов алгебраических чисел', *Math. Sbornik*, 77 (1968), pp. 423—436.

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