ON WARD'S PERRON-STIELTJES INTEGRAL

RALPH HENSTOCK

Introduction. In the paper **(5)**, Ward defines an integral of Perron type of a finite function f with respect to another finite function g, where g need not be of bounded variation. There arise two problems, (a) and (b) below, that have not been dealt with in **(5)**.

If $f = j$ at a countable number of points everywhere dense in (a, b), where f and j are both integrable with respect to g, then $f - j$ can be nonzero on a large set of points of (a, b). For example, if g is continuous and of bounded variation the countable number of points can be neglected in the integration and we can have $f \neq j$ everywhere else. But g is more rigidly fixed when we know its values on an everywhere dense set, if the integral exists. For example, if g is of bounded variation, and so continuous except at an at most countable set of points, we can only vary the values of g at a countable set of points. More generally, we have problem

(a) *If f is integrable with respect to g, and with respect to h, over the closed interval $[a, b]$, where $g = h$ at points everywhere dense in $[a, b]$, what are the properties of the difference $g - h$ and the set of points where the difference is not zero?*

This question is partially answered by Theorems 1 and 2, and we obtain the following result.

Let $\bar{E}_\epsilon$ be the closure of the set of u for which

$$|g(u) - h(u)| \geqslant \epsilon, \qquad a \leqslant u \leqslant b. \tag{1}$$

Then f must be *VBG* and continuous on[1] $\bar{E}_\epsilon$, and $mf(\bar{E}_\epsilon) = 0$.

However, if f is integrable with respect to g in $[a, b]$, and if $g - h$ satisfies (1) and is 0 at an everywhere dense set of points in $[a, b]$, it does not follow that f is integrable with respect to h in $[a, b]$. For example, take $g = 0$ and suppose that each set E_ϵ contains only a finite number of points and so has no limit-points. Then every function f is trivially *VBG* and continuous on $\bar{E}_\epsilon = E_\epsilon$, and $f(\bar{E}_\epsilon)$ contains only a finite number of points. But if the set of points where $h \neq 0$ does not satisfy Theorem 3 (9), (10), (11), with j replaced by h, it follows by Theorem 3 that there is a finite function f for which the Perron-Stieltjes integral of f with respect to h over $[a, b]$ does not exist. See the example of Theorem 5 (38) in §4.

There is another question of integrability, namely,

(b) *What are the properties of g in order that all bounded Baire[2] functions f are integrable with respect to g in $[a, b]$?*

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[1]I.e., when we use only the points of $\bar{E}_\epsilon$.

[2]A Baire (Borel-measurable) function is any function that can be obtained from continuous functions by using repeated limits.

Question (b) is partially answered in **(2)**, Theorem 2, and we give the complete answer in Theorem 3 of the present paper.

1. Notation.

We suppose that all functions considered are defined and finite in $a \leqslant u \leqslant b$, this interval being denoted by $[a, b]$. The existence of an integral or limit is taken to mean its existence as a finite number. If the limits exist,

$$f(u-) = \lim_{v \to u, a \leqslant v < u \leqslant b} f(v), \quad f(u+) = \lim_{v \to u, a \leqslant u < v \leqslant b} f(v).$$

Integral signs preceded by (LS), (PS), denote respectively the Lebesgue-Stieltjes and Perron-Stieltjes integrals, and we put

$$P(v, w) \equiv P(f, g; v, w) \equiv (PS) \int_v^w f(u)\, dg(u),$$

$f(E) \equiv \{f(u): u \in E\}$ where E is a set contained in $[a, b]$. A point v in $[a, b]$ is a point of *infinite variation* on $[a, b]$ of the function f if, for each open interval (ξ, η) containing v, the function f is not of bounded variation on

$$[\xi, \eta] \cap [a, b].$$

It follows that *the set W of points of infinite variation on $[a, b]$ of f is closed.* For if v is not in W there is an open interval (ξ, η) containing v, such that f is of bounded variation on

$$[\xi, \eta] \cap [a, b],$$

and then (ξ, η) is contained in CW.

The symbols E', $\bar{E}$, CE, mE denote respectively the derived set, the closure, the complement, and the measure of a set E in $[a, b]$. The *interior* of E is the largest open set contained in E.

2. The examination of question (a)

THEOREM 1. *If $P(f, g; a, b)$ and $P(f, h; a, b)$ exist, and if $g = h$ at points everywhere dense in $[a, b]$, then for all v, w in $a \leqslant v < w \leqslant b$,*

$$P(f, g; v, w) = P(f, h; v, w) + [f(g - h)]_v^w.$$

Proof. It is enough to assume that $h \equiv 0$, so that $g = 0$ at points everywhere dense in $[a, b]$. Let M_1 and M_2 be a major and a minor function, in Ward's sense, of f with respect to g in $[a, b]$ and take u in $[a, b]$. Then there is a $\delta_1(u) > 0$ depending on u, M_1, M_2, such that

$$(2) \qquad [M_1]_u^\xi \geqslant f(u)[g]_u^\xi \geqslant [M_2]_u^\xi, \qquad 0 \leqslant \xi - u \leqslant \delta_1(u),$$

$$(3) \qquad [M_1]_u^\xi \leqslant f(u)[g]_u^\xi \leqslant [M_2]_u^\xi, \qquad 0 \geqslant \xi - u \geqslant -\delta_1(u).$$

As in **(2)**, §2, the proof of Theorem 1, we can prove that in each $[v, w]$ there is a finite number of points

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$v = \alpha_0 = u_1 < \alpha_1 < \ldots < \alpha_n = u_n = w, \alpha_{p-1} \leqslant u_p \leqslant \alpha_p \quad (p = 2, \ldots, n-1),$

such that

$$g(\alpha_p) = 0 (p = 1, \ldots, n-1), \quad \alpha_p - \alpha_{p-1} < \delta_1(u_p) \qquad (p = 1, \ldots, n).$$

Thus (2), (3) are satisfied with $u = u_p$, $\xi = \alpha_p$, and $u = u_p$, $\xi = \alpha_{p-1}$, respectively, and we obtain

$$[M_1]_v^w = \sum_{p=1}^{n} [M_1]_p \geqslant [fg]_v^w \geqslant \sum_{p=1}^{n} [M_2]_p = [M_2]_v^w,$$

where $[M]_p$ stands for

$$M(\alpha_p) - M(\alpha_{p-1}).$$

Thus as $P(v, w)$ exists, the Theorem must be true for $h \equiv 0$, and so generally.

THEOREM 2. *If, for all u in $a \leqslant u \leqslant b$,*

$$P(f, g; a, u) = [fg]_a^u, \tag{4}$$

then (5) *f is VBG and continuous on $\bar{E}_\epsilon$, and* (6) *$mf(\bar{E}_\epsilon) = 0$, where E_ϵ is the set of u for which*

$$|g(u)| \geqslant \epsilon, \quad a \leqslant u \leqslant b, \ \epsilon > 0.$$

COROLLARY. *If* (4) *is true, and if $\bar{E}_\epsilon$ contains an interval $[\xi, \eta]$ for some $\epsilon > 0$, then f is constant in $[\xi, \eta]$.*

From Theorem 2 Corollary we can easily prove Theorem 1 of **(2)**.

To prove Theorem 2 let $a \leqslant u < v \leqslant b$ and let M_3, M_4 be arbitrary major and minor functions of f with respect to g in Ward's sense, and write $\chi_1 \equiv M_3 - M_4$. Then χ_1 is monotone increasing. Now, for fixed u and for sufficiently small and positive $v - u$, both functions

$$f(u)[g]_u^v, P(u, v)$$

lie between

$$[M_3]_u^v, \ [M_4]_u^v,$$

so that

$$|P(u, v) - f(u)[g]_u^v| \leqslant [\chi_1]_u^v.$$

Substituting in the value of $P(u, v)$ from (4) we obtain

$$|g(v)[f]_u^v| \leqslant [\chi_1]_u^v.$$

Hence there is a $\delta_2(u) > 0$ such that if

$$u \in E_\epsilon', v \in E_\epsilon, \ 0 < v - u < \delta_2(u),$$

we have

$$|[f]_u^v| \leqslant \epsilon^{-1}[\chi_1]_u^v. \tag{7}$$

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Similarly for $v < u$. If

$$w \in \bar{E}_\epsilon, \quad 0 < w - u < \delta_2(u),$$

then there is a v satisfying

$$v \in E_\epsilon, \quad 0 < v - u < \delta_2(u),$$

and arbitrarily near to w, so that by (7),

$$|[f]_u^w| \leqslant |[f]_u^v| + |[f]_v^w| \leqslant \epsilon^{-1}[\chi_1]_u^v + \epsilon^{-1}|[\chi_1]_v^w|,$$

$$(8) \qquad |[f]_u^w| \leqslant \epsilon^{-1}[\chi_1]_u^{w+} \leqslant \epsilon^{-1}[\chi_1]_a^b,$$

$$\limsup_{w \to u} |[f]_u^w| \leqslant \epsilon^{-1}[\chi_1]_a^b, \ \lim_{w \to u} f(w) = f(u),$$

as $\chi_1(b) - \chi_1(a)$ is arbitrarily small.

Similar results hold for

$$w < u, u \in E_\epsilon', w \in \bar{E}_\epsilon, w \to u,$$

so that f is continuous when we only use the points of the derived set of E_ϵ. As the other points of $\bar{E}_\epsilon$ are isolated, f is continuous on $\bar{E}_\epsilon$.

To show that f is *VBG* on $\bar{E}_\epsilon$ we use the method of the first part of the proof of (**5**, p. 592, Lemma 6) and we employ only points of $\bar{E}_\epsilon$. The relevant inequality is the first one in (8).

To prove (6) we first add $\theta(u - a)$ to $\chi_1(u)$ if necessary, to ensure that χ_1 is strictly increasing. The constant $\theta > 0$ can be arbitrarily small. Then as in (**5**, p. 581, Lemma 3) we prove from the first inequality of (8), and the similar inequality when $w < u$, that

$$m^*f(\bar{E}_\epsilon) \leqslant 2\epsilon^{-1}[\chi_1]_a^b,$$

where m^* denotes outer measure. The factor 2 occurs because of the $w+$ in (8). As the right-hand side is arbitrarily small we obtain (6).

To prove the Corollary we note that by (5), f is continuous on $[\xi, \eta]$. Thus if $f([\xi, \eta])$ contains two distinct points it contains the whole interval between the points. This is impossible by (6).

3. The integrability of Perron-Stieltjes integrals. In this section we prove two theorems, completely answering question (b). We begin with a lemma needed in the proof of the converse of Theorem 3.

LEMMA. *Let F be a sequence $\{I_n\}$ of open intervals, and let H_p be the set of points of $[a, b]$ lying in at most p intervals of F. Then all the intervals I_n covering the points of H_p can be put into at most $3p$ sets of non-overlapping intervals.*

We can define a sequence $\{\xi_q\}$ of points of H_p such that their closure contains H_p. Each interval I_n covering a point of H_p will then also cover at least one ξ_q, and conversely. Thus we need only consider the intervals covering the ξ_q.

We put the qth interval of the sequence $\{I_n\}$ that covers ξ_1 into the set S_q. Then $1 \leqslant q \leqslant p$, as ξ_1 lies in H_p. Suppose that the intervals I_n covering $\xi_1, \ldots, \xi_{r-1}$ have been arranged into sets $S_q (1 \leqslant q \leqslant 3p)$ of non-overlapping intervals, and let ξ_r lie between ξ_s and ξ_t for $s < r$, $t < r$, with no $\xi_q (q < r)$ between ξ_s and ξ_t. Then there are at most p intervals I_n covering ξ_s, and at most p intervals I_n covering ξ_t, so that at least p of the sets $S_1, \ldots, S_{3p}$, say $T_1, \ldots, T_p$, will be free from intervals I_n that cover ξ_s or ξ_t, and so will contain no interval lying in (ξ_s, ξ_t). The intervals I_n covering ξ_r that have not already been put into sets S_q, cannot cover ξ_s nor ξ_t, and so must lie between ξ_s and ξ_t. We can therefore put these intervals into some or all of the sets $T_1, \ldots, T_p$.

Similarly if

$$\xi_r < \min_{q<r} \xi_q \text{ or } \xi_r > \max_{q<r} \xi_q,$$

in which case one of ξ_s, ξ_t is missing. Hence the result is true for $\xi_1, \ldots, \xi_r$. It is true for ξ_1 and hence true in general.

THEOREM 3. *If, for a given function j, for all bounded Baire functions f defined in $[a, b]$, and for all u in $[a, b]$, the integral $P(f, j; a, u)$ exists equal to*

$$[fj]_a^u,$$

then the set of points u in $a < u < b$, where $j(u) \neq 0$, can be divided into two sequences $\{u_n\}$ and $\{d_n\}$, with the properties

$$\sum_{n=1}^{\infty} |j(u_n)| < \infty; \tag{9}$$

(10) *surrounding each d_n there is an open interval $I(d_n) \equiv (\underline{d}_n, \bar{d}_n)$ contained in (a, b) such that each point of $[a, b]$ can lie in an at most finite number of the $I(d_n)$;*

(11) *there is a monotone increasing bounded function χ such that*

$$\chi(\bar{d}_n+) - \chi(d_n) \geqslant |j(d_n)|, \ \chi(d_n) - \chi(\underline{d}_n-) \geqslant |j(d_n)|.$$

Conversely, if j satisfies (9), (10), (11), *and if f is bounded in $[a, b]$, then $P(f, j; a, u)$ exists and is equal to*

$$[fj]_a^u,$$

for all u in $a \leqslant u \leqslant b$.

To begin the proof of the first part of Theorem 3 we replace g by j in Theorem 2, obtaining from (5) that f is continuous on $\bar{E}_\epsilon$, where E_ϵ is the set in which $|j| \geqslant \epsilon$. But, for each u in $[a, b]$, the set of bounded Baire functions f includes the function equal to 0 in $[a, u)$, equal to 1 at u, and equal to 2 in $(u, b]$. Hence each point of $\bar{E}_\epsilon$ must be isolated, and E_ϵ is finite. This is true for each $\epsilon > 0$. Hence taking $\epsilon^{-1} = 1, 2, \ldots$, we obtain

(12) $j \neq 0$ *only at a countable set of points* $\{w_n\}$,

(13) $j(w_n) \to 0$ *as* $n \to \infty$.

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Also, as E_1 is finite,

(14) *j is bounded.*

We now wish to find a strictly increasing function χ and a function $\delta > 0$' defined for all u in $a \leqslant u \leqslant b$, such that for $u - \delta < w < u < v < u + \delta$, $a \leqslant w < v \leqslant b$,

$$[\chi]_u^v \geqslant |j(v)|, \tag{15}$$

$$[\chi]_w^u \geqslant |j(w)|. \tag{16}$$

There is in Ward's sense a major function $P(f, j; a, u) + \chi_2(u)$ of f with respect to j in $[a, b]$, where χ_2 is monotone increasing and bounded in $[a, b]$, with $\chi_2(a) = 0$. Thus, if we substitute in the value of $P(f, j; a, u)$, we find that for $a \leqslant u \leqslant b$ and for some $\delta_3 = \delta_3(u) > 0$, using Ward's definition of a major function,

$$[\chi_2]_u^v \geqslant j(v)[f]_v^u \qquad (u < v < u + \delta_3,\ a < v \leqslant b), \tag{17}$$

$$[\chi_2]_w^u \geqslant j(w)[f]_u^w \qquad (u > w > u - \delta_3,\ a \leqslant w < b). \tag{18}$$

We now take $f = -\operatorname{sgn} j$, where $\operatorname{sgn} a \equiv |a|/a (a \neq 0)$, $\operatorname{sgn} 0 = 0$. Then if χ_3, δ_4 are the corresponding χ_2, δ_3, and if the u of (17) does not lie in $\{w_n\}$, so that $j(u) = 0, f(u) = 0$, we obtain, for $u < v < u + \delta_4$, $a < v \leqslant b$,

$$[\chi_3]_u^v \geqslant |j(v)|. \tag{19}$$

Similarly let χ_4, δ_5 be the corresponding χ_2, δ_3 when for f we take $\operatorname{sgn} j$, and let the u of (18) lie outside the sequence $\{w_n\}$ so that $j(u) = 0, f(u) = 0$. Then

$$[\chi_4]_w^u \geqslant |j(w)|, \qquad u > w > u - \delta_5,\ a \leqslant w < b. \tag{20}$$

By (13), $j(w_n \pm) = 0$. Thus if we put

$$\chi_5(u) = \sum_{w_n < u} 2^{-n} \ (u \notin \{w_n\}) = \chi_5(w_p -) + 2^{-2p} \qquad (u = w_p,\ p = 1, 2, \ldots)$$

we obtain

$$\chi_5(w_p +) - \chi_5(w_p) = 2^{-2p} > 0 = |j(w_p +)|,$$

$$\chi_5(w_p) - \chi_5(w_p -) = 2^{-2p} > 0 = |j(w_p -)|,$$

and there is a number $\delta_p = \delta(w_p)$ such that $\chi_5(u)$ satisfies (15) and (16) at $u = w_p$, with χ replaced by χ_5 and δ by δ_p.

Using (19), (20) also, we see that to obtain (15), (16) for all u in $a \leqslant u \leqslant b$ and a strictly increasing function χ, we need only take

$$\chi(u) \equiv \chi_3(u) + \chi_4(u) + \chi_5(u) + u - a.$$

We now define the points d_n in (a, b) as those for which

$$|j(d_n)| > \chi(d_n +) - \chi(d_n), \quad |j(d_n)| > \chi(d_n) - \chi(d_n -). \tag{21}$$

The other points $\{u_n\}$ of $\{w_n\}$ then give

$$\sum_{n=1}^{\infty} |j(u_n)| \leqslant \sum_{n=1}^{\infty} \{\chi(u_n+) - \chi(u_n-)\} \leqslant [\chi]_a^b < \infty,$$

so that (9) is satisfied.

If $u < d_n < u + \delta(u)$ for some u, d_n, we have (15) with $v = d_n$. Let $\underline{d}_n$ be the upper bound of all $u < d_n$ satisfying (15) for fixed $v = d_n$. If there is no such u, put $\underline{d}_n = a$. Then

$$\chi(d_n) - \chi(\underline{d}_n-) \geqslant |j(d_n)|, \tag{22}$$

while if $d_n > u > \underline{d}_n$, we have

$$\chi(d_n) - \chi(u) < |j(d_n)|. \tag{23}$$

By (14), j is bounded, so that we can take a convenient finite value for $\chi(a-)$ to fit the cases when $\underline{d}_n = a$. From (21), (22), $\underline{d}_n < d_n$.

Similarly we can define $\bar{d}_n > \underline{d}_n$ such that

$$\chi(\bar{d}_n+) - \chi(d_n) \geqslant |j(d_n)|, \tag{24}$$

while if $d_n < u < \bar{d}_n$, we have

$$\chi(u) - \chi(d_n) < |j(d_n)|. \tag{25}$$

Results (22), (24) prove (11). We now suppose that (10) is false, so that a point u of $[a, b]$ lies in an infinity of the open intervals

$$I(d_n) \equiv (\underline{d}_n, \bar{d}_n) \subseteq (a, b).$$

Obviously $u \neq a$, $u \neq b$. Also by (23), (25), (13),

$$\chi(\bar{d}_n-) - \chi(\underline{d}_n+) \leqslant 2|j(d_n)| \to 0$$

as $n \to \infty$. Hence as χ is strictly increasing, $\underline{d}_n \to u$ and $\bar{d}_n \to u$, for the subsequence of n for which $\underline{d}_n < u < \bar{d}_n$. Hence the corresponding subsequence of $\{d_n\}$ also tends to u, so that for certain $v \to u$,

$$|\chi(v) - \chi(u)| < |j(v)|.$$

This result contradicts (15) or (16). Hence (10) is true, and the first part of Theorem 3 has been proved.

We now prove the converse. Let the discontinuities of χ in $[a, b]$ occur at the points $v_n (n = 1, 2, \ldots)$. Then we have

$$\sum_{n=1}^{\infty} \{\chi(v_n+) - \chi(v_n-)\} \leqslant [\chi]_{a-}^{b+} < \infty,$$

so that, given $\epsilon > 0$, there is an integer n_0 such that

$$\sum_{n=n_0}^{\infty} \{\chi(v_n+) - \chi(v_n-)\} < \epsilon. \tag{26}$$

Then there is an integer n_1 such that, for $n > n_1$, d_n is not one of the points $v_q (q = 1, \ldots, n_0 - 1)$.

We now let F in the Lemma be the family of intervals $I(d_n)$, and we take p so large that

(27) $$m\chi\{[a, b] - H_p\} < \epsilon.$$

This is possible since by (10),

$$[a, b] = \bigcup_{p \geqslant 0} H_p.$$

By the Lemma there are $3p$ sets S_q of non-overlapping intervals $I(d_n)$ that together cover $H_p - H_0$. There is an integer $t > n_1$, and depending on ϵ, such that for each q in $1 \leqslant q \leqslant 3p$,

(28) $$\sum\{\chi(\bar{d}_n+) - \chi(\underline{d}_n-)\} < \epsilon/(3p),$$

where the sum is taken over those intervals of S_q with $n > t$, as the sum for $n > 0$ is not greater than $\chi(b) - \chi(a)$. The integer t can also be chosen, by (9), so that

(29) $$\sum_{n>t} |j(u_n)| < \epsilon.$$

Let S be the set formed from those intervals of the S_q with $n > t$ and $1 \leqslant q \leqslant 3p$. Then

$$\{[a, b] - H_p\} \cup S$$

is a union of intervals. For if u lies in $[a, b] - H_p$ let J be the intersection of the first $(p + 1)$ intervals $I(d_n)$ covering u. Then J is open and contains u, and

$$J \subseteq [a, b] - H_p.$$

We add an at most countable number of points, if necessary, to obtain from $\{[a, b] - H_p\} \cup S$ a union U of open non-abutting intervals, and we put

(30) $$\chi_6(u) \equiv \sum\nolimits_1\{\chi(\beta+) - \chi(\alpha-)\} + \epsilon(u - a)/(b - a) + \sum\nolimits_2 2|j(u_n)|,$$

where $\sum_1$ denotes the summation over the intervals (α, β) of $U \cap (a, u)$, changing $\beta+$ to β if $\beta = u$; and $\sum_2$ denotes the summation over all $n > t$ such that $u_n < u$, adding $|j(u_p)|$ if $p > t$ and $u = u_p$. Then χ_6 is strictly increasing, and from (26), (27), (28), (29),

(31) $$[\chi_6]_a^b < 6\epsilon.$$

Now, by definition, the points of H_0 are not covered by any interval $I(d_n)$. If $n > t$ and if $I(d_n)$ covers a point of $H_p - H_0$, then $I(d_n)$ will lie in one of the S_q, and so in S, and so in U. It follows that $\chi(d_n) - \chi(\underline{d}_n-)$ will occur in $\sum_1$ for $u = d_n$. If $n > t$ and if $I(d_n)$ does not cover a point of $H_p - H_0$, then $I(d_n)$ will lie entirely within $[a, b]-H_p$, and so in U, and again, $\chi(d_n)-\chi(\underline{d}_n-)$ will occur in $\sum_1$ for $u = d_n$. Thus by (30),

(32) $$\chi_6(d_n) - \chi_6(\underline{d}_n-) \geqslant \chi(d_n) - \chi(\underline{d}_n-) \geqslant |j(d_n)| \quad (n > t).$$

Similarly for the result with $\bar{d}_n+$, so that χ_6 satisfies (11) for all $n > t$.

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Now each point u of $[a, b]$ lies in an at most finite number of the $I(d_n)$, say $I(\xi_1), \ldots, I(\xi_r)$, where $\xi_1, \ldots, \xi_r$ depend on u. Let the sequence $\{\eta_n\}$ include all points of the sequences $\{u_n\}$, $\{d_n\}$, $\{\underline{d}_n\}$, $\{\bar{d}_n\}$, and let u be outside $\{\eta_n\}$. We take $\delta_6 = \delta_6(u) > 0$ so that $(u - \delta_6, u + \delta_6)$ does not include

$$u_1, \ldots, u_t, \ d_1, \ldots, d_t, \ \xi_1, \ldots, \xi_r.$$

Then by (32), for $u < d_n < \min(b, u + \delta_6)$,

$$\chi_6(d_n) - \chi_6(u) \geqslant \chi(d_n) - \chi(\underline{d}_n-) \geqslant |j(d_n)|,$$

since $\underline{d}_n > u$. If u_n lies in $u < u_n < \min(b, u + \delta_6)$ then $n > t$, and by (30),

$$\chi_6(u_n) - \chi_6(u) \geqslant |j(u_n)|.$$

If v is neither in $\{u_n\}$ nor in $\{d_n\}$ then for $u < v < \min(b, u + \delta_6)$,

$$\chi_6(v) - \chi_6(u) > 0 = |j(v)|.$$

Hence, if u is outside $\{\eta_n\}$,

(33) $$\chi_6(v) - \chi_6(u) \geqslant |j(v)|, \qquad u < v < \min(b, u + \delta_6).$$

Similarly for all v in $u > v > \max(a, u - \delta_6)$. To deal with the case when $u = \eta_n$ for some n, we put

$$\chi_7(u) = \chi_6(u) + \sum_{\eta_n < u} \epsilon 2^{-n} \qquad (u \notin \{\eta_n\}),$$
$$\chi_7(\eta_p) = \chi_7(\eta_p -) + \epsilon . 2^{-2p} \qquad (p = 1, 2, \ldots).$$

As in the part of the proof that follows (20), we obtain a strictly increasing function χ_7 satisfying (33) for all u, and, for suitable $\delta_7 > 0$, for

$$u < v < \min(b, u + \delta_7),$$

and similarly for $v < u$. By (31),

(34) $$[\chi_7]_a^b < 7\epsilon.$$

Now suppose that $|f| \leqslant A$. We put

$$M_5(u) \equiv [fj + 2A\,\chi_7]_a^u.$$

Then from (33),

$$[M_5]_u^v - f(u)[j]_u^v = [f]_u^v j(v) + 2A[\chi_7]_u^v$$
$$\geqslant [f]_u^v j(v) + 2A|j(v)| \geqslant 0 (u < v < \min(b, u + \delta_7)).$$

The inequalities are reversed when $u > v > \max(a, u - \delta_7)$, so that M_5 is a major function, in Ward's sense, for f with respect to j in $[a, b]$. Similarly

$$M_6(u) \equiv [fj - 2A\,\chi_7]_a^u$$

is a minor function, and by (34),

$$M_5(b) - M_6(b) = 4A[\chi_7]_a^b < 28A\epsilon.$$

By choice of $\epsilon > 0$ this can be made arbitrarily small. Hence there exists

$$P(f, j; a, u) = [fj]_a^u$$

proving the converse in Theorem 3.

THEOREM 4. *If, for a given function g, and for all bounded Baire functions f in* $[a, b]$, *the integral* $P(f, g; a, b)$ *exists, then*

(35) $g(u-)$ *exists in* $a < u \leqslant b$, $g(u+)$ *exists in* $a \leqslant u < b$, *and both are of bounded variation in those ranges; and the function j satisfies Theorem* 3(9), (10), (11), *where*

(36) $j(a) = g(a) - g(a+)$, $j(b) = g(b) - g(b-)$,

$$j(u) = g(u) - \tfrac{1}{2}\{g(u+) + g(u-)\} \quad (a < u < b).$$

Conversely, if g satisfies (35), *and if the j defined by* (36) *satisfies Theorem* 3(9), (10), (11), *and if f is a bounded Baire function in* $[a, b]$, *then* $P(f, g; a, b)$ *exists and is equal to*

$$\{g(b) - g(b-)\} f(b) + \{g(a+) - g(a)\} f(a) + \sum_{a<u<b} f(u)\{g(u+) - g(u-)\}$$
$$+ (LS) \int_a^b f(u)\, dg_c(u),$$

where

$$g_c(v) = g(v-) - \sum_{a<u<v} \{g(u+) - g(u-)\} (a < v \leqslant b),\ g_c(a) = g(a+).$$

The result (35) is proved in **(2)**, Theorem 2, using only the hypotheses of the present Theorem 4. From (35) we see that $g - j$ is of bounded variation in $[a, b]$, so that $P(f, g - j; a, b)$ exists. By hypothesis $P(f, g; a, b)$ exists. Hence so does $P(f, j; a, b)$. Also, from (35),

$$\lim_{w \to u-} g(w-) = g(u-),\ \lim_{w \to u-} g(w+) = g(u-),$$

so that from (36), $j(u-) = 0$. Similarly $j(u+) = 0$. If E_ϵ is the set in $a \leqslant u \leqslant b$ where $j \geqslant \epsilon > 0$, and if E_ϵ has a limit-point ξ, then

$$\limsup_{w \to \xi} j(w) \geqslant \epsilon.$$

This contradicts $j(\xi-) = 0 = j(\xi+)$, so that E_ϵ has no limit-points and so must contain only a finite number of points. Thus taking $\epsilon = n^{-1} (n = 1, 2, \ldots)$, the set where $j > 0$ is at most countable. Similarly the set where $j < 0$ is at most countable. Hence by Theorem 1,

$$P(f, j; a, u) = [fj]_a^u$$

so that the first part of Theorem 3 completes the first part of Theorem 4.

To prove the converse in Theorem 4 we need only use the converse in Theorem 3 and the fact that $g - j$ is of bounded variation in $[a, b]$, and (**4**, pp. 208–209, Theorem 8.1)).

4. The points of infinite variation of j. We now suppose that

(37) $$j(u-) = 0 \ (a < u \leqslant b),\ j(u+) = 0 (a \leqslant u < b).$$

Let T_1 be the union of the interiors of all closed intervals J contained in $[a, b]$, such that $P(f, j; J)$ exists for all bounded Baire functions f, adding one or both of a, b to T_1 according as one or both of $[a, a+\epsilon]$, $[b-\epsilon, b]$ are intervals J for some $\epsilon > 0$. Also put $T = CT_1 \cap [a, b]$. Let W be the set of points of infinite variation of j.

THEOREM 5. *If J is a closed interval, there is a function j satisfying* (37), *such that*

(38) $$J = W, J = T.$$

If Q is a closed nowhere dense set, there is a function j satisfying (37), *such that*

(39) $$T = W = Q,$$

and there is another function j satisfying (37), *such that*

(40) $$T = \phi,\ W = Q,$$

where ϕ is the empty set.

We begin by supposing that

(41) *the set of points $\{v_n\}$ in $[a, b]$ can be put into one-one correspondence with the points $(2q+1)2^{-p}$ $(0 \leqslant q < 2^{p-1};\ p = 1, 2, \ldots)$, the order of the points being preserved.*

Then we define $j(v_n) = p^{-1}$ when v_n corresponds to $(2q+1)2^{-p}$, and $j(u) = 0$ when u is outside $\{v_n\}$. Such a j satisfies (37), as only a finite number of $j(v_n)$ are greater than any given positive ϵ. If a χ exists satisfying Theorem 3(10), (11), we can suppose that

(42) $$[\chi]_a^b = B,\ [\chi]_u^v \geqslant v - u,$$

for all $a \leqslant u < v \leqslant b$. Then the set of intervals $I(d_n)$ for which

$$\chi(\bar{d}_n +) - \chi(\underline{d}_n -) \geqslant 2/p$$

must be such that any non-overlapping and non-abutting subset has at most $\frac{1}{2}pB$ members. Hence any non-overlapping subset has at most pB members. The points of $\{v_n\}$ that are not in $\{d_n\}$ are points $\{u_n\}$ satisfying Theorem 3(9). It follows that for some integer r, there is a point d_{01} in $\{d_n\}$ with

$$\chi(\bar{d}_{01}+) - \chi(\underline{d}_{01}-) \geqslant 2/r$$

such that $I(d_{01})$ contains at least two different points ξ_1, ξ_2 of $\{v_n\}$ corresponding to points $(2q+1)2^{-r}$ with the given r. Hence

$$Q_1 \equiv I(d_{01}) \cap \{v_n\} \cap (\xi_1, \xi_2)$$

is not empty, as there are points of $\{v_n\}$ between each two points of $\{v_n\}$ by (41). Since ξ_1, ξ_2 lie at a positive distance from the ends of $I(d_{01})$, and since

$$\bar{d}_n - \underline{d}_n \leqslant \chi(\bar{d}_n+) - \chi(\underline{d}_n-) \to 0$$

as $n \to \infty$, by (42), (10), and the bounded variation of χ, there is an n_2 such that if $n > n_2$ and $d_n \in Q_1$ then

$$I(d_n)' \subseteq I(d_{01}).$$

We can now repeat the construction, defining $d_{02}, d_{03}, \ldots$, and

$$I(d_{01}) \supseteq I(d_{02}) \supseteq \ldots \supseteq I(d_{0n}) \supseteq \ldots.$$

As $\{d_{0n}\}$ is a subsequence of $\{d_n\}$ we have $\bar{d}_{0n} - \underline{d}_{0n} \to 0$ as $n \to \infty$, and hence for a point u in (a, b), $I(d_{0n}) \to u$. This u lies in an infinity of the intervals $I(d_n)$, contrary to (10). Hence in this case there is no χ satisfying Theorem 3(10), (11), so that for some bounded Baire function f, $P(f, j; a, b)$ cannot exist.

A similar result is true for each interval J containing points of $\{v_n\}$ in its interior, by (41). Hence

$$T \supseteq \{v_n\}', \tag{43}$$

since by (41) each point of $\{v_n\}'$ is the limit-point of a sequence of intervals of T.

To prove (38) let J be the interval $[\alpha, \beta]$. Then the points

$$v_n = \alpha + (\beta - \alpha)(2q + 1)2^{-p} \qquad (0 \leqslant q \leqslant 2^{p-1}; p = 1, 2, \ldots)$$

will satisfy (41), and by (43),

$$\{v_n\}' = J = T.$$

To prove (39) we take the points v_n to be the centres of the intervals I_n complementary to Q in $[a, b]$. That $\{v_n\}$ so defined satisfies (41), can be shown by (**3**, p. 57, Proposition 20). Then by (43),

$$T = \{v_n\}' = Q,$$

and (39) is proved.

To prove (40) let d_{1n} be the centre of the nth interval $J_n \equiv (\alpha_n, \beta_n)$ complementary to Q in $[a, b]$. Next, let d_{2n1} and d_{2n2} be the centres of (α_n, d_{1n}) and (d_{1n}, β_n), respectively, calling these two points the *points of the second stage*. We continue this process of continued bisection to the stage n^2. If d_{pnq} is a point of the pth stage in J_n put $j(d_{pnq}) = n^{-2}\, 2^{-p}$, with $(\underline{d}_{pnq}, \bar{d}_{pnq})$ as the $(p - 1)$th stage interval with centre d_{pnq}. If this is done for $1 \leqslant p \leqslant n^2$ $(n = 1, 2, \ldots)$ with $j = 0$ elsewhere, and if

$$\chi(\bar{d}_{pnq}) - \chi(\underline{d}_{pnq}) \equiv n^{-2}\, 2^{-p}$$

we have

$$\chi(\beta_n) - \chi(\alpha_n) = n^{-2}/2,$$

and the construction of a strictly increasing χ satisfying the required conditions is possible. Each point of $[a, b]$ lies in an at most finite number of the $I(d_{pnq})$, as it lies in at most n^2 in the interval J_n. Finally, over all the points d_{pnq} in J_n,

$$\sum |j(d_{pnq})| = \tfrac{1}{2}.$$

Thus T is empty and $W = Q$, proving (40).

THEOREM 6. *Let j satisfy* (37), *with T, W as defined just before Theorem* 5. *Then:*
(44) *T is perfect;*
(45) *$W \supseteq T$;*
(46) *The interior of W is contained in T;*
(47) *If $Q \subseteq R$ are two perfect sets in $[a, b]$ with the same interior, there is a j such that $T = Q$, $W = R$;*
(48) *In order that T should be empty, it is necessary but not sufficient that the set of points $\{d_n\}$ of Theorem* 3 *should be scattered.*[3]

COROLLARY 1. *If W is at most countable then T is empty and $P(f, j; a, b)$ exists.*

COROLLARY 2. *No structural property of W can be both necessary and sufficient for T to be empty.*

By construction, T is closed. Thus to prove (44) we have only to show that T has no isolated points. Suppose on the contrary that v is an isolated point of T. Then there are points α, β, such that $\alpha < v < \beta$, with $[\alpha, v)$ and $(v, \beta]$ in T_1. Putting

$$v_n = v - (v - \alpha)/(n + 1),$$

we see that

$$P_n = P(f, j; v_n, v_{n+1})$$

exists for each n and each bounded Baire function f. By hypothesis $j = 0$ except at an at most countable set of points, so that by Theorem 1,

$$P_n = f(v_{n+1})\, j(v_{n+1}) - f(v_n)\, j(v_n).$$

Hence for each $\epsilon > 0$ there is an increasing function χ_8 such that

$$[fj]_\alpha^u + \chi_8(u), \quad [fj]_\alpha^u - \chi_8(u)$$

are a major and a minor function, respectively, in $\alpha \leqslant u < v$, in Ward's sense, with

$$\chi_8(v_{n+1}) - \chi_8(v_n) \leqslant \epsilon\, 2^{-n}, \quad \chi_8(u) - \chi_8(\alpha) \leqslant 2\epsilon.$$

If we set $\chi_8(v) - \chi_8(v-) = \epsilon$, then as f is bounded, say by A, and $j(v-) = 0$, we have

$$[\chi_8]_u^v \geqslant \epsilon \geqslant 2A\,|j(u)| \geqslant [f]_v^u j(u)$$

[3]"Zerstreute" (F. Hausdorff), "separierte" (G. Cantor), "clairsemé" (A. Denjoy).

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for $v - \delta_8 < u < v$ and some $\delta_8 > 0$. Hence

$$[fj + \chi_8]_u^v \geqslant f(v)[j]_u^v, \text{ and } [fj]_\alpha^u + \chi_8(u)$$

is a major function in $[\alpha, v]$. Similarly

$$[fj]_\alpha^u - \chi_8(u)$$

is a minor function in $[\alpha, v]$, and

$$[\chi_8]_\alpha^v \leqslant 3\epsilon.$$

Thus $P(\alpha, v)$ exists. Similarly $P(v, \beta)$ exists, so that by (**5**, pp. 585–586), property I, $P(\alpha, \beta)$ exists, and v does not lie in T, contrary to hypothesis.

If j is of bounded variation in the closed interval J then $P(f, j; J)$ exists. Hence (45) is true. Further, if W contains an interval $[\xi, \eta]$ let J be a sub-interval. If $P(f, j; J)$ exists for each bounded Baire function f, then by Theorem 1, and then Theorem 3(10), the set of points $\{d_n\}$ in J has the Denjoy property (see, e.g., (**1**), chap. III, p. 140). Hence it is scattered, and so is nowhere dense in J. It follows that W must be nowhere dense in J, as the points $\{u_n\}$ of Theorem 3 add nothing to W. This contradicts the fact that J is contained in W, so that $[\xi, \eta]$ is contained in T, and T contains the interior of W, proving (46).

To prove (47) we first take the closure J_n of the nth interval of the interior of Q, and construct a function j_n satisfying (37), (38) with $J = J_n$. Then we construct a function j_0 satisfying (37), (39), with the Q there replaced by the present Q less its interior. Finally we construct a function j_{-1} satisfying (37), (40), with the Q there replaced by the closure of $R - Q$. Then

$$\sum_{n=-1}^{\infty} j_n$$

satisfies the conditions of (47).

For (48), if T is empty then by Theorems 1 and 3(10), the set of points $\{d_n\}$ in $[a, b]$ has the Denjoy property, and so is scattered. But for the function satisfying (37), (39), the set of points $\{d_n\}$ in $[a, b]$ is also scattered, so that (48) follows.

Corollary 1 follows from (44), (45), and Corollary 2 from (47).

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Queen's University,
Belfast