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COMPLEX BLOW-UP IN BURGERS' EQUATION: AN ITERATIVE APPROACH

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We show that for a given holomorphic noncharacteristic surface $\mathcal{S} \in \mathbb{C}^2$, and a given holomorphic function on $\mathcal{S}$, there exists a unique meromorphic solution of Burgers' equation which blows up on $\mathcal{S}$. This proves the convergence of the formal Laurent series expansion found by the Painlevé test. The method used is an adaptation of Nirenberg's iterative proof of the abstract Cauchy-Kowalevski theorem.

1. Introduction

A partial differential equation (PDE) is said to have the Painlevé property if all solutions are single-valued around all noncharacteristic holomorphic movable singularity manifolds, where movable means that the manifold's location depends on initial conditions. In practice, a necessary condition of the property is usually checked through formal power series expansion (see [11]). Here we show, through an iterative method in $\mathbb{C}^2$, that such series converge for Burgers' equation

$$u_t + uu_x = u_{xx}. \tag{1}$$

The Painlevé property has become a widely used indicator for integrability (see [**2, 3**]), meaning exact solvability through the inverse scattering method [**4, 1**] or linearisability through a transformation of variables. Burgers' equation is regarded as integrable because it can be linearised (to the Heat equation) by the Cole-Hopf transformation [**6, 5**]. Hence, according to Ablowitz, Ramani and Segur [**2, 3**] it should possess the Painlevé property. To check that it does, Weiss, Tabor and Carnevale [**11**] proposed that one should formally expand all solutions around an arbitrary noncharacteristic singularity manifold given by $\Phi(x,t) = 0$ in a power series with a leading term

$$u = \sum_{n=0}^{\infty} u_n(x,t)\Phi^{n+\alpha}, \tag{2}$$

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where α is to be found.

The expansion may be simplified by using the noncharacteristic nature of the singularity manifold $\Phi = 0$ which implies $\Phi_x \neq 0$. By the implicit function theorem (rescaling Φ if necessary) we have

$$\Phi = x - \xi(t), \tag{3}$$

near $\Phi = 0$, where $\xi(t)$ is an arbitrary function of t. Replacing x by $\xi(t)+\Phi$ throughout the series (2) we get a series in powers of $x-\xi(t)$ with coefficients u_n that are functions of t alone. Formal expansion then shows that $\alpha = -1$ and that the coefficient $u_2(t)$ is arbitrary. Hence the series (2) formally represents a meromorphic general solution described by two arbitrary functions of one variable, namely $\xi(t)$ and $u_2(t)$, near the singularity manifold.

Although widely used, there are two obvious deficiencies in this procedure. First, convergence is ignored. Second, the procedure yields only necessary consequences of the Painlevé property and makes no statement about whether these are sufficient.

In this paper, we overcome the first deficiency. Our aim is to develop a method that will generalise to all integrable PDEs. Here, we present a method that does generalise. An announcement of its generalisation to the Korteweg-deVries equation was made in [**8**]. Although Burgers' equation may be solved through the Heat equation, we present the details of our method for Burgers' equation here because of its value as a more transparent nonlinear example than the Korteweg-deVries equation.

The method we use is a generalisation of one given for the Painlevé equations (six classical nonlinear second-order ODEs) by Joshi and Kruskal [**7**]. They showed that each Painlevé equation could be recast as an integral equation suitable for iteration near movable singularities. Furthermore, the iteration of this equation has a fixed point which gives a meromorphic solution in a neighbourhood of each movable singularity.

In Section 2, we recast Burgers' equation as an integral equation that is suitable for iteration near a movable singularity to prove the following theorem.

Theorem 1. *Let S be a holomorphic surface in $\mathbb{C}^2$ given by $\{t = \xi(x)\}$. Then locally there exists a solution of Burgers' equation*

$$u_x + uu_t = u_{tt}, \tag{4}$$

which has the form

$$u(t,x) = -\frac{2}{t-\xi(x)} + h(t,x), \tag{5}$$

near S where $h(t,x)$ is holomorphic. Moreover,

$$\lim_{t\to\xi(x)} \left\{ u_t(t,x) - \frac{1}{2}\left(u(t,x) - \xi'(x)\right)^2 \right\} \tag{6}$$

is a holomorphic function of x, which can be given arbitrarily in advance.

Note that in keeping with the PDE literature, we have taken Burgers' equation to be given by (4). That is, the roles of t and x have been interchanged. Also, note that throughout the paper, (t, x) refers to a point in $\mathbb{C}^2$.

Our proof was influenced by the iteration proof of the abstract Cauchy-Kowalevski theorem given by Nirenberg [**10**]. After the completion of our work, we learnt of a different approach developed by Kichenassamy and Littman [**9**] for nonlinear Klein-Gordon equations.

2. Proof of the Theorem

In this section, we convert Burgers' equation to an integral equation suitable for iteration near $\mathcal{S}$, and prove the theorem stated above.

Let $f(x)$ be any analytic function. We begin by fixing our notation. Assume (without loss of generality) that the origin lies on $\mathcal{S}$. Let D be an open neighbourhood of the origin in $\mathbb{C}^2$ where

$$\widetilde{f}(t,x) := t\xi''(x) + f(x)$$

is holomorphic. We can straighten the surface $\mathcal{S}$ locally into the x-plane $\{t = 0\}$ by using a biholomorphism $(t,x) \mapsto (t - \xi(x), x) =: (\widetilde{t}, \widetilde{x})$, $u(x,t) \mapsto \widetilde{u}(\widetilde{t}, \widetilde{x})$. Notice that this changes Burgers' equation into

$$\widetilde{u}_{\widetilde{t}\widetilde{t}} = \big(\widetilde{u} - \xi'(x)\big)\widetilde{u}_{\widetilde{t}} + \widetilde{u}_{\widetilde{x}}. \tag{7}$$

It is sufficient to find a solution $\widetilde{u}$ having the form

$$\widetilde{u} = -\frac{2}{\widetilde{t}} + \widetilde{h},$$

where $\widetilde{h}$ is holomorphic such that

$$\lim_{\widetilde{t}\to 0}\left\{\widetilde{u}_{\widetilde{t}} - \frac{1}{2}\big(\widetilde{u} - \xi'(x)\big)^2\right\} = f(x).$$

In the following, we shall assume that $\mathcal{S}$ is already locally given by the plane $\{t = 0\}$. So Burgers' equation will be assumed to be (7).

To obtain a suitable integral equation, integrate (7) as though only the dominant terms that is, $\widetilde{u}_{\widetilde{t}\widetilde{t}}$, $\widetilde{u}\widetilde{u}_{\widetilde{t}}$, were present. Then, dropping the tildes, we get

$$u_t = \frac{1}{2}\left(u - \xi'(x)\right)^2 + \int_0^t dt\, u_x + f(x). \tag{8}$$

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Change variables to the reciprocal

$$U = \frac{1}{u - \xi'(x)}. \tag{9}$$

Then if U does not vanish in some neighbourhood off the x-plane, (8) gives

$$-U_t = FU \tag{10}$$

where

$$FU(t) := \frac{1}{2} + U(t)^2 \left(\int_0^t \partial_x \left(\frac{1}{U(\tau)} \right) d\tau + \widetilde{f}(t,x) \right), \tag{11}$$

is well defined. Integrate (10) once more to get

$$U = \mathcal{F}U, \tag{12}$$

where

$$\mathcal{F} := -\int_0^t FU(\tau)\, d\tau. \tag{13}$$

Conversely, if we find a fixed point U of the operator $\mathcal{F}$ then the corresponding $\widetilde{u} := 1/U + \xi'(t)$ will solve (7).

We shall study the iteration of the operator $\mathcal{F}$ for functions U of the form

$$U = -\frac{t}{2} + O(|t|^2).$$

Note that substitution of such a function into (12) reproduces a function of the same form.

Let $\mathcal{O}_0$ be an open neighbourhood of the origin in $\mathbb{C}$ and $d > 0$ be a real number. Then for $0 < s \leqslant 1$, define

$$\mathcal{O}_s := \{x \mid \text{ dist } (x, \mathcal{O}_0) < sd\}.$$

We assume $\mathcal{O}_0$ and d small enough that D contains the disk $\{0\} \times \mathcal{O}_1$. Define, for any number $a > 0$

$$D_a := \{(t,x) \in \mathbb{C}^2 \mid \exists\, 0 \leqslant s \leqslant 1 \text{ such that } |t| < a(1-s) \text{ and } x \in \mathcal{O}_s\}, \tag{14}$$

and assume a small enough that D_a is a subset of D. For any real number K and integer n, let

$$O_K^n(D_a) := \{U : D_a \to \mathbb{C} \mid U \text{ is holomorphic and } \quad \forall (t,x) \in D_a \;\; |U(t,x)| \leqslant K\,|t|^n\}.$$

These spaces denote remainder terms in Taylor expansions. Their union will be written as

$$O^n(D_a) := \bigcup_K O^n_K(D_a).$$

The function spaces in which we shall work are given by

$$B^K_a := \{U : D_a \to \mathbb{C} \mid U \text{ is holomorphic and } U = -t/2 + O^2_K(D_a)\},$$

equipped with the *sup*-norm on D_a. Our aim is to find a number $a > 0$ and a holomorphic function $U \in B^1_a$ that solves the fixed point equation (12). We accomplish this by showing that the sequence $\{U_n\}$ of iterates defined recursively by

$$U_0 = -\frac{t}{2}, \quad U_{n+1} = \mathcal{F}U_n,$$

converges to the desired fixed point of $\mathcal{F}$.

In general, our (Newton) iteration method consists of two stages, one linear, and the other nonlinear, where the linear part is given by the iteration of the Fréchet derivative of $\mathcal{F}$. However, for Burgers' equation it is sufficient to take this derivative to be zero. (This is not the case for the Korteweg-deVries equation.)

Our proof relies on the following lemmas. Proofs of these are given in subsections at the end of this section.

LEMMA 1. *Suppose a and K are given positive numbers such that $a < \min\{1/6, 1/(6K)\}$. If $U \in B^K_a$ then there is a holomorphic function $g : D_a \to \mathbb{C}$ such that*

$$U(t,x)\Big(-\frac{2}{t} + g(t,x)\Big) = 1$$

wherever $U \neq 0$. Moreover, $|g|$ is bounded by $6K$.

LEMMA 2. *Let $n \geqslant 0$, $a > 0$ and $K > 0$ be given numbers, $0 < \varepsilon < 1$, and $0 < a^* \leqslant a(1-\varepsilon)$. Assume that the holomorphic function $g : D_a \to \mathbb{C}$ satisfies for all $(t,x) \in D_a$*

$$|g(t,x)| \leqslant K\,|t|^n.$$

Then for all $(t,x) \in D_{a^*}$ we get

$$\left|\int_0^t \partial_x g(\tau,x)\,d\tau\right| \leqslant \frac{a}{d}\ln\left(\frac{1}{\varepsilon}\right) K\,|t|^n.$$

LEMMA 3. *Let $n \geqslant 1$, and suppose a, a^*, K, L are given positive numbers which satisfy*

$$K \geqslant \sup_D\{|\tilde{f}|, |f|\}, \quad a < \min\{1/6, 1/(6K), d\},$$
$$a^n < 1/(12L), \quad a^* \leqslant a\Big(1 - 2^{-(n+2)}\Big).$$

Assume that U_1, U_2 *are elements of* B_a^K *and their difference satisfies*

$$v := U_2 - U_1 \in O_L^{n+1}(D_a).$$

Then we have

$$\mathcal{F}U_2 - \mathcal{F}U_1 \in O_{10L}^{n+2}(D_{a^*}).$$

The last lemma is the key to the proof of our theorem. We shall apply it to the sequence $\{U_n\}$ in the sense that if the iterates U_{n-1} and U_n already agree up to order $n+1$, then the next pair of iterates U_n and U_{n+1} will agree up to order $n+2$.

PROOF OF THE THEOREM: Let

$$K \geqslant \sup_D\{|\tilde{f}(t,x)|, |f(t,x)|\}.$$

Note that this implies

$$\sup_D |t\xi''(x)| \leqslant 2K.$$

Now assume

$$0 < a_0 < \min\{1/11, 1/(11K), d\}.$$

(As always, a_0 is assumed to be sufficiently small such that $D_{a_0} \subset D$.) Moreover, define a sequence $\{a_n\}$ recursively by

$$a_n := a_{n-1}\Big(1 - 2^{-(n+2)}\Big).$$

We start the iteration in D_{a_0} with

$$U_0 := -\frac{t}{2}, \quad v_0 := \mathcal{F}U_0 - U_0.$$

Note that for all $(t,x) \in D_{a_0}$, v_0 is bounded by

$$\begin{aligned}|v_0| &= \Big|\int_0^t \big(t\xi''(x) + f(x)\big)\frac{t^2}{4}\Big| \\ &\leqslant \frac{5K\,|t|^3}{24} \leqslant \frac{Ka_0\,|t|^2}{4}.\end{aligned}$$

Let $L = Ka_0/4\,(\leqslant 1/44)$. We have $U_0 \in B_{a_0}^0$, $v_0 \in O_L^2(D_{a_0})$.

Now for the inductive step, suppose we have

$$U_{n-1} \in B_{a_{n-1}}^1, \quad v_{n-1} \in O_{L_{n-1}}^{n+1}(D_{a_{n-1}}),$$

where $L_{n-1} = 10^{n-1}L$. Define

(15) $$U_n := U_{n-1} + v_{n-1} = U_0 + \sum_{j=0}^{n-1} v_j.$$

We now show that $U_n \in B^1_{a_n}$. First note that $D_{a_n} \subset D_{a_j}$, $j = 0, \dots, n-1$. The induction hypothesis gives $v_j \in O^{j+2}_{L_j}(D_{a_n})$. That is, for all $0 \leqslant j \leqslant n-1$

(16) $$|v_j| \leqslant \left(\frac{10}{11}\right)^j L\, |t|^2 .$$

Hence we get

$$|U_n - U_0| \leqslant 11L\, |t|^2 ,$$

which implies that $U_n \in B^1_{a_n}$ because $11L < 1$.

Now we apply Lemma 3 (with a, a^*, L, v replaced by a_{n-1}, a_n, L_{n-1}, v_{n-1} respectively) to get an estimate on v_n. Note that the hypothesis

$$a^n_{n-1} \leqslant \frac{1}{12L_{n-1}}$$

follows from $a_{n-1} \leqslant a_0 \leqslant \min\bigl(1/(11), 1/(11K)\bigr)$ and the definition of L. Hence we get

$$v_n \in O^{n+2}_{L_n}(D_{a_n}).$$

The sequence $\{U_n\}$, $n = 0, 1, 2, \dots$, produced by the iteration process above, is contained in B^1_a where

(17) $$a := \lim_{n\to\infty} a_n = a_0 \prod_{n=0}^{\infty}\left(1 - 2^{-(n+2)}\right) > 0.$$

Consider now the limit of the sequence $\{U_n\}$. From (15), (at $n+1$) and (16), we have

(18) $$U_{n+1} - U_n = v_n \in O^2_{L(10/11)^n}(D_a).$$

Hence it follows that $\{U_n\}$ is a convergent sequence and that the limit

$$U := \lim_{n\to\infty} U_n$$

lies in B^1_a. Writing $\|\ \|$ for the *sup*-norm on D_a, we get for any positive integer n,

(19) $$\|\mathcal{F}U - U\| = \|\mathcal{F}U - \mathcal{F}U_n + U_{n+1} - U\|$$

(20) $$\leqslant \|\mathcal{F}U - \mathcal{F}U_n\| + \|U_{n+1} - U\| .$$

So by continuity of $\mathcal{F}$ we can conclude that a fixed point for $\mathcal{F}$

$$\mathcal{F}U = U$$

exists in B^1_a. $\square$

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2.1 Proof of Lemma 1.

Proof: The number a, in the definition of D_a, was assumed to be sufficiently small that U does not vanish off the x-plane. Let $(t,x) \in D_a$, with $t \neq 0$. Then we can define g to be

$$g(t,x) := \frac{1}{U(t,x)} + \frac{2}{t}.$$

The bounds on a, K, and $|t|$ give $|U(t,x)| > |t|/3$. Therefore,

$$\left|\frac{1}{tU(t,x)}\right| < \frac{3}{|t|^2}$$

and so

$$|g(t,x)| = \left|\frac{t + 2U(t,x)}{tU(t,x)}\right| \leqslant 6K.$$

Since g is then bounded and holomorphic off the x-plane, by Kistler's theorem (see Osgood [11]), it can be extended to all of D_a. ∎

2.2 Proof of Lemma 2.

Proof: Let $(t,x) \in D_a^*$, and $|\tau| < |t|$. We put

$$s' := 1 - \frac{|t|}{a^*}, \tag{21}$$

$$s(\tau) = 1 - \frac{|\tau|}{a}. \tag{22}$$

Note that $s(\tau) > s'$ and $D_{a^*} \subset D_a$ by the assumed properties of a^*, τ and t. Similarly, $x \in O_{s'}$, $\{\tau\} \times O_{s(\tau)} \subset D_a$. So we can apply the Cauchy estimate

$$\|\partial_x g(\tau,.)\|_{s'} \leqslant \frac{\|g(\tau,.)\|_{s(\tau)}}{d(s(\tau) - s')}$$

together with the hypothesis on g to get

$$\left|\int_0^t \partial_x g(\tau,x)\,d\tau\right| \leqslant K \left|\int_0^t \frac{|\tau|^n\,d\tau}{d(s(\tau) - s')}\right|,$$

because $x \in O_{s'}$. Using the substitution $\tau = rt$ we get

$$\frac{1}{s(\tau) - s'} \leqslant \frac{a^*}{|t|\,(1 - (1-\varepsilon)r)}$$

and so

$$\left|\int_0^t \partial_x g(\tau,x)\,d\tau\right| \leqslant \frac{a^*}{d} K\,|t|^n \int_0^1 \frac{r^n\,dr}{(1 - (1-\varepsilon)r)} \tag{23}$$

$$\leqslant \frac{a^*}{d} K\,|t|^n \frac{\ln(1/\varepsilon)}{1 - \varepsilon}. \tag{24}$$

∎

2.3 PROOF OF LEMMA 3.

PROOF: In the following, we shall drop references to D_a, that is, $O_K^n(D_a)$ will be written as O_K^n. Also wherever convenient, we shall denote an element of O_K^n by the set symbol O_K^n itself. Since $U_1 \in B_a^K$ we have, for all $(t,x) \in D_a$,

$$\frac{|t|}{2} - K|t|^2 \leqslant |U_1| \leqslant \frac{|t|}{2} + K|t|^2 .$$

By the given hypotheses on a, K, and t, we then get

$$\frac{|t|}{3} \leqslant |U_1| \leqslant \frac{2|t|}{3},$$

which implies $v/U_1 \in O_{3L}^n$. Now using $a^n \leqslant 1/(12L)$, we have

$$\frac{v}{U_1}\left(1 + 3La^n + 9L^2a^{2n} + 27L^3a^{3n} + \dots\right) \in O_{4L}^n,$$

implying that

$$\frac{1}{U_2} = \frac{1}{U_1}\frac{1}{1+v/U_1} = \frac{1}{U_1}(1 + O_{4L}^n).$$

By similar calculations, we get

$$v^2 \in O_L^{n+2}, \quad U_2^2 = U_1^2 + O_{3L}^{n+2}.$$

So by using Lemma 1, we get

$$FU_2 - FU_1 = O_{4/9}^2 \int_0^t \partial_x O_{12L}^{n-1} + O_{3L}^{n+2}\left(\int_0^t \partial_x O_{6K}^0 + \tilde{f}\right).$$

So far all estimates have been obtained in D_a. Now we apply Lemma 2, and thereby restrict our domain to D_{a^*}, to estimate the terms differentiated with respect to x in the above equations. To apply Lemma 2, note that $\varepsilon = 2^{-(n+2)}$ and that $a < d$. Then for any given integer $N \geqslant 0$ and given $k > 0$, we get

$$\int_0^t \partial_x O_k^N(D_a) \leqslant (n+2)O_k^N(D_{a^*}).$$

Recalling that $\left|\tilde{f}\right| < K$ and using the definition of $\mathcal{F}U$, we get

$$\mathcal{F}U_2 - \mathcal{F}U_1 = \int_t^0 \left\{FU_2(\tau) - FU_1(\tau)\right\} d\tau \tag{25}$$

$$= \int_t^0 \left\{(n+2)O_{16L/3}^{n+1} + (n+2)O_{18KL}^{n+2} + O_{3KL}^{n+2}\right\} d\tau \tag{26}$$

$$= \int_t^0 \left\{(n+2)O_{25L/3}^{n+1} + O_L^{n+1}\right\} d\tau \tag{27}$$

where the last line is obtained by using $aK < 1/6$ and the integrands after the first line are evaluated on D_{a^*}. Integration gives the desired result

$$\mathcal{F}U_2 - \mathcal{F}U_1 \in O_{10L}^{n+2}(D_{a^*}).$$

∎

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