Using $\frac{1}{2}R \geqslant r$ this becomes $27R^4 \geqslant 16\Delta^2$ or $\frac{1}{2}tR \geqslant \sqrt{\Delta}$, as claimed. Also, applying the AM-GM inequality to $\frac{1}{r_a}$, $\frac{1}{r_b}$ and $\frac{1}{r_c}$ we get

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} \geqslant 3\sqrt[3]{\frac{1}{r_a}\cdot\frac{1}{r_b}\cdot\frac{1}{r_c}}$$

which by Lemmas 9 and 11 gives $\Delta^2 \geqslant 27r^4$ or $\sqrt{\Delta} \geqslant tr$. So $\frac{t}{2}R \geqslant \sqrt{\Delta} \geqslant tr$. [WEIFFTTIE].

Theorem 9: We have $4s^3 \geqslant 27\Delta R$. [WEIFFTTIE].

Proof: Since $2s = a + b + c \geqslant 3\sqrt[3]{abc}$, the result follows at once by Lemma 10.

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108.34 One sharpening of the Garfunkel-Bankoff inequality and some applications

Garfunkel-Bankoff inequality

For a triangle *ABC* we use the notation $\sum \tan^2 \frac{A}{2}$ and $\prod \sin \frac{A}{2}$ for the cyclic sum and the cyclic product respectively. Then we have

Theorem 1: In any triangle *ABC* holds

$$\sum \tan^2 \tfrac{A}{2} \geqslant 2 - 8 \prod \sin \tfrac{A}{2} + \left(1 - 8 \prod \sin \tfrac{A}{2}\right) \prod \tan^2 \tfrac{A}{2}. \qquad (1)$$

Proof: By the well-known identities

$$\sum \tan^2 \frac{A}{2} = \frac{(4R + r)^2}{s^2} - 2, \quad \prod \sin \frac{A}{2} = \frac{r}{4R}, \quad \prod \tan \frac{A}{2} = \frac{r}{s}$$

where R, r and s are the circumradius, inradius and semiperimeter of the triangle, inequality (1) is transformed to

$$\frac{(4R + r)^2}{s^2} - 2 \geqslant 4 - \frac{2r}{R} + \frac{r^2}{s^2}\left(1 - \frac{2r}{R}\right)$$

and this is equivalent to

$$s^2 \leqslant \frac{R(4R + r)^2 - r^2(R - 2r)}{2(2R - r)}. \tag{2}$$

To prove the last inequality we use the right-hand side of the fundamental inequality of a triangle [1, 5.10], [2]

$$2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R(R - 2r)} \leqslant s^2 \tag{3}$$
$$\leqslant 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)}.$$

Simple algebra gives

$$\frac{R(4R + r)^2 - r^2(R - 2r)}{2(2R - r)} = \frac{8R^3 + 4R^2r + r^3}{2R - r}.$$

We prove that

$$2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)} \leqslant \frac{8R^3 + 4R^2r + r^3}{2R - r}.$$

We put $t = \frac{r}{R}, 0 < t \leqslant \frac{1}{2}$. Then the last inequality can be rewritten as

$$2 + 10t - t^2 + 2(1 - 2t)\sqrt{1 - 2t} \leqslant \frac{8 + 4t + t^3}{2 - t}$$

and it is true since

$$\left[\frac{8 + 4t + t^3}{2 - t} - (2 + 10t - t^2)\right]^2 - \left[2(1 - 2t)\sqrt{1 - 2t}\right]^2$$
$$= \frac{8t^3(1 - 2t)^2}{(2 - t)^2} \geqslant 0.$$

The classic Garfunkel-Bankoff inequality for a triangle *ABC* is [3]

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geqslant 2 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{4}$$

By the simple inequality $1 - 8\prod\sin\frac{1}{2}A \geqslant 0$, it is stronger than the well-known inequality $\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geqslant 1$.

It is obvious that Theorem 1 gives a sharpening of the Garfunkel-Bankoff inequality (4).

Refinements of Kooi's and Finsler-Hadwiger inequality

As applications of Theorem 1 we obtain refinements of Kooi's and the Finsler-Hadwiger inequality. Garfunkel-Bankoff inequality (4) is equivalent to the well-known Kooi's inequality [3, 4, 5]

$$s^2 \leqslant \frac{R(4R + r)^2}{2(2R - r)}, \tag{5}$$

and they are both equivalent to [6]

$$a^2 + b^2 + c^2 \geqslant 4\Delta\sqrt{3 + \frac{R - 2r}{R}} + (a - b)^2 + (b - c)^2 + (c - a)^2. \tag{6}$$

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Kooi's inequality has important applications in triangle geometry; for example, its variant is used in the proof of the inequality [7]

$$2r \leqslant IL_A + IL_B + IL_C \leqslant R$$

which bounds the sum of the distances of the incentre from the inarc centres of the triangle [8]. The astute reader may have noticed that the inequality (2) in the proof of Theorem 1

$$s^2 \leqslant \frac{R(4R + r)^2 - r^2(R - 2r)}{2(2R - r)}$$

is a refinement of Kooi's inequality (5). This refinement is better than the refinement given in [6]

$$s^2 \leqslant \frac{R(4R + r)^2}{2(2R - r)} - \frac{r^2(R - 2r)}{4R},$$

since obviously

$$\frac{R(4R + r)^2 - r^2(R - 2r)}{2(2R - r)} \leqslant \frac{R(4R + r)^2}{2(2R - r)} - \frac{r^2(R - 2r)}{4R}.$$

It is interesting to note that (by Euler's inequality $R \geqslant 2r$) the inequality (6) is a sharpening of the famous Finsler-Hadwiger inequality [9, 10, 11]

$$a^2 + b^2 + c^2 \geqslant 4\Delta\sqrt{3} + (a - b)^2 + (b - c)^2 + (c - a)^2.$$

So it is natural to ask if the sharpening of the Garfunkel-Bankoff inequality (1) has some similar form which is a sharpening of inequality (6).

Indeed, by the well-known identity $ab + bc + ca = s^2 + r(4R + r)$, we have

$$a^2 + b^2 + c^2 - \left[(a - b)^2 + (b - c)^2 + (c - a)^2\right] = 4r(4R + r).$$

Hence (2) is equivalent to the following refinement of the Finsler-Hadwiger inequality

$$a^2 + b^2 + c^2 \geqslant 4\Delta\sqrt{3 + \frac{R - 2r}{R} + \frac{r^2(R - 2r)}{Rs^2}} + (a - b)^2 + (b - c)^2 + (c - a)^2,$$

which is an improvement of (6).

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108.35 Analogues of the 12 pentagons theorem for other families of polyhedra

A well-known result in the theory of buckyballs is the "12 pentagons" theorem. This states that if a convex polyhedron consists entirely of pentagonal and hexagonal faces where three meet at each vertex, then there are necessarily exactly 12 pentagons (with no such restriction on the number of hexagons). The truncated icosahedron comprising 12 pentagons and 20 hexagons gives the structure of buckminsterfullerene, C_{60}, but less symmetrical closed higher fullerenes such as C_{2n}, $35 \leqslant n \leqslant 45$ have also been found; here, the subscripts in C_{60} and C_{2n} specify the number of vertices of the polyhedron.

The aim of this short Note is to point out analogues for the 12 pentagons theorem for two other families of polyhedra. Suppose that a polyhedron consists of A regular a-sided polygonal faces and B regular b-sided faces, where $a \neq b$ and three faces meet at each vertex. Then, if F, V, E denote the number of faces, vertices and edges, we have

$$A + B = F \text{ and } Aa + Bb = 2E = 3V \text{ (meaning that } V \text{ is always even).}$$