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Quasi-isometry and Plaque Expansiveness

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Abstract. We show that a partially hyperbolic diffeomorphism is plaque expansive (a form of structural stability for its center foliation) if the strong stable and unstable foliations are quasi-isometric in the universal cover. In particular, all partially hyperbolic diffeomorphisms on the 3-torus are plaque expansive.

A diffeomorphism f of a compact Riemannian manifold M is called *partially hyperbolic*[1] if there are constants $\lambda < \hat{\gamma} < 1 < \gamma < \mu$, $C \geq 1$, and a Tf-invariant splitting of TM such that for every $x \in M$, $T_xM = E^s_x \oplus E^c_x \oplus E^u_x$, where

$$\begin{aligned} \|df^n(x)v^s\| &\leq C\lambda^n\|v^s\| && \text{for } v^s \in E^s_x,\ n > 0;\\ C^{-1}\hat{\gamma}^n\|v^c\| \leq \|df^n(x)v^c\| &\leq C\gamma^n\|v^c\| && \text{for } v^c \in E^c_x,\ n > 0;\\ C^{-1}\mu^n\|v^u\| &\leq \|df^n(x)v^u\| && \text{for } v^u \in E^u_x,\ n > 0. \end{aligned}$$

It is known that the unstable, center, and stable subbundles E^u, E^c, and E^s are Hölder continuous and that there are unique Hölder continuous foliations W^u and W^s tangent to E^u and E^s, respectively [4, 8, 9]. In general, E^c, $E^{cu} = E^c \oplus E^u$, and $E^{cs} = E^c \oplus E^s$ do not integrate to foliations, but when they are integrable, the system may be called *dynamically coherent* [1, 6].

When there is a foliation tangent to E^c, we may ask whether that foliation is plaque expansive. The notion of plaque expansiveness was first introduced by Hirsch, Pugh, and Shub, and they showed cases where the property holds [8]. While the definition involves the plaquations of foliations (hence, the name), an equivalent definition can be given without their explicit mention.

Let $f\colon M \to M$ be a diffeomorphism with an invariant foliation W. Let $d(x, y)$ be the distance given by the Riemannian metric on M; if x and y lie on the same leaf of W, let $d_W(x, y)$ denote the distance determined by pulling the metric from M back to W.

An *ϵ-pseudo orbit* of f that respects W is a bi-infinite sequence $\{x_n\}$ in M such that for all $n \in \mathbb{Z}$, $f(x_{n-1})$, and x_n lie on the same leaf of W and $d_W(f(x_{n-1}), x_n) < \epsilon$. The diffeomorphism f is *plaque expansive* with respect to W if for every $\epsilon_0 > 0$ there exists $\epsilon > 0$ such that the following holds.

If $\{x_n\}$ and $\{y_n\}$ are ϵ-pseudo orbits of f that respect W and $d(x_n, y_n) < \epsilon$ for all $n \in \mathbb{Z}$, then x_0 and y_0 lie on the same leaf of W and $d_W(x_0, y_0) < \epsilon_0$.

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[1] This definition of partial hyperbolicity is sometimes called *absolute* partial hyperbolicity, in constrast to *relative* partial hyperbolicity, where the values $\lambda < \hat{\gamma} < 1 < \gamma < \mu$ are not true constants but may vary depending on the point $x \in M$.

Intuitively, to say f is plaque expansive means that it acts to pull apart points that lie on different leaves, and a modest amount of sliding the points along the leaves cannot overcome this pulling.

For a partially hyperbolic system, it is not known if the existence of a center foliation necessarily implies its plaque expansiveness. When $f: M \to M$ is partially hyperbolic and the center foliation W^c_f is plaque expansive, it enjoys a form of structural stability. Every f' C^1-close to f also has a center foliation $W^c_{f'}$, and (f, W^c_f) and $(f', W^c_{f'})$ are *leaf conjugate*, *i.e.,* there exists a homeomorphism $h: M \to M$ such that if L is a leaf of W^c_f, then $h(L)$ is a leaf of $W^c_{f'}$ and $hf(L) = f'h(L)$ [8].

The distance along a leaf can, in general, be quite large when compared to the absolute distance on the manifold M. In some cases the situation is better on the universal cover $\tilde{M}$. A foliation W of a simply-connected Riemannian manifold $\tilde{M}$ is *quasi-isometric* if there are $a, b > 0$ such that

$$d_W(x, y) \le a \cdot d(x, y) + b$$

for any x, y on a leaf on W [7].

Theorem *Let f be a partially hyperbolic diffeomorphism of a compact Riemannian manifold M. Suppose the stable W^s and unstable W^u foliations of f are quasi-isometric in the universal cover $\tilde{M}$. Then the distributions E^c, E^{cs}, and E^{cu} integrate uniquely to plaque expansive foliations.*

Remark This theorem is inspired by the proof of dynamical coherence under the same hypotheses due to M. Brin [1]. One great advantage of establishing plaque expansiveness for a partially hyperbolic diffeomorphism f is that perturbations of f are also plaque expansive and therefore dynamically coherent. In many cases, however, one can show that the hypothesis of quasi-isometry is stable under perturbation, so plaque expansiveness is not needed to establish stable dynamical coherence. The result is still useful, though, in establishing that f is leaf conjugate to its neighbors, and engenders hope of answering the open question of whether all dynamically coherent, partially hyperbolic systems are plaque expansive.

Proof That the distributions are uniquely integrable is shown by Brin [1]. We will prove that W^{cs} is plaque expansive. The case for W^{cu} is similar, and then it follows from the definition that the intersection W^c of the foliations W^{cs} and W^{cu} is also plaque expansive.

Given $\epsilon > 0$ small, let $\{x_n\}$ and $\{y_n\}$ be ϵ-pseudo orbits respecting W^{cs} such that for all $n \in \mathbb{Z}$, $d(x_n, y_n) < \epsilon$. There exist paths $\alpha_n, \beta_n: [0, 1] \to M$ of length at most ϵ and tangent to E^{cs} such that

$$\alpha_n(0) = f(x_{n-1}), \quad \alpha_n(1) = x_n,$$
$$\beta_n(0) = f(y_{n-1}), \quad \beta_n(1) = y_n.$$

Because x_0 and y_0 are close together, by sliding y_0 along its W^{cs} leaf, we may assume,

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without loss of generality, that x_0 and y_0 lie on the same local unstable leaf.[2] To establish plaque expansiveness, we can then show that $x_0 = y_0$.

The diffeomorphism f lifts from M to its universal cover $\tilde{M}$ where, by abuse of notation, we still call it f. Lift x_0 and y_0 to $\tilde{x}_0, \tilde{y}_0 \in \tilde{M}$ so that the two points still lie close together. Then, inductively for $n > 0$, lift the paths α_n, β_n on M to paths $\tilde{\alpha}_n, \tilde{\beta}_n$ on $\tilde{M}$ such that $\tilde{\alpha}_n(0) = f(\tilde{x}_{n-1})$ and $\tilde{\beta}_n(0) = f(\tilde{y}_{n-1})$, and define $\tilde{x}_n := \tilde{\alpha}_n(1)$ and $\tilde{y}_n := \tilde{\beta}_n(1)$. Because the lengths of α_n and β_n are small and $\tilde{M}$ is locally identified with M, it follows that $d(\tilde{x}_n, \tilde{y}_n) = d(x_n, y_n) < \epsilon$.

As f is partially hyperbolic (on both M and $\tilde{M}$), there are constants $1 < \gamma < \mu$ and $C \geq 1$ such that

$$\|df^n(x)v^{cs}\| \leq C\gamma^n \|v^{cs}\| \quad \text{for } v^{cs} \in E^{cs}_x \text{ and } n > 0,$$

and

$$C^{-1}\mu^n \|v^u\| \leq \|df^n(x)v^u\| \quad \text{for } v^u \in E^u_x \text{ and } n > 0.$$

Consequently, as the $\tilde{\alpha}_n$ are tangent to E^{cs},

$$\text{length}(f^k \circ \tilde{\alpha}_n) \leq C\gamma^k \, \text{length}(\tilde{\alpha}_n),$$

so

$$d\big(f^k(f(\tilde{x}_n)), f^k(\tilde{x}_{n+1})\big) < C\gamma^k \epsilon,$$

and

$$\begin{aligned} d(f^n(\tilde{x}_0), \tilde{x}_n) &\leq \sum_{k=0}^{n-1} d(f^{k+1}(\tilde{x}_{n-k-1}), f^k(\tilde{x}_{n-k})) \\ &< \sum_{k=0}^{n-1} C\gamma^k \epsilon = C\frac{\gamma^n - 1}{\gamma - 1}\epsilon. \end{aligned}$$

Similarly, $d(f^n(\tilde{y}_0), \tilde{y}_n) < C\frac{\gamma^n-1}{\gamma-1}\epsilon$, so

$$\begin{aligned} d(f^n(\tilde{x}_0), f^n(\tilde{y}_0)) &\leq d(f^n(\tilde{x}_0), \tilde{x}_n) + d(\tilde{x}_n, \tilde{y}_n) + d(\tilde{y}_n, f^n(\tilde{y}_0)) \\ &< \left(2C\frac{\gamma^n - 1}{\gamma - 1} + 1\right)\epsilon. \end{aligned}$$

On the other hand, $\tilde{x}_0$ and $\tilde{y}_0$ lie on the same unstable leaf, so

$$d_u(f^n(\tilde{x}_0), f^n(\tilde{y}_0)) \geq C^{-1}\mu^n d_u(\tilde{x}_0, \tilde{y}_0),$$

[2] Because W^{cs} and W^u are uniformly transverse, there is a constant $0 < c < \frac{1}{2}$ such that if $d(x_0, y_0) < c\epsilon$, then there is a point z_0 on the unstable leaf of x_0, and the center-stable leaf of y_0 and $d_u(x_0, z_0)$, $d_{cs}(y_0, z_0)$, and $d_{cs}(f(y_0), f(z_0))$ are each less than $\epsilon/2$. Therefore, a $c\epsilon$-pseudo orbit is turned into an ϵ-pseudo orbit by replacing y_0 with z_0.

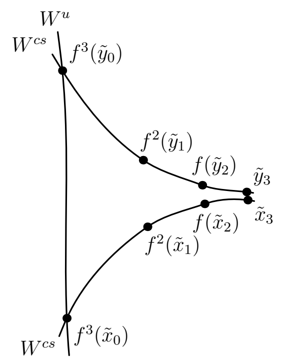

Figure 1: The invariant manifolds through $f^n(\tilde{x}_0)$ and $f^n(\tilde{y}_0)$ for $n = 3$.

where d_u is distance measured along the unstable leaf. By quasi-isometry

$$d(f^n(\tilde{x}_0), f^n(\tilde{y}_0)) \geq (d_u(f^n(\tilde{x}_0), f^n(\tilde{y}_0)) - b)/a \geq (C^{-1}\mu^n d_u(\tilde{x}_0, \tilde{y}_0) - b)/a.$$

Since $\gamma < \mu$, these two estimates are irreconcilable for large $n > 0$ unless $d_u(\tilde{x}_0, \tilde{y}_0) = 0$. This means that $\tilde{x}_0 = \tilde{y}_0$, so $x_0 = y_0$, and plaque expansiveness is proved. ■

M. Brin, D. Burago, and S. Ivanov have shown that all partially hyperbolic diffeomorphisms on the 3-torus are dynamically coherent [2, 3, 5]. Since this is proved by establishing quasi-isometry as in the hypotheses of the preceeding theorem, it yields the following.

Corollary *All partially hyperbolic systems on the 3-torus are plaque expansive.*

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