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TRIPLE FACTORISATIONS: GROUP THEORETIC AND GEOMETRIC APPROACHES

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1. Introduction

Triple factorisations of groups G of the form $G = ABA$, for proper subgroups A and B, are fundamental in the study of Lie type groups, as well as in geometry. This thesis [1] is devoted to investigating such factorisations in the context of (permutation) group theory and geometry. The main results of this thesis appear in three papers (one is published [2] and others are under review). Some results in this thesis regarding triple factorisations of general linear groups, in particular the results in algebraic groups, are unpublished.

2. Group theoretic approach

A major part of this thesis introduces and develops a general framework for studying triple factorisations $G = ABA$ for finite groups G, especially *nondegenerate* ones where $G \neq AB$. We identify two necessary and sufficient conditions, called the geometric and restricted movement criteria, for subgroups A, B to satisfy $G = ABA$, in terms of the G-actions on the set Ω_A of right cosets of A and on the set Ω_B of right cosets of B.

THEOREM 2.1 [2, Theorem 1.1]. *The following are equivalent.*

(a) *$\mathcal{T} = (G, A, B)$ is a triple factorisation.*

(b) *(Geometric criterion) The set $\{Ab \mid b \in B\}$ intersects nontrivially each A-orbit in Ω_A.*

(c) *(Restricted movement criterion) The set $\{Ba \mid a \in A\}$ has restricted movement in the G-action on Ω_B.*

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The geometric criterion seems to be better known and it is the major tool used by M. Guidici and J. P. James (in a paper in preparation) to analyse triple factorisations $S_n = ABA$ of finite symmetric groups with A and B conjugate maximal subgroups. It is also the central tool in studying triple factorisations in this thesis and current studies. To my knowledge the second criterion is new, and we use it to analyse triple factorisations $\mathrm{GL}(V) = BAB$ with A (maximal) parabolic and B the stabiliser of a certain decomposition of V (see [1, Ch. 8]). Moreover, the restricted movement criterion leads to an order (upper) bound for $|G|$ in terms of $|A|$ and $|B|$ which is sharp precisely for the point–line incidence geometries of flag-transitive projective planes (see [2, Theorem 1.2]).

There are many ways to construct new triple factorisations from an original triple factorisation $\mathcal{T} = (G, A, B)$. For example, for a normal subgroup N of G, the *quotient* of $\mathcal{T}$ is $\mathcal{T}/N = (G/N, AN/N, BN/N)$; for overgroups $A \leq C \leq G$ and $B \leq D \leq G$, $\mathcal{T}' = (G, C, D)$ is called a *lift* of $\mathcal{T}$; for $H \leq G$, sometimes $\mathcal{T}$ restricts to a triple factorisation $\mathcal{T}|_H = (H, A \cap H, B \cap H)$, but not always. Although each quotient and lift of $\mathcal{T}$ is a triple factorisation, the same is not true for restrictions. Note that even for quotients and lifts, the properties of nondegeneracy or nontriviality need not be preserved (see [2, Sections 4–6]). We gave several conditions under which $\mathcal{T}/N$ inherits nondegeneracy. These lead us in particular to faithful G-actions on Ω_A and/or Ω_B, which is useful as we may then express problems about triple factorisations in the language of permutation groups. Faithfulness allows us to apply results in [4, 5] on subsets with restricted movement to get a nontrivial improvement to the trivially obtained upper bound $|G| \leq |A|^2|B|/|A \cap B|$ (see [2, Theorem 1.4]).

We study, in particular, *imprimitive triple factorisations* $\mathcal{T} = (G, A, B)$ where G acts imprimitively on Ω_A. Then $A < H < G$, for some H. If there is a maximal such subgroup H for which the lift (G, H, B) remains nondegenerate, then much can be learned from the lift. On the other hand, if the lift is degenerate, then $G = HB$, and H determines a block Δ for G in Ω_A containing A, and a block system Σ, together with induced permutation groups $G_0 := H^{\Delta}$ and $G_1 := G^{\Sigma}$ on Δ and Σ, respectively. We prove the following theorem.

Theorem 2.2 [2, Theorem 1.5]. *There exist triple factorisations $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_0 \wr \mathcal{T}_1$ of G_0, G_1 and $G_0 \wr G_1$, respectively, such that $\mathcal{T}_0 \wr \mathcal{T}_1$ is nondegenerate and either $\mathcal{T}_1$ is nondegenerate, or $\mathcal{T}_0$ is nondegenerate and $(\mathcal{T}_0 \wr \mathcal{T}_1)|_G$ is a nondegenerate lift of $\mathcal{T}$. Moreover, if B is maximal in G, then $(\mathcal{T}_0 \wr \mathcal{T}_1)|_G = \mathcal{T}$.*

This suggests that for understanding triple factorisations of finite groups (and their associated geometrical structures) a fundamental problem is to study/understand *primitive triple factorisations* $\mathcal{T} = (G, A, B)$ in which A is maximal and core-free in G (see [2, Section 7.2]).

3. Geometric approach

Each triple factorisation $G = ABA$ gives rise to a *collinearly complete* coset geometry $\mathsf{Cos}(G; A, B)$ (with A the stabiliser of a point p and B the stabiliser of a

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line ℓ incident with p) in which 'each pair of points lies on at least one line', and vice versa (see [3, Lemma 3]). Interchanging the roles of points and lines leads us to a dual completeness concept: a geometry is *concurrently complete* if 'each pair of lines is incident with at least one point'. In addition to the well-known examples (linear spaces, symmetric designs and projective spaces), we construct infinitely many new collinearly and/or concurrently complete spaces (see [1, Ch. 6–8]).

3.1. Parabolic triple factorisations and associated geometries. Let V be a vector space of dimension n over a field $\mathbb{F}$, and let m and k be positive integers less than n. Suppose that j is an integer such that $\max\{0, m + k - n\} \le j \le \min\{m, k\}$. An (m, k, j)-*projective space* or *subspace–subspace geometry*, denoted by $\mathsf{Proj}^{j}_{(m,k)}(V)$, is a point–line incidence structure whose point (respectively, line) set $\mathbb{P}$ (respectively, $\mathbb{L}$) is the set of m-subspaces (respectively, k-subspaces) of V, and a point $U \in \mathbb{P}$ is incident with a line $W \in \mathbb{L}$ if and only if $\dim(U \cap W) = j$. These new rank-two geometries are related to *projective spaces* and *Grassmannian geometries* (see [1, Section 6.2]). We obtained a necessary and sufficient condition for which $\mathsf{Proj}^{j}_{(m,k)}(V)$ is collinearly complete.

THEOREM 3.1 [1, Theorem 1.2.1]. *The projective space $\mathsf{Proj}^{j}_{(m,k)}(V)$ is collinearly complete if and only if $j \le k/2 + \max\{0, m - (n/2)\}$.*

It is of importance to know if an incidence structure and its dual share the same (geometric) property. We then investigated the conditions under which $\mathsf{Proj}^{j}_{(m,k)}(V)$ has both, one, or neither of the completeness properties (see [1, Theorems 1.2.2 and 6.8 and Corollary 6.7]).

Since a collinearly (respectively, concurrently) complete $\mathsf{Proj}^{j}_{(m,k)}(V)$ gives rise to a *parabolic triple factorisation* $\mathrm{GL}(V) = ABA$ (respectively, $\mathrm{GL}(V) = BAB$) in which A and B are (maximal) parabolic subgroups of $\mathrm{GL}(V)$, we may translate our geometric results into group theoretic results about parabolic triple factorisations of $\mathrm{GL}(V)$ (see [1, Theorem 6.2 and Section 6.5]).

3.2. Subspace-bisection triple factorisations and associated geometries. This thesis is also devoted to studying the *subspace-bisection* triple factorisation $\mathrm{GL}(V) = ABA$ and its dual $\mathrm{GL}(V) = BAB$ where A is (maximal) parabolic and $B = \mathrm{GL}(V)_{\{V_1,V_2\}}$ is the (setwise) stabiliser of a decomposition $V = V_1 \oplus V_2$ with $\dim(V_1) = \dim(V_2)$ (see [1, Theorems 7.2 and 8.1]).

The associated rank-two geometry of a subspace-bisection triple factorisation is called a *subspace-bisection* geometry or an (m, k, J)-*projective* space, where J is the pair (k_1, k_2) of dimensions $k_j = \dim(U \cap V_j)$ for an m-subspace U, and $k_1 \le k_2$. For this geometry, denoted by $\mathsf{Proj}^{J}_{(m,k)}(V)$, the point set $\mathbb{P}$ is the set of all m-subspaces of V, the line set $\mathbb{L}$ is the set of all bisections $\{V_1, V_2\}$ of V such that $V = V_1 \oplus V_2$ and $\dim(V_1) = \dim(V_2) = k$, and the incidence between $U \in \mathbb{P}$ and $\{V_1, V_2\} \in \mathbb{L}$ is given by $(\dim(U \cap V_1), \dim(U \cap V_2)) = (k_1, k_2)$ or (k_2, k_1). Theorem 3.2 gives in geometrical language the existence of subspace-bisection geometries with specified collinear and concurrent completeness (see [1, Theorem 8.2]).

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TABLE 1. Completeness properties of $\mathsf{Proj}^{J}_{(m,k)}(V)$ for $k \geq 2$.

Conditions on m, k, and q	Collinearly complete	Concurrently complete
$q \geq 5$, or $\lvert m-k \rvert > 1$, or $q = 2$, $(m,k) = (1,1), (1,2), (3,2)$, or $q = 3$, $\lvert m-k \rvert = 1$, or $q = 4$, $\lvert m-k \rvert = 1$, $m = k \leq 2$	Yes	Yes
$m < \dfrac{n+4}{4}$ or $m > \dfrac{3n-4}{4}$	Yes	No
$\dfrac{n+4}{4} \leq m \leq \dfrac{3n-4}{4}$	No	No

THEOREM 3.2. *Let m and k be positive integers, and let V be a vector space of dimension $n = 2k$ over a finite field $\mathbb{F}$ of size q with $m < n$. If $n \geq 4$ and (m, k, q) satisfies the conditions of one of the lines of Table 1, then there exists a J such that $\mathsf{Proj}^{J}_{(m,k)}(V)$ has the completeness properties of the last two entries in that line of Table 1.*

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